ENGR-233: Applied Advanced Calculus Winter 2014

Midterm test solutions

Variant A.

Problem 1. Find the parametric equation of the line of intersection of the two planes:

$P_1 : x + y - 8z = 4$ and $P_2 : 3x - y + 4z = 0$.

Solution: Let us eliminate the variables x and y from the equations. Adding the equations for P_1 and P_2 we get the equation $4x - 4z = 4$, or $x - z = 1$; so, $x = z + 1$. Substituting this into the equation for P_2 we get $y = 3x + 4z = 3z + 3 + 4z = 7z + 3$. So, we have expressed both x and y in terms of z . Hence, the parametric equation of the line of intersection of P_1 and P_2 are: $x = t + 1,\ y = 7t + 3,\ x = t$.

Problem 2. Position vector of a moving particle is given by $\boldsymbol{r}(t) = \left(3t^2 + 1, 2t^2 - 7t + 3, (t-1)^2\right)$.

(a) At what time(s) does the particle pass the xz -plane?

(b) What are the particle (i) coordinates, (ii) velocity, (iii) speed, (iv) acceleration at $t = 2$?

Solution: (a) We have to find t such that $y = 2t^2 - 7t + 3 = 0$; so, we have to solve the quadratic equation $2t^2 - 7t + 3 = 0$. Its solution:

$$t = \frac{7 \pm \sqrt{49 - 24}}{4} = \frac{7 \pm 5}{4} \ ; \quad t_1 = 1/2,\ t_2 = 3 \ .$$

(b) $\boldsymbol{r}(t) = \left(3t^2 + 1, 2t^2 - 7t + 1, (t-1)^2\right)$;

$$\dot{\boldsymbol{r}}(t) = (6t, 4t - 7, 2t - 2) \ ; \quad \ddot{\boldsymbol{r}}(t) = (6, 4, 2) \ .$$

$$\boldsymbol{r}(2) = (13, -3, 1) \ ; \quad \dot{\boldsymbol{r}}(2) = (12, 1, 2) \ ; \quad \|\dot{\boldsymbol{r}}(2)\| = \sqrt{149} \ ; \quad \ddot{\boldsymbol{r}}(2) = (6,4,2) \ .$$

Problem 3. Find the directional derivative of $F(x, y, z)=15x^2e^{-z}+3y^2$ in the direction $\boldsymbol{u}=(4,-4,2)$ at the point $(1,2,0)$.

Solution. $\dfrac{\partial f}{\partial x}=30xe^{-z}$; $\dfrac{\partial F}{\partial y}=6y$; $\dfrac{\partial F}{\partial z}=-15x^2e^{-z}$.

So, $\dfrac{\partial F}{\partial x}(1,2,0)=30$, $\dfrac{\partial F}{\partial y}(1,2,0)=12$, $\dfrac{\partial F}{\partial z}(1,2,0)=-15$.

Now, $\|\boldsymbol{u}\|=\sqrt{16+16+4}=\sqrt{36}=6$; so, the normalized vector

$\boldsymbol{v}=\dfrac{\boldsymbol{u}}{\|\boldsymbol{u}\|}=(2/3,-2/3,1/3)$. Then,

$$D_{\boldsymbol{u}}F=\nabla F\cdot\boldsymbol{v}=30\cdot\frac{2}{3}+12\cdot\left(\frac{-2}{3}\right)+(-15)\cdot\frac{1}{3}=7 .$$

Problem 4. Let $\boldsymbol{F}=\left(x(x^2+y^2+z^2)^m, y(x^2+y^2+z^2)^m, z(x^2+y^2+z^2)^m\right)$.
(a) Find $\nabla\cdot\boldsymbol{F}$; (b) Find m such that $\nabla\cdot\boldsymbol{F}=0$ for $x^2+y^2+x^2>0$.

Solution. $\nabla\cdot\boldsymbol{F}=\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}+\dfrac{\partial R}{\partial z}$

$$\begin{aligned}
&=\frac{\partial}{\partial x}\left(x(x^2+y^2+z^2)^m\right)+\frac{\partial}{\partial y}\left(y(x^2+y^2+z^2)^m\right)+\frac{\partial}{\partial z}\left(z(x^2+y^2+z^2)^m\right)\\
&=(x^2+y^2+z^2)^m+x\cdot m(x^2+y^2++z^2)^{m-1}\cdot 2x\\
&+(x^2+y^2+z^2)^m+y\cdot m(x^2+y^2+z^2)^{m-1}\cdot 2y\\
&+(x^2+y^2+z^2)^m+z\cdot m(x^2+y^2+z^2)^{m-1}\cdot 2z\\
&=3(x^2+y^2+z^2)^m+2m(x^2+y^2+z^2)(x^2+y^2+z^2)^{m-1}\\
&=(3+2m)(x^2+y^2+z^2)^m .
\end{aligned}$$

(b) $\nabla\cdot\boldsymbol{F}=0$ if $3+2m=0$, i.e. $m=-3/2$.

Problem 5. Let

$$\boldsymbol{F}(x,y,z)=(a\cos y+b\sin z, c\cos z+d\sin x, e\cos x+f\sin y) .$$

(a) Find $\nabla\times\boldsymbol{F}$; (b) Find the values of a, b, c, d, e, f such that $\nabla\times\boldsymbol{F}\equiv\boldsymbol{F}$.

Solution.

(a) $$\nabla\times\boldsymbol{F}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a\cos y+b\sin z & c\cos z+d\sin x & e\cos x+f\sin y \end{vmatrix}$$

$$=(f\cos y+c\sin z, b\cos z+e\sin x, d\cos x+a\sin y) .$$

(b) Comparing the coefficients we conclude that $\nabla\times\boldsymbol{F}\equiv\boldsymbol{F}$ if $a=f, b=c, e=d$.

Problem 6. Find the work done by the force $\boldsymbol{F}(x,y,z)=(x-y, x^2, -z)$ moving a particle along **a line segment** from a point $P(1,2,3)$ to a point $Q(2,1,2)$.

Hint: Find the parametric equation of the line connecting P and Q , then evaluate the integral.

Solution. The parametric equations of the segment connecting P and Q is $\boldsymbol{r}(t)=(1+t, 1-t, 3-t)$ (0<=t<=1). Then $\boldsymbol{r}'(t)=(1,-2,-1)$. Then the work done by the force $\boldsymbol{F}$ on the segment is

$$W=\int_0^1 (2\mathrm{t}, (1-t)^{2,}-3+t)\cdot(1,-1,-1)\,dt=\int_0^1 (2t-1-2t-t^2+3-t)\,dt$$

$$=\frac{-1}{3}-\frac{1}{2}+2=\frac{7}{6} .$$

Problem 7. Let $\boldsymbol{F}(x,y,z)=\left(ye^{xy},\,xe^{xy}-\sin(y+z),\,3z^2-\sin(y+z)\right)$.
(a) Show that $\int_C \boldsymbol{F}\cdot d\,\boldsymbol{r}$ is independent of the path;
(b) Compute the integral for any path C from the point $A(2,-1,1)$ to the point $B(3,2,-2)$.

Solution. Let us check the conditions for the path independence.

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=e^{xy}+xye^{xy}-e^{xy}-yxe^{xy}\equiv 0 \ ;$$

$$\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}=0-0\equiv 0 \ ;$$

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=-\cos(y+z)+\cos(y+z)\equiv 0 \ .$$

Hence, there exists a function $\varphi(x,y,z)$ such that $\boldsymbol{F}=\nabla\varphi$, i.e.

$$\frac{\partial\varphi}{\partial x}=P=ye^{xy} \ ,$$

$$\frac{\partial\varphi}{\partial y}=Q=xe^{xy}-\sin(y+z) \ ,$$

$$\frac{\partial\varphi}{\partial z}=R=3z^2-\sin(y+z) \ .$$

From the first equation we have

$$\varphi=\int ye^{xy}\,dx=y\cdot\frac{1}{y}e^{xy}+g(y,z)=e^{xy}+g(y,z);$$

Then,

$$\frac{\partial \varphi}{\partial y} = xe^{xy} + \frac{\partial g}{\partial y} \equiv xe^{xy} - \sin(y+z);$$

Hence

$$\frac{\partial g}{\partial y} = -\sin(y+z); \quad g = \int(-\sin(y+z))\,dy = \cos(y+z) + h(z) \;;$$

Then,

$$\varphi = e^{xy} + \cos(y+z) + h(z) \;; \quad \frac{\partial \varphi}{\partial z} = -\sin(y+z) + h'(x) = 3z^2 - \sin(y+z);$$

$h'(z) = 3z^2;\; h(z) = z¿3 + C; \quad \varphi(x,y,z) = e^{xy} + \cos(y+z) + z^3$. YYou can check yourself that $\partial\varphi \frac{¿}{\partial x} = P, \; \frac{\partial \varphi}{\partial y} = Q, \; \frac{\partial \varphi}{\partial z} = R$.

Then,

$$\int_A^B \boldsymbol{F} \cdot d\,\boldsymbol{r} = \varphi(B) - \varphi(A) = \varphi(3,2,-2) - \varphi(2,-1,1) = e^6 + \cos(0) - 8 - e^2 - 1 + 1$$

$$= e^6 - e^2 - 9 \;.$$

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Variant B

Problem 1. Find the parametric equation of the line of intersection of two planes:

P_1: $x + 3y + 5z = 0$ and P_2: $x + y - 3z = 6$.

Solution. Let us eliminate the variables x and y from the equations of the planes. Subtract the second equation from the first one:

$$2y + 8z = -6; \quad y = -4z = 3.$$

Now express x in terms of y,z from the first equation:

$$x=-y+3z+6=4z+3+3z+6=7z+9 .$$

So, the parametric equations of the line are

$$x=7t+9,\ y=-4t-3,\ z=t .$$

Problem 2. Position vector of a moving particle is given by $\boldsymbol{r}(t)=\left(2t^2-5t+2, 2t^2+1, (t+1)^2\right)$.

(a) At what time(s) does the particle pass the yz -plane?

(b) What are the particle (i) coordinates, (ii) velocity, (iii) speed, and ((iv) acceleration at $t=1$?

Solution. (a) We have to solve the equation $2t^2-5t+2=0$; ;

$t_1=\frac{1}{2},\ t_2=2$.

(b) (i) Position $\boldsymbol{r}(1)=(-1,3,4)$; (ii) Velocity $\boldsymbol{r}'(t)=(4t-5, 4t, 2t+2)$, so $\boldsymbol{r}'(1)=(-1,4,4)$; (iii) Speed $\boldsymbol{r}'(1)=\sqrt{1+16+16}=\sqrt{33}$;
(iv) Acceleration $\boldsymbol{r}''(t)=(4,4,2)$, so $\boldsymbol{r}''(1)=(4,4,2)$

Problem 3. Find the directional derivative of $F(x,y,z)=7y^2e^{-x}+3z^2$ in the direction $\boldsymbol{u}=(3,6,-2)$ at the point $(0,1,7)$.

Solution. $\frac{\partial F}{\partial x}=-7y^2e^{-x}$, $\frac{\partial F}{\partial y}=14ye^{-x}$, $\frac{\partial F}{\partial z}=6z$; so,

$\frac{\partial F}{\partial x}(0,1,7)=-7,\ \frac{\partial F}{\partial y}(0,1,7)=14,\ \frac{\partial F}{\partial z}(0,1,7)=42$. Next,

$$\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}=\frac{(3,6,-2)}{\sqrt{9+36+4}}=\frac{(3,6,-2)}{7}=(\frac{3}{7},\frac{6}{7},-\frac{2}{7}) \ .$$

Hence, $D_{\boldsymbol{u}}F=\frac{3}{7}\cdot(-7)+\frac{6}{7}\cdot 14+-\frac{2}{7}\cdot 42=-3+12-12=-3$.

Problem 4. Let

$$\boldsymbol{F}(x,y,z)=\left(x(x^2+y^2+z^2-1),y(x^2+y^2+z^2-1),z(x^2+y^2+z^2-1)\right) \ .$$

(a) Find $\nabla\cdot\boldsymbol{F}$; (b) Find $\|\boldsymbol{r}\|$ such that $\nabla\cdot\boldsymbol{F}=0$ where $\boldsymbol{r}=(x,y,z)$.

Solution. (a)

$$\nabla\cdot\boldsymbol{F}=(x^2+y^2+z^2-1)+x\cdot 2x+(x^2+y^2+z^2-1)+y\cdot 2y$$

$$+(x^2+y^2+z^2-1)+z\cdot 2\mathrm{z}=5(x^2+y^2+z^2)-3 \ .$$

(b) $\nabla\cdot\boldsymbol{F}=0$ if $5(x^2+y^2+z^2)=3;$ $r^2=x^2+y^2+z^2=\frac{3}{5}$; $r=\sqrt{\frac{3}{5}}$.

Problem 5. Let $\boldsymbol{F}(x,y,z)=\left(-y(x^2+y^2)^m,x(x^2+y^2)^m,0\right)$.

(a) Find $\nabla\times\boldsymbol{F}$; (b) Find m such that $\nabla\times\boldsymbol{F}=0$ for $x^2+y^2>0$.

Solution. (a)

$$\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -y(x^2+y^2)^m & x(x^2+y^2)^m & 0\end{vmatrix}$$

$$=0\cdot\boldsymbol{i}+0\cdot\boldsymbol{j}+\left[\left(x^2+y^2\right)^m+x\cdot m\left(x^2+y^2\right)^{m-1}\cdot 2x\right.$$

$$\left.+\left(x^2+y^2\right)^m+y\cdot m\left(x^2+y^2\right)^{m-1}\cdot 2y\right]\boldsymbol{k}$$

$$=\left[2\left(x^2+y^2\right)^m+2m\left(x^2+y^2\right)^{m-1}\left(x^2++y^2\right)\right]\boldsymbol{k}=(2m+2)\left(x^2+y^2\right)^m\boldsymbol{k} \quad .$$

(b) $\nabla\cdot\boldsymbol{F}=0$ if $2+2m=0$, i.e. $m=-1$.

Problem 6. Find the work done by the force $\boldsymbol{F}(x,y,z)=(xyz,-\cos(yz),xz)$ moving a particle along a **line segment** from a point $P(1,1,1)$ to a point $Q(-2,1,3)$. **Hint:** find the parametric equation of a line connecting P and Q , then evaluate the integral.

Solution. The segment connecting the points P and Q has the parametric equation $\boldsymbol{r}(t)=(1,1,1)+t(-3,0,2)=(1-3t,1,1+2t)$ $(0\le t\le 2)$; the velocity vector is $\boldsymbol{r}'(t)=(-3,0,2)$. The field along this segment, $\boldsymbol{F}(\boldsymbol{r}(t))=((1-3t)(1+2t),-\cos(1+2t),(1-3t)(1+2t))$. The work along this segment,

$$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1 \boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\,dt$$

$$=\int_0^1 (-3(1-3t)(1+2t)+2(1-3t)(1+2t))\,dt=-\int_0^1 (1-3t)(1+2t)\,dt$$

$$=-\int_0^1 (1-t-6t^2)\,dt=-1+\frac{1}{2}+2=\frac{3}{2} \quad .$$

Problem 7. Let $\boldsymbol{F}(x,y,z)=(y,x+z,y)$.

(a) Show that $\int_C \boldsymbol{F} \cdot d\,\boldsymbol{r}$ is independent of the path;

(b) compute the integral for any path C from the point $A(2,1,4)$ to the point $B(8,3,1)$.

Solution. (a) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0;$ $\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 1 - 1 = 0$;
$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0 - 0 = 0$. So, the integral is path-independent.

(b) Let us find the potential function $\varphi(x, y, z)$. It satisfies equations
$\frac{\partial \varphi}{\partial x} = y$, $\frac{\partial \varphi}{\partial y} = x + z$, $\frac{\partial \varphi}{\partial z} = y$. So,

$$\varphi = \int y\,dx = sy + g(y, z) \;;$$

$$\frac{\partial \varphi}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \;; \quad \frac{\partial g}{\partial y} = z; \;\; g = yz + h(z); \;\; \varphi = xy + yz + h(z).$$

$$\frac{\partial \varphi}{\partial z} = y + h(z) = y; \;\; h(z) = 0; \;\; \varphi(x, y, z) = xy + yz \;.$$

Now, the work

$$W = \int_A^B \boldsymbol{F} \cdot d\,\boldsymbol{r} = \varphi(B) - \varphi(A) = \varphi(8,3,1) - \varphi(2,1,4) = 27 - 6 = 21.$$