1. (12 pts total)

(1a) (7 points) Determine whether this integral

$$\int_0^5 \frac{dx}{(5-x)^{\frac{1}{3}}}$$

is convergent or divergent, and evaluate it if it is convergent.

(1b) (5 points) Use the Comparison Theorem to determine whether the following integral is convergent or divergent:

$$\int_1^\infty \frac{e^{-x}}{\sqrt{x}}dx$$

(1a)

$$\int_0^5 \frac{dx}{(5-x)^{\frac{1}{3}}} = \lim_{B \to 5^-} \int_0^B \frac{dx}{(5-x)^{\frac{1}{3}}} = \left\{ \begin{array}{l} \text{Substitution} \\ 5-x = u \\ dx = -du \end{array} \right\}$$

$$= \lim_{B \to 5^-} \int_5^{5-B} \frac{-du}{u^{\frac{1}{3}}} = \lim_{B \to 5^-} \int_{5-B}^5 \frac{du}{u^{\frac{1}{3}}} =$$

$$= \lim_{B \to 5^-} \frac{3}{2} u^{\frac{2}{3}} \Big|_{5-B}^5 = \lim_{B \to 5^-} \frac{3}{2}\left(5^{\frac{2}{3}} - (5-B)^{\frac{2}{3}}\right) =$$

$$= \frac{3}{2} 5^{\frac{2}{3}}$$

Yes, the integral is convergent, its value is

Extra space for Question 1

1B (i) Note that $\frac{1}{\sqrt{x}} \le 1$ for $x \geqslant 1$

(ii) integral $\int_1^\infty e^{-x}\,dx$ is convergent,

We've done this in class, but here is the proof:

$$\int_1^\infty e^{-x}\,dx = \lim_{B \to +\infty} \int_1^B e^{-x}\,dx = \lim_{B\to+\infty} -e^{-x}\Big|_1^B =$$

$$= \lim_{B \to +\infty} (e^{-1} - e^{-B}) = e^{-1}$$

(iii) Thus $0 \le \frac{e^{-x}}{\sqrt{x}} \le e^{-x}$ for $x \geqslant 1$

and $\int_1^\infty e^{-x}\,dx$ converges.

Applying Comparison Theorem we conclude that $\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ converges.

2. (9 pts) Find the exact area of the surface obtained by rotating the curve

$$y = \sqrt{x+1}, \quad 0 \le x \le 3,$$

around the x-axis

$$S = \int_0^3 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx =$$

$$= \int_0^3 2\pi \sqrt{x+1} \sqrt{1+\left(\frac{1}{2\sqrt{x+1}}\right)^2}\, dx =$$

$$= 2\pi \int_0^3 \sqrt{(x+1)\left(1+\frac{1}{4(x+1)}\right)}\, dx =$$

$$= 2\pi \int_0^3 \sqrt{x+1+\frac{1}{4}}\, dx = 2\pi \int_0^3 \sqrt{x+\frac{5}{4}}\, dx =$$

$$= \left\{ \begin{array}{c} \text{Substitution} \\ u = x + \frac{5}{4} \\ dx = du \end{array} \right\} = 2\pi \int_{\frac{5}{4}}^{3+\frac{5}{4}} u^{\frac{1}{2}}\, du =$$

$$= 2\pi \, \frac{2}{3} u^{\frac{3}{2}} \Big|_{\frac{5}{4}}^{\frac{17}{4}} = \frac{4\pi}{3}\left(\left(\frac{17}{4}\right)^{\frac{3}{2}} - \left(\frac{5}{4}\right)^{\frac{3}{2}}\right)$$

The answer could be simplified to

$$\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

3. (10 pts) Solve the initial value problem:

$$x\frac{dy}{dx} = y + x^2\sin(x), \quad y(\pi) = 0.$$

$$\frac{dy}{dx} - \frac{1}{x}y = x\sin(x)$$

$$I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln(x)} = \frac{1}{x}$$

multiply by $I(x)$: $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \sin(x)$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \sin(x)$$

$$\Rightarrow \frac{1}{x}y = \int \sin(x)\,dx + C$$

$$\frac{1}{x}y = -\cos(x) + C$$

$$y = x(-\cos(x) + C)$$

use $y(\pi) = 0$ to find C:

$$y(\pi) = \pi(-\cos(\pi) + C) = 0 \Leftrightarrow$$

$$\pi(-(-1) + C) = 0 \Leftrightarrow \boxed{C = -1}$$

Answer: $y = x(-\cos(x) - 1)$

4. (9 pts) The transport of a substance across a capillary wall in lung physiology has been modeled by the differential equation

$$\frac{dh}{dt} = -\frac{R}{V}\left(\frac{h}{k+h}\right),$$

where h is the hormone concentration in the bloodstream, t is time, R is the maximum transport rate, V is the volume of the capillary, and k is a positive constant that measures the affinity between the hormones and the enzymes that assist the process. Solve this differential equation and find a relationship between h and t.

Hint: keep in mind that R, V and k are constants.

This is a separable equation.

$$\frac{(k+h)\,dh}{h} = -\frac{R}{V}\,dt$$

$$\int\left(\frac{k}{h}+1\right)dh = -\int \frac{R}{V}\,dt$$

$$\boxed{k\ln(h) + h = -\frac{R}{V}t + C}$$

Answer: