York University,
Department of Mathematics and Statistics
Math 1014
March 8, 2017
Test 2 (Sections M, N, O, Q)

Solutions

Last name ....

First name ....

Student Id ....

Email ....

Instructions:

1) NO calculators or other aids.
2) There 5 questions on pages numbered 2-6
3) Show all work for full credit
4) **Write only on pages, where there is a barcode. No other pages will be seen or marked. Check the backs of sheets for questions.**

1. (10 pts.) Find the exact arc length of the curve
$36y^2 = (x^2 - 4)^3$, $2 \le x \le 3$, $y \ge 0$.

Since $y \ge 0$, we have

$$6y = (x^2-4)^{3/2}$$

$$\Rightarrow y = \frac{1}{6}(x^2-4)^{3/2} \qquad \frac{dy}{dx} = \frac{1}{4}(x^2-4)^{1/2}(2x).$$

So the arc length is

$$L = \int_2^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_2^3 \sqrt{1+\frac{x^2}{4}(x^2-4)}\,dx$$

$$= \int_2^3 \sqrt{1+\frac{x^4}{4}-x^2}\,dx$$

$$= \int_2^3 \sqrt{\left(1-\frac{1}{2}x^2\right)^2}\,dx$$

$$\cancel{= \int_2^3 \left(1-\frac{1}{2}x^2\right)dx}$$

$$\cancel{= \left(x-\frac{1}{6}x^3\right)\Big|_2^3}$$

$$\cancel{= \left(3-\frac{27}{6}\right)-\left(2-\frac{8}{6}\right)}$$

$$\cancel{= 1-\frac{19}{6} = \frac{6-19}{6}}$$

$$= \int_2^3 \left(\frac{1}{2}x^2-1\right)dx$$

$$= \left(\frac{1}{6}x^3-x\right)\Big|_2^3$$

$$= \left(\frac{27}{6}-3\right)-\left(\frac{8}{6}-2\right)$$

$$= \frac{19}{6}-1 = \frac{13}{6}.$$

2. (10 pts.) The given curve is rotated about the y-axis. Write the integral that represents the surface area of the resulting surface:

$$y = \frac{1}{3}x^{\frac{3}{2}} \text{ where } 0 \le x \le 12$$

DO NOT evaluate the integral!

$$A = \int_0^{12} 2\pi x \, ds$$

$$= \int_0^{12} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\frac{dy}{dx} = \frac{1}{2} x^{1/2}$$

So,

$$A = \int_0^{12} 2\pi x \sqrt{1 + \frac{x}{4}} \, dx.$$

3. a) **(10 pts)** Solve the following differential equation:

$$\frac{dz}{dt} + e^{-t+z} = 0$$

$$\frac{dz}{dt} = -e^{-t+z}$$

$$e^{-z}\,dz = -e^{-t}\,dt$$

Integrating:

$$-e^{-z} = e^{-t} + C.$$

So $e^{-z} = -e^{-t} + C.$

Taking logs:

$$-z = \ln|-e^{-t} + C|.$$

$$z = -\ln|-e^{-t} + C|.$$

3b) (10 pts) Solve the initial-value problem

$$t\frac{du}{dt} = t^2 + 3u \text{ where } t > 0, u(2) = 4.$$

$$\frac{du}{dt} \underbrace{-\frac{3}{t}}_{P(t)}u = \underbrace{t}_{Q(t)}$$

$$I(t) = e^{\int P(t)dt} = e^{-3\ln t} \text{ , since } t>0$$

$$= \frac{1}{t^3}$$

$$\frac{d}{dt}\left(I(t)u\right) = I(t)\,Q(t)$$

$$\Rightarrow d\left(I(t)u\right) = I(t)\,Q(t)$$

Integrating:

$$I(t)u = \int \frac{t}{t^3}dt = \int \frac{1}{t^2}dt = -\frac{1}{t} + C.$$

$$\Rightarrow u = t^3\left(-\frac{1}{t} + C\right)$$

$$u(2) = 4 = 2^3\left(-\frac{1}{2} + C\right) = -4 + 8C \Rightarrow C = 1.$$

So the solution is $u(t) = t^3\left(1 - \frac{1}{t}\right)$.

$$= t^3 - t^2$$

4. (10 pts) A curve C is defined by the parametric equations $x = t^2$ and $y = t^3 - 3t$.

(a) Show that C has two tangents at the point (3, 0) and find their equations.

(b) Determine where the curve is concave upward or downward.

(a) $(x, y) = (3, 0)$ when

$x = t^2 = 3 \Leftrightarrow \boxed{t = \pm\sqrt{3}}$

and $y = t^3 - 3t = 0 \Leftrightarrow t(t^2-3) = 0 \Leftrightarrow t = 0$ or $\boxed{t = \pm\sqrt{3}}$

So when $t = -\sqrt{3}$ and $t = \sqrt{3}$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3t^2-3}{2t}$$

$$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \text{Tangent line is } y = \sqrt{3}x - 3\sqrt{3}.$$

$$\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{6}{-2\sqrt{3}} = -\sqrt{3} \Rightarrow \text{Tangent line is } y = -\sqrt{3}x + 3\sqrt{3}.$$

(b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{6t(2t) - 2(3t^2-3)}{4t^2} \cdot \frac{1}{2t}$$

Continued →

Extra space

$$\frac{d^2y}{dx^2} = \frac{6t^2+6}{8t^3} = \frac{3}{4}\cdot\left(\frac{t^2+1}{t^3}\right)$$

No zeroes, discontinuous at $t=0$.

	$\frac{d^2y}{dx^2}$	
$t<0$	$-$	Curve is concave down
$t>0$	$+$	Curve is concave up.