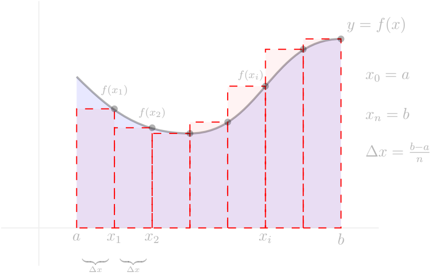
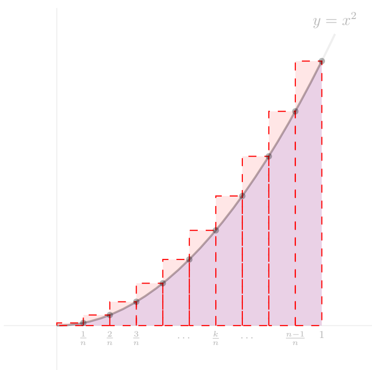
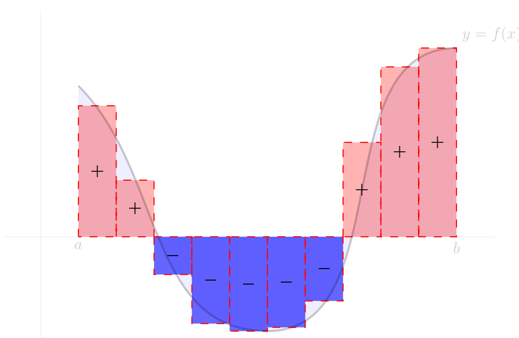
INTEGRAL CALCULUS





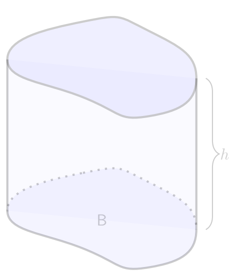
Calculus 1(B)
MAT136H1S

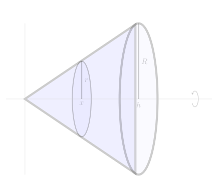


SPRING 2016

BEATRIZ NAVARRO LAMEDA
NIKITA NIKOLAEV

SOME ILLUSTRATIONS ARE TAKEN FROM
Single Variable Calculus, Early Transcendentals Version
BY JAMES STEWART





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§0 Prelude

In MAT135 you spent most part of the course studying the operation of *differentiation*. In MAT136 we will study the reverse operation called **anti-differentiation**. Roughly speaking, the main question we want to answer is this: given a function f, can we find a function F whose derivative F' is f? We will discover the answer to this question is equivalent to other interesting but seemingly unrelated questions, such as "What is the area of a given geometric shape?" or "How much distance has a given particle travelled?"

§1 Antiderivatives

Definition 1.1 (Antiderivative)
A function F is an **antiderivative** of a function f on an interval I if

$$F'(x) = f(x) \qquad \text{for all } x \text{ in I.}$$

Example 1.2 (finding antiderivative)

Consider the function $f(x) = 3x^2$. By definition, an antiderivative of f is a function F such that $F'(x) = 3x^2$. It is easy to guess such a function: differentiating x^3, we get exactly $3x^2$. Thus, the function $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$.

Notice, however, that the function $G(x) = x^3 + 5$ is *also* an antiderivative of $f(x) = 3x^2$, because $G'(x) = (x^3 + 5)' = 3x^2 + 0 = 3x^2$. In fact, there is nothing special about the added constant 5: for *any* constant real number $C \in \mathbb{R}$, the function $x^3 + C$ is an antiderivative of $f(x) = 3x^2$.

So the function $f(x) = 3x^2$ from this example has *many* antiderivatives — as many antiderivatives as there are real numbers! This is true in general: given a function f, suppose F and G are two antiderivatives of f. Then by definition $F'(x) = f(x)$ and $G'(x) = f(x)$, so $F'(x) = G'(x)$. Recall that one of the consequences of the Mean Value Theorem states that if two functions have the same derivatives then they must differ by a constant. Thus,

$$F(x) - G(x) = C \qquad \text{for some } C \in \mathbb{R}, \text{ for all } x\ .$$

This gives the following result.

Theorem 1.3 (General form of antiderivative)
If F is an antiderivative of f on an interval I, then ***the most general antiderivative*** *of f on I is*

$$F(x) + C$$

where $C \in \mathbb{R}$ is an arbitrary constant.

In other words, if F is an antiderivative of f, then any antiderivative of f is of the form $F(x) + C$ for some constant $C \in \mathbb{R}$. Antiderivatives are only unique up to an additive constant, and there is no such thing as *the* antiderivative of f. By choosing a *specific* constant C (e.g., 5), we select a *specific* antiderivative of f.

Example 1.4 (antiderivatives: most general v. specific)

In example 1.2, we guessed an antiderivative of $f(x) = 3x^2$ to be x^3. The most general antiderivative of f is therefore of the form $x^3 + C$ for any $C \in \mathbb{R}$. Choosing specific values for the arbitrary constant C, we get specific antiderivatives of f.

For instance, what is the specific antiderivative F of $f(x) = 3x^2$ such that $F(0) = 1$? We already know the form of the most general antiderivative of f, so $F(x) = x^3 + C$; we just need to find the specific constant C such that F satisfies the desired condition:

$$F(0) = 1 \quad \Longrightarrow \quad 0^3 + C = 1 \quad \Longrightarrow \quad C = 1 \,.$$

Thus, $F(x) = x^3 + 1$.

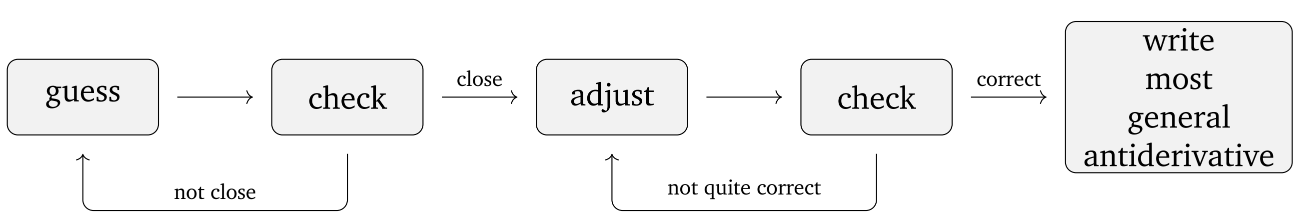
At the moment, except for guesswork, we lack actual methods for finding antiderivatives of functions. The following example demonstrates how knowledge of Differential Calculus aids in making the right guess.

Example 1.5 (finding the most general antiderivative)

Find the most general antiderivative of $f(x) = (3x + 5)^7$.

Solution.
We want to find $F(x)$ such that $F'(x) = (3x + 5)^7$. From Differential Calculus, we know the general principle that *the derivative of a polynomial is a polynomial of one degree less*. So our initial guess is that an antiderivative is something like $(3x+5)^8$. Let's check it: $\big((3x+5)^8\big)' = 24(3x+5)^7$. It's exactly what we want except for the factor 24, so we are very close! We should adjust our initial guess by a factor of $\frac{1}{24}$. Let's check it: $\big(\frac{1}{24}(3x+5)^8\big)' = \frac{24}{24}(3x+5)^7 = (3x+5)^7$, which is precisely $f(x)$. Thus, the most general antiderivative of $f(x) = (3x + 5)^7$ is of the form $F(x) = \frac{1}{24}(3x+5)^7 + C$ for an arbitrary constant C. □

Here's the general strategy in the form of a flow diagram:



Example 1.6 (antidifferentiation is linear)

Show that if F is an antiderivative of f, and G is an antiderivative of g, then
(1) An antiderivative of $cf(x)$ is $cF(x)$ for any constant c.
(2) An antiderivative of $f(x) + g(x)$ is $F(x) + G(x)$.

Solution.
Using the properties of the derivative,

(1) $(cF(x))' = cF'(x) = cf(x)$,

(2) $\big(F(x)+G(x)\big)' = F'(x)+G'(x) = f(x)+g(x)$. □

In the following table we list antiderivatives of some commonly encountered functions. Knowing and referring to these formulas can help you make an educated guess for an antiderivative of a given function. For example, in example 1.5, the first line in the left column was what guided us.

FUNCTION	ANTIDERIVATIVE
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\cosh x$	$\sinh x + C$
$\sinh x$	$\cosh x + C$

FUNCTION	ANTIDERIVATIVE
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x + C$
$-\dfrac{1}{1+x^2}$	$\cot^{-1} x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x + C$

Exercise 1.7: Check that each formula is true by differentiating the given antiderivative.

Example 1.8 (finding all antiderivatives)

Find all functions g such that $g'(x) = \cos x + \sqrt{x}$.

Solution.
Since antidifferentiation is linear, we compute an antiderivaitve of $\cos x$ and of $\sqrt{x}$ separately and add them. Write $g'(x) = \cos x + x^{1/2}$, and use our table above to get:

$$g(x) = \sin x + \frac{1}{\frac{1}{2}+1}x^{1/2+1} + C = \sin x + \tfrac{2}{3}x^{3/2} + C \; .$$ □

Example 1.9 (finding specific second antiderivative)

Find a function $f(x)$ if $f''(x) = \sin x + e^x - x^2$, and $f(0) = 0$, $f'(0) = 2$.

Solution.
Notice that we are given the second derivative of f, so we need to compute an antiderivative twice. In this example, you will see the importance of remembering that antiderivatives are defined only

up to additive constants. Since $f'' = (f')'$, it follows that f' is an antiderivative of f'': using the table above,

$$f'(x) = -\cos x + e^x - \tfrac{1}{3}x^3 + C \ , \tag{1}$$

for an arbitrary constant C. Next, f is an antiderivative of f': using the table again,

$$f(x) = -\sin x + e^x - \tfrac{1}{12}x^4 + Cx + D \ , \tag{2}$$

for some other arbitrary constant D. At this point, you should stop and note the importance of writing the additive constant C in equation (1): had we forgotten to write "$+C$", the term "Cx" in equation (2) would never have appeared!

To find the specific function $f(x)$ that satisfies the desired conditions, we need to find the specific right constants C, D:

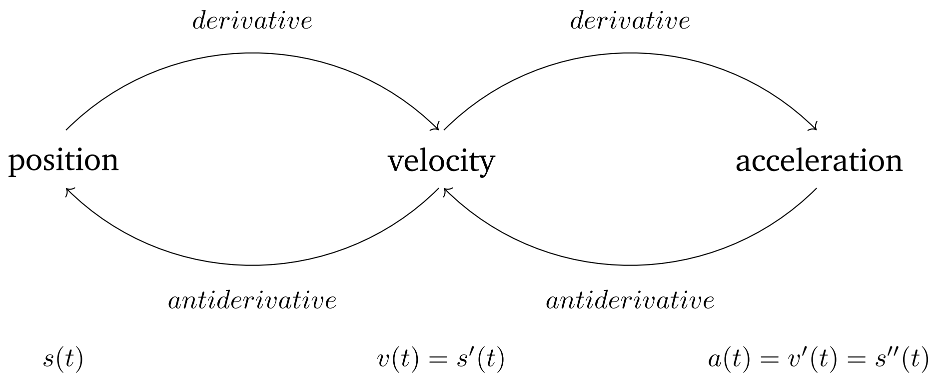
$$\begin{aligned} f'(0) = 2 \quad &\Longrightarrow \quad -\cos(0) + e^0 - \tfrac{1}{3}\cdot 0^3 + C = 2 \quad \Longrightarrow \quad C = 2 \ . \\ f(0) = 0 \quad &\Longrightarrow \quad \cdots \quad \Longrightarrow \quad D = -1 \ . \end{aligned}$$

Thus, we conclude that the desired function is

$$f(x) = -\sin x + e^x - \tfrac{1}{12}x^4 + 2x - 1 \ . \qquad \square$$

§1.1 Rectilinear Motion

Antiderivatives are particularly useful in analysing the motion of an object moving in a straight line. Since position, velocity and acceleration are all related by differentiation, they are also related by anti-differentiation:



Therefore, if the acceleration and the initial values of $s(0)$ and $v(0)$ are known, then the position function can be found by anti-differentiating twice.

Example 1.10 (finding position function of particle knowing acceleration)

A particle moves in a straight line and has acceleration given by $a(t) = 6t + \sin t$. Its initial velocity is $v(0) = -2$ cm/s and its initial displacement is $s(0) = 5$ cm. Find its position function $s(t)$.

Solution.
Since $v'(t) = a(t) = 6t + \sin t$, anti-differentiation gives

$$v(t) = 6\frac{t^2}{2} - \cos t + C = 3t^2 - \cos t + C \ .$$

Note that $v(0) = -1 + C$. Since $v(0) = -2$ as given, we have $C = -1$, so

$$v(t) = 3t^2 - \cos t - 1 \ .$$

Since $v(t) = s'(t)$, we anti-differentiate again to get

$$s(t) = 3\frac{t^3}{3} - \sin t - t + D = t^3 - \sin t - t + D \ .$$

This gives $s(0) = D$. Since $s(0) = 5$ as given, we must have $D = 5$ and the position is

$$s(t) = t^3 - \sin t - t + 5 \ .$$

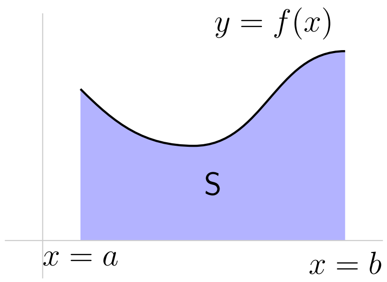
□

§2 Integrals

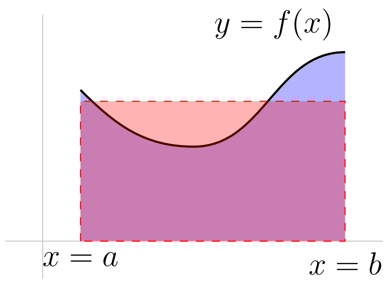
§2.1 The Area Problem

The most common interpretation of the *integral* is as a tool to compute the area under the graph of a function, so this is where we begin.

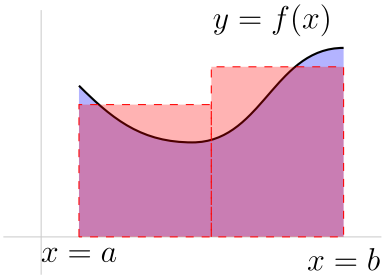
Suppose we are given a function $f(x)$ defined on a region (a, b). Let S be the blue region under the graph of f, as you can see on the right. Imagine that our lives depended on finding the area A of this region. We know how to compute areas of some simple shapes like triangles and rectangles, because we know simple exact formulas. The shape of this region looks rather complicated, however: certainly complicated enough that we don't have an obvious formula to compute its area. But we must do whatever it takes to save our lives, and the time is running out!

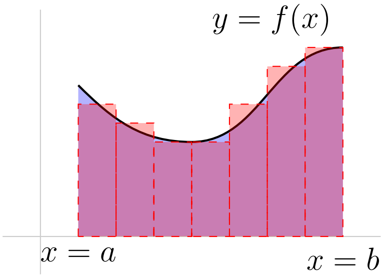


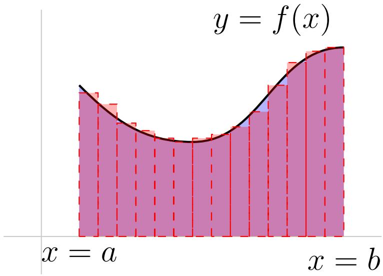
In a desperate attempt, the least we can do is find the area of S approximately. Let's slap a rectangle on this region. We can easily compute the area R_1 of this rectangle, and the answer is kind of close to the actual area of S. The discrepancy between R_1 and the true value A is just the area of the light pink region and the leftover blue region, but maybe that's no big deal.

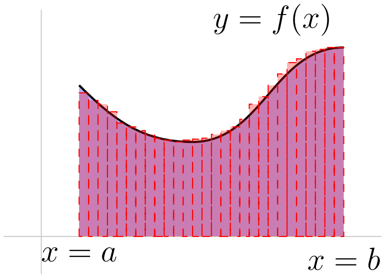


But wait! What if we take two rectangles, each of smaller width, together covering S more closely. If we just compute the areas of these two rectangles and add them, then the approximate result R_2 is closer to the true value A. If we take even more rectangles, we can arrange their sizes to cover S even more closely.









As you can see in the sequence of pictures above, the more rectangles we take, the easier it is to make the discrepancy relatively small. In other words, the greater the number of rectangles n is, the closer the approximation R_n is to the true area A.

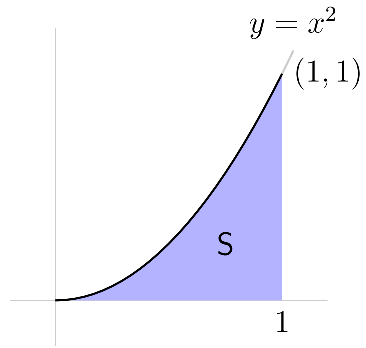
Hold on! This sounds familiar. We have taken Differential Calculus, so we realise immediately that what we're doing is actually just *taking a limit* of these approximations R_n as $n \to \infty$. We expect that

$$\boxed{\text{Area}(\mathsf{S}) = \lim_{n\to\infty} R_n}\ .$$

If we can make this work, we're guaranteed to keep our lives!

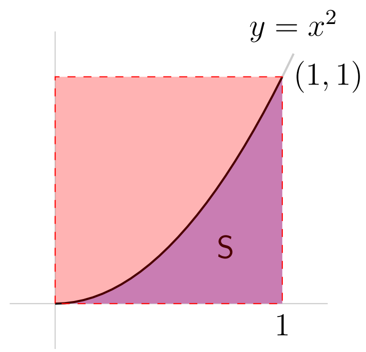
§2.1.1 Approximating the Area: Example

Suppose the function f is given by $f(x) = x^2$, and we must find the area $A := \text{Area}(\mathsf{S})$ of the region S under the graph of f between 0 and 1. Guided by the previous observations, we'll use rectangles to estimate this area. The procedure is to use more and more rectangles to cover S, computing the approximation R_n in each step; we'll then take the limit of R_n as $n \to \infty$. It'll be especially easy to find this limit if we can find a formula for R_n in terms of n. If we hope to find a formula for R_n, we should choose rectangles carefully and methodically: let's choose the height of each rectangle to be the value of the function $f(x) = x^2$ at the right end point of the subinterval.



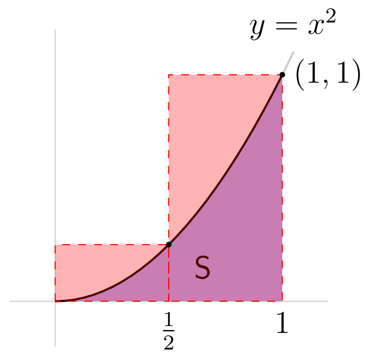
First, let's choose just one rectangle with base the full interval $[0, 1]$ and height $f(1) = 1^2 = 1$. The area of the chosen rectangle is $1 \cdot 1 = 1$, so

$$R_1 = 1.$$



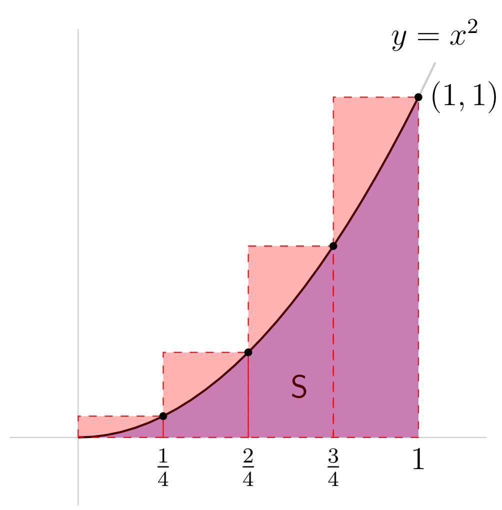
Let's choose two rectangles with bases $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ and heights $f(\frac{1}{2}) = \frac{1}{4}$ and $f(1) = 1$, respectively. Then

$$R_2 = \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}1^2 = 0.625.$$

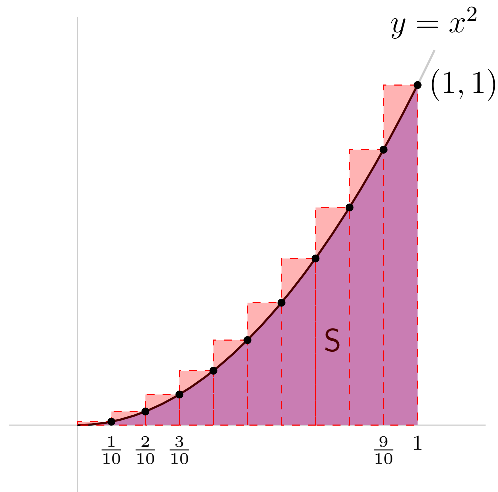


If we choose four rectangles, we get

$$\begin{aligned} R_4 &= \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{2}{4}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}1^2 \\ &= \sum_{k=1}^{4} \frac{1}{4}\left(\frac{k}{4}\right)^2 = \frac{15}{32} = 0.46875. \end{aligned}$$

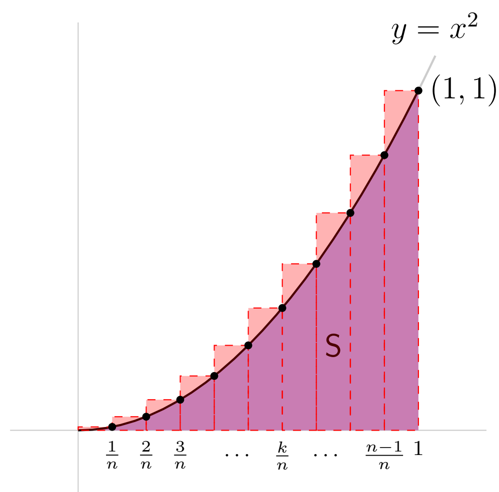


Repeat this calculation with a larger and larger number of rectangles. For example:



$$R_{10} = \frac{1}{10}\left(\frac{1}{10}\right)^2 + \frac{1}{10}\left(\frac{2}{10}\right)^2 + \ldots + \frac{1}{10}\left(\frac{9}{10}\right)^2 + \frac{1}{10}1^2$$
$$= \sum_{k=1}^{10} \frac{1}{10}\left(\frac{k}{10}\right)^2 = \frac{77}{200} = 0.385.$$
$$R_{25} = \sum_{k=1}^{25} \frac{1}{25}\left(\frac{k}{25}\right)^2 = \frac{221}{625} = 0.3536.$$
$$R_{100} = \sum_{k=1}^{100} \frac{1}{100}\left(\frac{k}{100}\right)^2 = \frac{6767}{20000} = 0.33835.$$
$$R_{1000} = \sum_{k=1}^{1000} \frac{1}{1000}\left(\frac{k}{1000}\right)^2 = \frac{667667}{2000000} = 0.3338335.$$

The finite sum approximations become closer and closer to the actual value of A as the number of rectangles increases. If you stare at the formulas above, it's easy to guess the formula for R_n:



$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2 + \frac{1}{n}1^2$$
$$= \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^2$$
$$= \frac{1}{n^3}\sum_{k=1}^{n} k^2$$
$$= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

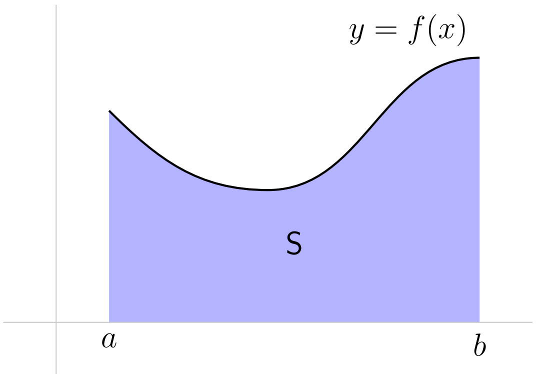
Taking the limit we have

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{2}{6} = \frac{1}{3} = 0.3333\ldots.$$

§2.1.2 Approximating the Area: General

Let f be a nonnegative continuous function defined on an interval $[a, b]$. To compute the area A of the region S under the graph of f, we divide the interval $[a, b]$ into n equal subintervals of length

$$\Delta x := \frac{b-a}{n}.$$



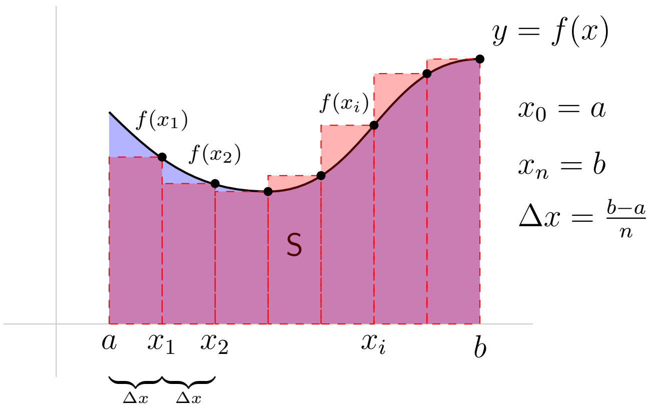
Collecting all the observations we've made to this point, we conclude that

$$R_n = f(x_0)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\Delta x,$$

and the quantity

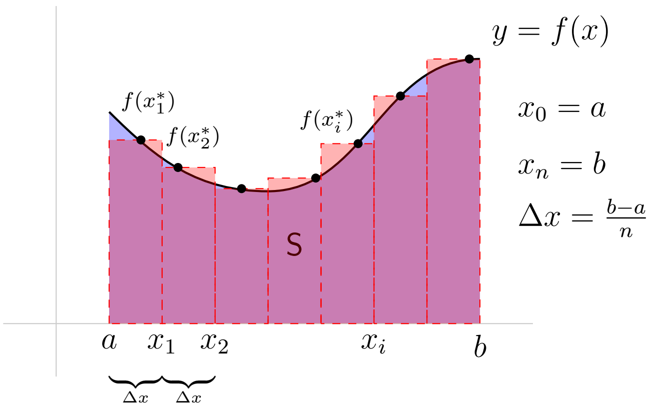
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

represents the area of the region S.



It turns out that instead of using the right endpoints, we can take the height of the i-th rectangle to be the value of f at *any* number $x_i^* \in [x_{i-1}, x_i]$; the numbers $x_1^*, x_2^*, \ldots, x_n^*$ are called a **sample points**. As such, a more general expression for the area of S is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$



§2.2

The Definite Integral

We summarise all the observations we've made in the following important definition.

Definition 2.1 (Definite Integral)
Let f be a function defined for $a \leqslant x \leqslant b$. Divide the interval $[a, b]$ into n subinterval of width $\Delta x = \frac{b-a}{n}$. Let $x_0, x_1, \ldots, x_n$ be the endpoints of these subintervals, where $x_0 = a$ and $x_n = b$. Choose any sample points $x_1^*, x_2^*, \ldots, x_n^*$, where $x_i^* \in [x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x\,,$$

provided that this limit exists and gives the same value for all possible choices of sample points. If this limit exists, we say that f is **integrable on $[a, b]$**.

Let us discuss this definition in great detail to make sure we understand exactly what is being said.

Notice that the definite integral is a *number*, and NOT a function of x; i.e., it does *not* depend on x. Even though "x" appears in the notation, it is not really *variable* – we sometimes say that "the variable x is *integrated out*".

To be absolutely clear, "f is integrable on $[a, b]$" means that the definite integral $\int_a^b f(x)\,\mathrm{d}x$ exists.

The fact that the limit exists is NOT automatic, and it always needs to be checked. Indeed, there are functions which are NOT integrable.

Remark 2.2 (example of a non-integrable function)

A common interesting example that people give of a function that is not integrable is

$$f(x) := \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}\,. \end{cases}$$

It turns out that $f(x)$ is not integrable on the interval $[0, 1]$, but it's a little too involved to show this, and we don't need it.

Despite the fact that such nasty functions exist and we should be aware of this fact, most functions we will ever encounter in this course are integrable.

The part that says "gives the same value for all possible choices of sample points" is also not automatic; however, for us, this technicality will never be a problem. In fact, if a given function f is already known to be integrable, then it turns out that it doesn't matter what sample points you choose:

Proposition 2.3
If f is integrable on $[a, b]$, then the limit in the definition of the definite integral exists and gives the same value for any choice of sample points x_i^*.

(We will very soon learn a very powerful criterion to determine whether a given function is integrable.) In practical terms, this means the following: suppose I give you an integrable function f, and ask you to compute the definite integral of f over some interval using its definition; since f is integrable, you can make the most convenient choice of sample points to make the direct calculation, and you are guaranteed that your answer would be the same if you made any other choice of sample points.

Although we defined the definite integral by dividing the interval $[a, b]$ into subintervals of equal width, it turns out that we can also use subintervals of unequal width. If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0. We can achieve this if $\max \Delta x_i \to 0$. In this case, the definition becomes

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The sums appearing in these limits have a special name.

Definition 2.4 (Riemann Sum)
The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

appearing in the definition is called a **Riemann sum** of f.

Thus, by definition, the definite integral of f is the limit of Riemann sums of f.

§2.2.1 Example: The Distance Problem

The considerations above show that the definite integral of a function f calculates the area under the graph of f, at least if f is nonnegative.

If an object moves with constant velocity, then

$$\text{(distance)} = \text{(velocity)} \times \text{(time)}.$$

However, if the velocity is not constant but instead depends on time t, this formula is no longer valid. If the velocity is changing only a little over the total time travelled, this formula can produce an approximate answer than differs from the true answer only very little. But if the velocity is changing significantly, this formula is totally false. This is what is called the *distance problem*.

Suppose that an object moves in a line with velocity function $v(t)$ without changing direction (i.e., $v(t) \geqslant 0$). We wish to find how far the object travelled from time $t = a$ to $t = b$. A good strategy is to divide the time interval $[a, b]$ into many subintervals that are small enough that during each time subinterval, the velocity is changing only very little. Summing the distances travelled over all time subintervals, we get an approximate value for the distance travelled:

$$d_n = \sum_{i=1}^{n} v(t_i^*)\Delta t \ ,$$

where n is the number of time subintervals, each of width $\Delta t := \frac{b-a}{n}$, and $t_1^*, \ldots, t_n^*$ are sample time instances at which we measure the velocity of our object. Then increasing the number of time subintervals gives better and better approximations to the actual distance travelled d. Therefore,

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_i^*)\Delta t$$

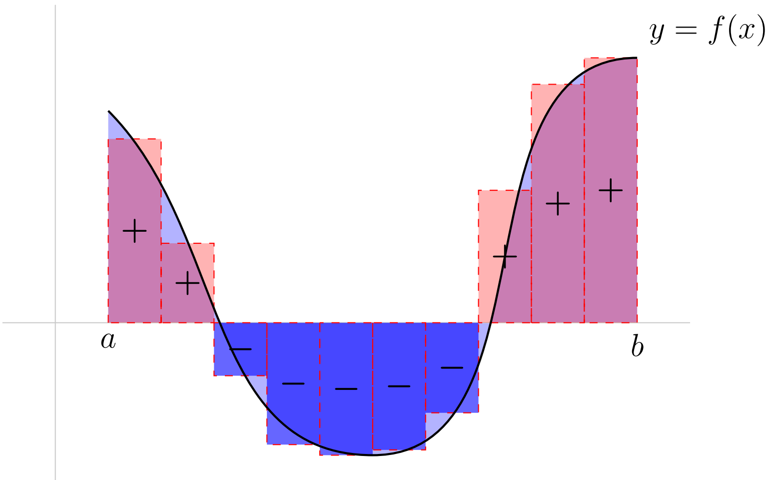
Comparing this equality with the definition of the definite integral, we can see immediately that the correct interpretation of the distance problem is as computing the definite integral of the function $v(t)$ from a to b:

$$d = \int_a^b v(t)\mathrm{d}t.$$

Before we end the discussion of the distance problem, let's observe one other interesting connection. As we've just discovered, the problem of finding the distance travelled given the velocity function leads naturally to definite integrals. On the other hand, in the section about rectilinear motion, we discussed the fact that the operation that relates velocity to distance is the operation of anti-differentiation. This begs the question: are the operations of integration and anti-differentiation connected? We'll discover an answer very soon.

§2.2.2 Negatively and Positively Weighted Area

We motivated the definition of the definite integral by using a non-negative function. Here we explain the meaning of the Riemann sums for functions that take both positive and negative values. We simply count the area of rectangles that lie below the horizontal axis with a *negative* sign:

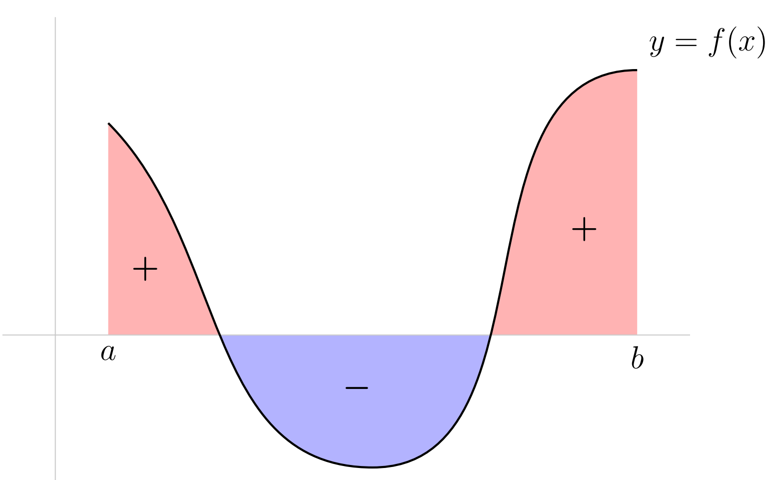


Be careful: area is a non-negative quantity, so it is incorrect to say that "a rectangle has negative area". Instead, it is best to think that the areas of the blue rectangles are *weighted negatively*. Thus, the correct interpretation of a Riemann sum for a function f that is not necessarily non-negative is as follows:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = (\text{area of rectangles above axis}) - (\text{area of rectangles below axis}).$$

Taking the limit as $n \to \infty$, the correct interpretation of the definite integral of f is as a *weighted area* under the graph of f:

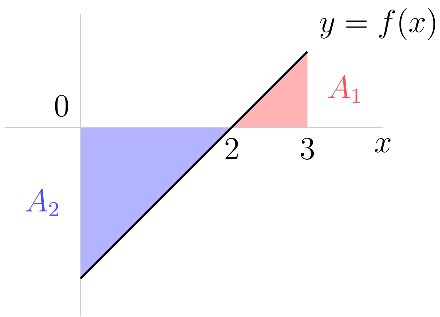
$$\int_a^b f(x)\mathrm{d}x = (\text{area above axis}) - (\text{area below axis}).$$



Example 2.5 (computing definite integral by finding area)

Evaluate the following integral: $\displaystyle\int_0^3 (x-2)\mathrm{d}x.$

Solution.
The graph of $f(x) = x-2$ is depicted on the right. Evaluating the integral amounts to computing the area under the graph of f, weighted positively if it lies above the x-axis (red), and weighted negatively if it lies below (blue). We use the formula for the area of a triangle to calculate the areas of the two regions:



$$\int_0^3 (x-2)\mathrm{d}x = A_1 - A_2 = \tfrac{1}{2}(1\cdot 1) - \tfrac{1}{2}(2\cdot 2) = -\frac{1}{2}.$$ □

§2.3 Evaluating Integrals

To use the definition of the definite integral in a calculation, we need to choose sample points to write down the Riemann sums. We have already mentioned that once we know that a given function f is integrable, any choice of sample points gives the same result (see proposition 2.3). The following very important theorem (which we assume without proof) gives a very powerful criterion to determine whether a given function is integrable.

Theorem 2.6 (Integrability of Functions)
If f has only finitely many jump-discontinuities, then f is integrable.

In particular, any continuous function has zero jump-discontinuities, so the theorem applies:

Corollary 2.7 (Continuous ⇒ Integrable)
If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

To simplify the calculation of the definite integral using its definition in terms of Riemann sums, we often choose subintervals to be of equal length and sample points to be the right end-points of subintervals.

Proposition 2.8
If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \tag{3}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Example 2.9 (expressing limit as integral)

Express the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n+2i} \tag{4}$$

as an integral.

Solution.
In order to be able to use proposition 2.8, we would like our expression (4) to look like the expression on the right-hand side of equation (3). The idea is to factor n out from the denominator like so:

$$\frac{2}{n+2i}=\frac{2}{n(1+\frac{2i}{n})}.$$

Then the limit (4) becomes

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n+2i}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{2i}{n}}\cdot\frac{2}{n}$$

Comparing with equation (3), this suggests that $\Delta x=\frac{2}{n}$. Since $\Delta=\frac{b-a}{n}$, we have $b-a=2$; so we can try taking $a=0$ and $b=2$. Then

$$x_i=a+i\Delta x=0+i\frac{2}{n}=\frac{2i}{n},$$

which is exactly what is appearing in the fraction $\frac{1}{1+\frac{2i}{n}}$. So our function must be

$$f(x)=\frac{1}{1+x}.$$

Therefore

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n+2i}=\int_0^2\frac{1}{1+x}\mathrm{d}x.$$ □

At this point we do not have many methods for evaluating definite integrals. We basically have only two options:

(1) We can write the integral as a limit of Riemann sums and compute the limit.

(2) We can interpret the integral in terms of areas we already know how to compute.

The following two examples demonstrate these two techniques in action.

Example 2.10 (computing integral using definition)

Use the definition of definite integral to evaluate $\displaystyle\int_0^2(2x^2+3)\mathrm{d}x.$

Solution.
We subdivide the integral $[0,2]$ into n equal subintervals of width $\Delta x=\frac{2-0}{n}=\frac{2}{n}$. As sample points,

we'll take the right end-points of the subintervals: $x_i = 0 + i\frac{2}{n} = \frac{2i}{n}$. Therefore,

$$\begin{aligned}
\int_0^2 (2x^2+3)\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x && \text{(defn of integral)}\\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\tfrac{2i}{n}\right)\tfrac{2}{n} && \text{(sub in what we know)}\\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(\tfrac{2i}{n}\right)^2+3\right)\tfrac{2}{n} && \text{(sub in expression for } f\text{)}\\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{16}{n^3}i^2+\tfrac{6}{n}\right) && \text{(simplify)}\\
&= \lim_{n\to\infty} \left(\frac{16}{n^3}\sum_{i=1}^{n} i^2 + \frac{6}{n}\sum_{i=1}^{n} 1\right) && \text{(distribute the sum)}\\
&= \lim_{n\to\infty} \left(\frac{16}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{6}{n}\cdot n\right) && \text{(see Appendix A)}\\
&= \lim_{n\to\infty} \left(\tfrac{8}{3}\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right)+6\right) && \text{(factor } \tfrac{1}{n^3} \text{ in)}\\
&= \tfrac{8}{3}(1)(2)+6 && \text{(take the limit)}\\
&= \frac{34}{3}.
\end{aligned}$$

□

Example 2.11 (definite integral | using knowledge of areas of shapes)

Evaluate the definite integral: $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\mathrm{d}x.$

Solution.
Since $f(x) = \sqrt{4-x^2} \geqslant 0$, we can interpret this area as the area under the curve $y = \sqrt{4-x^2}$ between -2 and 2. Squaring both sides of this equality, we get $y^2 = 4-x^2$. Now, recall that $y^2+x^2=4$ is the equation of a circle of radius $\sqrt{4}=2$ centred at the origin. Therefore, the graph of f is a semicircle of radius 2:

$$\begin{aligned}
\int_{-2}^{2} \sqrt{4-x^2}\mathrm{d}x &= \text{area of semicircle of radius } 2\\
&= \tfrac{1}{2}(\text{area of circle of radius } 2)\\
&= \frac{\pi 2^2}{2}\\
&= 2\pi.
\end{aligned}$$

□

Example 2.12 (writing integral as limit of Riemann sums)

Write the integral $\int_1^3 xe^{-x^2}\mathrm{d}x$ as a limit of Riemann sums.

Solution.
We have $f(x) = xe^{-x^2}$, $a = 1$, $b = 3$, so

$$\Delta x = \frac{3-1}{n} = \frac{2}{n} \quad \text{and} \quad x_i = 1 + \frac{2i}{n}.$$

Therefore

$$\int_1^3 xe^{-x^2}\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\tfrac{2i}{n}\right)e^{-\left(1+\frac{2i}{n}\right)^2}\frac{2}{n}. \qquad \square$$

§2.4 Properties of the Integral

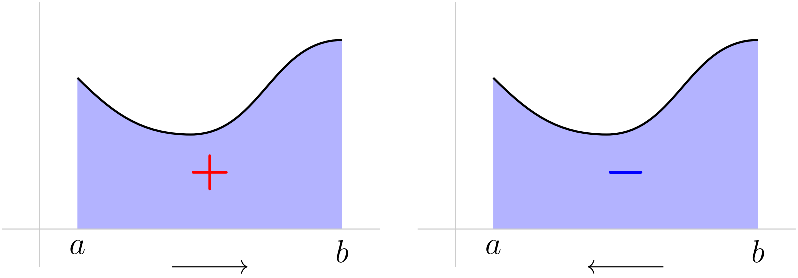
In defining the integral $\int_a^b f(x)\mathrm{d}x$ as a limit of Riemann sums, we assumed that $a < b$. In calculating the area under the graph of f, we moved from left to right across the interval $[a, b]$, filling it in with rectangles.

Notice that if we reverse a and b, then Δx changes from $\frac{b-a}{n}$ to $\frac{b-a}{n}$. But

$$\frac{b-a}{n} = -\frac{a-b}{n} = -\Delta x,$$

so reversing a and b amounts to replacing Δx by $-\Delta x$. Each term in a Riemann sum contains the factor Δx, so reversing a and b exchanges the sign of a Riemann sum. Finally, since a definite integral is the limit of Riemann sums, we obtain the following rule regarding the order of limits of integration:

$$\boxed{\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x.}$$



If $a = b$, then each Riemann sum vanishes because $\Delta x = 0$, so the integral over a zero width interval is zero:

$$\int_a^a f(x)\mathrm{d}x = 0.$$

We now explain some additional properties of the definite integral. These properties will be used all the time to evaluate and work with integrals.

Theorem 2.13 (Properties of the Definite Integral)
Let f and g be integrable functions, and let a, b, c be any real numbers.

(1) [integral of a constant] $$\int_a^b c\,\mathrm{d}x = c(b-a)$$

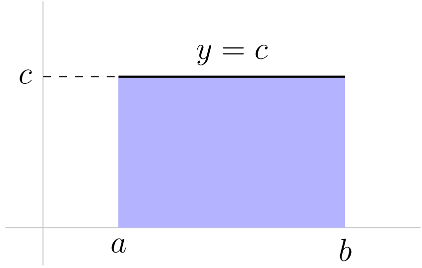
(2) [constant multiple] $$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

(3) [sum] $$\int_a^b \big(f(x)+g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

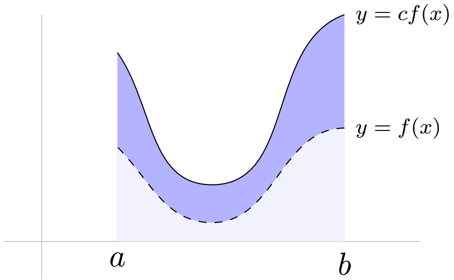
(4) [additivity] $$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

All these properties readily follow from the definition of the definite integral as a limit of Riemann sums, and we will give their formal proofs after the following remarks. All these properties have nice geometric interpretations.

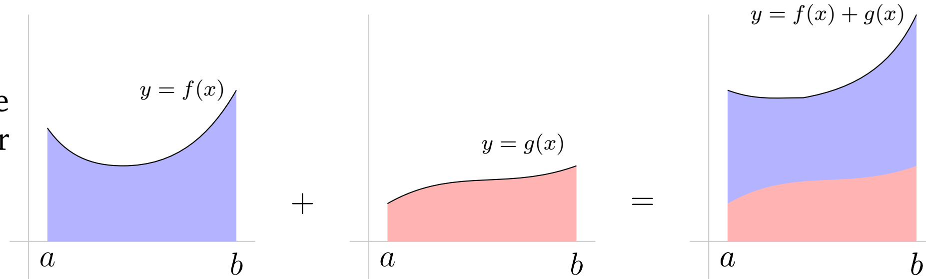
1. If $c > 0$, and $a < b$, this is just the area of the shaded rectangle with sides c and $b-a$, as depicted on the right. If $c < 0$, then $c(b-a) < 0$, but also the shaded rectangle would appear below the x-axis, and so its area would be negatively weighted, hence the formula still makes sense.



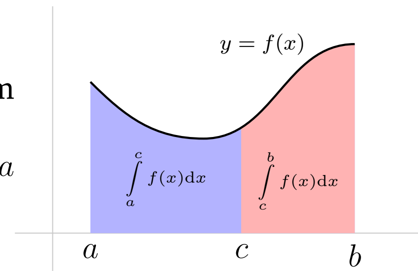
2. If $c > 0$, multiplying by c scales up/down the graph of f vertically by a factor of c. Thus, the height of each rectangle under the graph of f is scaled up/down by a factor of c. As a result, each rectangle's area is multiplied by c.



3. For $f, g \geqslant 0$, this means that the area under $f+g$ is the area under f plus the area under g.



4. If $f \geqslant 0$ and $a < c < b$, this rule means that the area under f from a to b is the area from a to c plus the area from c to b.
If, on the other hand, $a < b < c$, then the rule says that the area from a to b is the area from a to c minus the area from b to c.



Let us give a formal proof of (2).

Proof of theorem 2.13 part (2).

$$
\begin{aligned}
\int_a^b cf(x)\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} cf(x_i)\Delta x && \text{defn of integral}\\
&= \lim_{n\to\infty} c\sum_{i=1}^{n} f(x_i)\Delta x && c \text{ is a constant (algebra of finite sums)}\\
&= c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x && c \text{ is a constant}\\
&= c\int_a^b f(x)\mathrm{d}x. && \text{defn of integral} \quad \square
\end{aligned}
$$

Property (3) can be proved in a very similar way using the fact that the limit of the sum is the sum of the limits (given that the two limits exists).

Exercise 2.14: Give a formal proof of theorem 2.13 property (3).

Example 2.15 (computing definite integral using properties of integrals)

Given that

$$\int_{-1}^{1} f(x)\mathrm{d}x = 2, \qquad \int_{1}^{5} f(x)\mathrm{d}x = 3, \qquad \int_{-1}^{1} g(x)\mathrm{d}x = 5, \qquad \int_{-1}^{0} g(x)\mathrm{d}x = 1,$$

find each of the following integrals:

(a) $\int_{-1}^{1} \big(2f(x) + g(x)\big)\,\mathrm{d}x$

(b) $\int_{-1}^{5} f(x)\,\mathrm{d}x$

(c) $\int_{0}^{1} f(x)\,\mathrm{d}x$

(d) $\int_{0}^{1} g(x)\,\mathrm{d}x$

Solution.

(a) $\int_{-1}^{1} \big(2f(x) + g(x)\big)\,\mathrm{d}x = \int_{-1}^{1} 2f(x)\mathrm{d}x + \int_{-1}^{1} g(x)\,\mathrm{d}x = 2\int_{-1}^{1} f(x)\mathrm{d}x + \int_{-1}^{1} g(x)\,\mathrm{d}x = 2\cdot 2 + 5 = 9.$

(b) $\int_{-1}^{5} f(x)\,\mathrm{d}x = \int_{-1}^{1} f(x)\mathrm{d}x + \int_{1}^{5} f(x)\,\mathrm{d}x = 2 + 3 = 5.$

(c) Not enough information. (For example, although we know the integral of $f(x)$ over the interval $[-1, 1]$, we cannot assume that the integral over the interval $[0, 1]$ of half the length is half the integral over $[-1, 1]$.)

(d) By property (4), $\int_{-1}^{1} g(x)\,\mathrm{d}x = \int_{-1}^{0} g(x)\,\mathrm{d}x + \int_{0}^{1} g(x)\,\mathrm{d}x.$

Therefore, $\int_{0}^{1} g(x)\,\mathrm{d}x = \int_{-1}^{1} g(x)\,\mathrm{d}x - \int_{-1}^{0} g(x)\,\mathrm{d}x = 5 - 1 = 4.$ □

Proposition 2.16 (Comparison Properties of the Integral)
The following properties are true only if $a \leqslant b$.

(5) **[integral of a non-negative function]**

$$f(x) \geqslant 0 \text{ on } [a,b] \implies \int_a^b f(x)\,\mathrm{d}x \geqslant 0.$$

(6) **[domination]**

$$\text{If } f(x) \geqslant g(x) \text{ on } [a,b] \implies \int_a^b f(x)\,\mathrm{d}x \geqslant \int_a^b g(x)\,\mathrm{d}x.$$

(7) **[lower-upper estimate]**

$$\text{If } m \leqslant f(x) \leqslant M \text{ on } [a,b] \implies m(b-a) \leqslant \int_a^b f(x)\,\mathrm{d}x \leqslant M(b-a).$$

Again, these properties can be interpreted geometrically.

5. If $f \geqslant 0$, then $\int_a^b f(x)\mathrm{d}x$ represents the area under the graph of f, so this property simply expresses the fact that area is a non-negative quantity.

6. Assume $f, g \geqslant 0$. If $f \geqslant g$, then the graph of f is higher than the graph of g, so the area under the graph of f is larger than the area under the graph of g.

7. The area under the graph of f is greater than the area of a rectangle with height m and less than the area of a rectangle of height M.

Exercise 2.17: Give a formal proof of property (6). HINT: notice that $f - g \geqslant 0$, and use properties (2), (3) and (5).

Proof of property (7).
Since $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, by property (6) we get

$$\int_a^b m\,\mathrm{d}x \leqslant \int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b M\,\mathrm{d}x.$$

Then by property (1), we have

$$m(b-a) \leqslant \int_a^b f(x)\,\mathrm{d}x \leqslant M(b-a). \qquad \square$$

Example 2.18 (finding bounds for an integral)

Show that $1 \leqslant \int_0^1 \sqrt{2+\cos x}\,\mathrm{d}x \leqslant \sqrt{3}$.

Solution.
We know that $-1 \leqslant \cos x \leqslant 1$, therefore $1 \leqslant 2+\cos x \leqslant 3$. Hence $1 \leqslant \sqrt{2+\cos x} \leqslant \sqrt{3}$. So by property (7),

$$1 = 1(1-0) \leqslant \int_0^1 \sqrt{2+\cos x}\,\mathrm{d}x \leqslant \sqrt{3}(1-0) = \sqrt{3}.$$

□

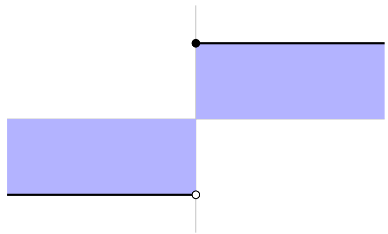
Example 2.19 (area under curve | piecewise-defined functions)

Let $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$ Find $\int_{-1}^{2} f(x)\,\mathrm{d}x$.

Solution.
By the definition of the absolute value, we have

$$\frac{|x|}{x} = \begin{cases} \frac{-x}{x} = -1, & \text{if } x < 0, \\ \frac{x}{x} = 1, & \text{if } x > 0. \end{cases}$$

(Notice that $\frac{|x|}{x}$ is undefined at $x = 0$.) The graph of f is depicted on the right. Note that the function has one jump discontinuity at $x = 0$, where it jumps from $y = -1$ to $y = 1$, so we know it is integrable. We can compute the integral using property (4):



$$\int_{-1}^{2} f(x)\,\mathrm{d}x = \int_{-1}^{0} f(x)\mathrm{d}x + \int_{0}^{2} f(x)\mathrm{d}x = \int_{-1}^{0} (-1)\mathrm{d}x + \int_{0}^{2} (1)\mathrm{d}x = (-1)\big(0-(-1)\big) + (1)(2-0) = 1.$$

□

Example 2.20 (non-integrable function)

The function $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ is not integrable.

Explanation.
This function has exactly one discontinuity, but it is not integrable. (The discontinuity is not a jump discontinuity, so this does not contradict the previously stated theorem regarding jump discontinuities.) Notice that f is unbounded near $x = 0$. This prevents the Riemann sums from tending to a finite limit.

□

§3 Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus establishes a link between the two branches of calculus that you have encountered — the Differential Calculus and the Integral Calculus — by relating the corresponding central concepts of study: differentiation and integration. Informally speaking, the essence of the Fundamental Theorem of Calculus is that differentiation and integration are inverse operations. In this section, we will understand exactly what this means and learn how to use this powerful fact.

The fundamental theorem of calculus deals with functions of the form

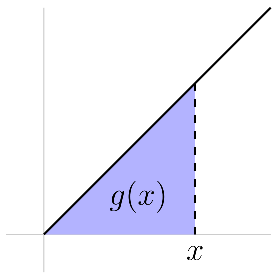
$$g(x) = \int_a^x f(t)\,\mathrm{d}t, \tag{5}$$

where f is a continuous function on $[a, b]$ and x varies between a and b. For example, if f is non-negative, then $g(x)$ can be interpreted as the area under the graph of f between a and x, where x varies from a to b. You can think of g as "the area so far" function.

Example 3.1 ("the area so far" function)

Let $f(t) = t$ and $a = 0$, then the function

$$g(x) = \int_0^x t\mathrm{d}t$$



represents the area under the curve in the picture on the right. Thus,

$$g(x) = \int_0^x t\,\mathrm{d}t = \tfrac{1}{2}x \cdot x = \tfrac{1}{2}x^2.$$

Observe that $g'(x) = x$; that is, $g' = f$. In other words, if g is defined as the integral of f by equation (5), then g is an anti-derivative of f. The first part of the Fundamental Theorem of Calculus says that this is true in general.

Theorem 3.2 (Fundamental Theorem of Calculus, Part 1 [FTC1])
If f is continuous on an interval $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,\mathrm{d}t \qquad \text{for } a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $g'(x) = f(x)$.

Roughly speaking, this says the following: when f is continuous, if we first integrate and then differentiate, we get f back.

Here is the geometric idea behind the Fundamental Theorem of Calculus. Let's assume that $f \geqslant 0$ on

$[a, b]$. To compute $g'(x)$, we use the definition of derivative as a limit:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

For $h > 0$, the numerator $g(x+h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of f between x and $x + h$. If h is small, this area is approximately the area of the rectangle of height $f(x)$ and width h:

$$g(x+h) - g(x) \approx f(x) \cdot h, \qquad \text{so} \qquad \frac{g(x+h) - g(x)}{h} \approx f(x).$$

Intuitively, we therefore expect

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x).$$

Example 3.3 (using FTC)

Find the derivative $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_2^x 3t \sin t \, \mathrm{d}t$.

Solution.
Since the function $f(t) = 3t \sin t$ is continuous, the Fundamental Theorem of Calculus 1 gives us

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_2^x 3t \sin t \, \mathrm{d}t = 3x \sin x.$$ □

Example 3.4 (using FTC | chain rule)

Find the derivative $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_2^{x^2} e^t \, \mathrm{d}t$.

Solution.
Notice that the upper limit of integration is x^2, not x, which means we need to apply chain rule. Let $u = x^2$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_2^{x^2} e^t \, \mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x} \int_2^{u} e^t \, \mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}u} \int_2^{u} e^t \, \mathrm{d}t \cdot \frac{\mathrm{d}}{\mathrm{d}x} u = e^u \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2xe^{x^2}.$$ □

Example 3.5 (using FTC | variable limits of integration)

(a) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_x^5 \sqrt{1+t^2}\,\mathrm{d}t$.

(b) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{2x}^{x^2} e^t\,\mathrm{d}t$.

Solution.

(a) Applying the properties of the integral, we transform this integral in a way that we can apply FTC1 directly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^5 \sqrt{1+t^2}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_5^x \sqrt{1+t^2}\,\mathrm{d}t\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\int_5^x \sqrt{1+t^2}\,\mathrm{d}t = -\sqrt{1+x^2}.$$

(b) For this integral, we apply various the properties of the integral, as well as chain rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{2x}^{x^2} e^t\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{2x}^{0} e^t\,\mathrm{d}t + \int_0^{x^2} e^t\,\mathrm{d}t\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_0^{2x} e^t\,\mathrm{d}t + \int_0^{x^2} e^t\,\mathrm{d}t\right)\\
&= -e^{2x}\frac{\mathrm{d}}{\mathrm{d}x}(2x) + e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\\
&= -2e^{2x} + 2xe^{x^2}.
\end{aligned}$$

□

Computing integrals from the definition as a limit of Riemann sums is usually rather difficult. The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

Theorem 3.6 (Fundamental Theorem of Calculus, Part 2 [FTC2])
If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a),$$

where F is any anti-derivative of f (i.e., $F' = f$).

We often use the following notation:

$$F(x)\Big|_a^b := F(b) - F(a), \qquad \text{or} \qquad \Big[F(x)\Big]_a^b := F(b) - F(a).$$

Thus, the equation of FTC2 can be written as

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b .$$

Proof.
Let $g(x) := \int_a^b f(t)\mathrm{d}t$, then from FTC1, $g'(x) = f(x)$, so g is an anti-derivative of f. If F is any other anti-derivative of f on $[a,b]$, we know that F and g differ by a constant:

$$F(x) = g(x) + C \qquad\qquad \text{for } a < x < b.$$

Since both F and g are continuous on the interval $[a,b]$, by taking one-sided limits as $x \to a^+$ and $x \to b^-$, we see that the equality $F(x) = g(x) + C$ also holds at the end-points $x = a$ and $x = b$. Therefore,

$$\begin{aligned} F(b) - F(a) &= g(b) + C - \big(g(a) + C\big) \\ &= g(b) - g(a) \\ &= \int_a^b f(t)\,\mathrm{d}t - \underbrace{\int_a^a f(t)\,\mathrm{d}t}_{=0} \\ &= \int_a^b f(t)\,\mathrm{d}t \end{aligned}$$

□

The theorem says the following: to calculate the definite integral of f over $[a,b]$, we need to do the following:

(1) Find an anti-derivative F of f,

(2) Calculate the number $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

Typically, step 1 is very complicated, and we will spend much of the rest of this course learning various techniques and tricks of finding anti-derivatives in special situations.

Example 3.7 (definite integral | using FTC)

Evaluate the integral $\int_2^5 \cos x \mathrm{d}x$.

Solution.
The function $f(x) = \cos x$ is continuous, and we know that $F(x) = \sin x$ is an anti-derivative of f. So by FTC2, we have

$$\int_2^5 \cos x\,\mathrm{d}x = \sin(5) - \sin(2).$$

□

Notice that FTC2 says we can use any anti-derivative F of f. We could alternatively have presented the following completely correct solution: $F(x) = \sin x + 7$ is an anti-derivative of $f(x) = \cos x$, so

$$\int_2^5 \cos x \, dx = \big(\sin(5) + 7\big) - \big(\sin(2) + 7\big) = \sin(5) - \sin(2).$$

In the solution above, we chose the anti-derivative $F(x) = \sin x$ only because it looks 'simplest'. But you must understand that 'simplest' here is merely an aesthetic preference, if you will. No anti-derivative is 'better' than any other.

Example 3.8 (area under curve)

Find the area under the curve $y = x^2$ from 0 to 1.

Solution.
An anti-derivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$, so using FTC2 we have

$$\text{Area} = \int_0^1 x^2 \, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

Compare this calculation with the one we did much earlier (see section 2.1.1): computing integrals via anti-derivatives is so much quicker! □

Example 3.9 (definite integral | danger: continuity is important)

What's wrong with the following calculation?

$$\int_{-2}^{1} \frac{1}{x^2} dx = \left.\frac{x^{-1}}{-1}\right|_{-2}^{1} = -1 + \tfrac{1}{2} = -\tfrac{1}{2}.$$

Solution.
Notice that we are integrating a positive function: $f(x) = \frac{1}{x^2} > 0$. So this calculation must be wrong, because the answer is negative (cf. property 5 of the indefinite integral). The Fundamental Theorem of Calculus applies only to continuous functions. In this example, it is inapplicable, because $f(x) = \frac{1}{x^2}$ is not continuous on the interval $[-2, 1]$: at $x = 0$, $f(x)$ has an infinite discontinuity (i.e., *not* a finite jump discontinuity). Therefore,

$$\text{the definite integral} \quad \int_{-2}^{1} \frac{1}{x^2} dx \quad \text{does not exist.}$$

□

§3.1 Differentiation and Integration are Inverse Operations

To conclude this section, let's put together and interpret the two parts of the Fundamental Theorem of Calculus.

Theorem 3.10 (Fundamental Theorem of Calculus)
Let f be a continuous function on $[a, b]$. Then

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_a^x f(t)\,\mathrm{d}t = f(x)$

(2) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$ *where F is any anti-derivative of f (i.e., $F' = f$).*

Part 1 says that if f is integrated and then the result is differentiated, we arrive back at the original function. At the same time, since $F'(x) = f(x)$, Part 2 can be written as

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

This tells us that if we take a function F, differentiate it and then integrate it, then the result is the original function F, but in the form $F(b) - F(a)$.

So the Fundamental Theorem of Calculus is the precise meaning of the statement that differentiation and integration are inverse operations.

§3.2 The Indefinite Integral

Let f be a continuous function on $[a,b]$, and let F be an anti-derivative of f (i.e., $F' = f$). Then the Fundamental Theorem of Calculus implies

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Big[F(x)\Big]_a^b$$

At the same time, if G is another anti-derivative of f, then Fundamental Theorem of Calculus again implies

$$\int_a^b f(x)\,\mathrm{d}x = \Big[G(x)\Big]_a^b.$$

Recall that anti-derivatives differ by a constant, so $G(x) = F(x) + C$ for some constant C. Thus

$$\int_a^b f(x)\,\mathrm{d}x = \Big[G(x)\Big]_a^b = \Big[F(x)+C\Big]_a^b = \big(F(b)+C\big) - \big(F(a)+C\big) = F(b) - F(a) = \Big[F(x)\Big]_a^b.$$

Therefore, the Fundamental Theorem of Calculus expresses a relationship between the integral and *the most general* anti-derivative of f. We express this fact by writing

$$\int f(x)\,\mathrm{d}x := F(x) + C \qquad \text{where } F'(x) = f(x).$$

Basically, all we've done is drop the limits of integration. From this point of view, the most general anti-derivative is called the *indefinite integral*, whose limits of integration are not specified (to be contrasted with the *de*finite integral of f, whose limits of integration *are* specified).

Definition 3.11 (Indefinite Integral)
Let f be a continuous function on $[a,b]$. The **indefinite integral of f** is the most general anti-derivative of f:

$$\int f(x)\,\mathrm{d}x := F(x) + C \qquad \text{where } F'(x) = f(x).$$

The constant C is called the **constant of integration**.

Example 3.12 (indefinite integral)

We can write

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

because $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$.

Remark 3.13 (Important Distinction)

Let's take a moment to reflect on some important concepts that we have introduced so far: definite integral, antiderivative, and indefinite integral. These concepts looks very similar, but they are very different in nature. It is important that you keep this distinction clear in your mind.

A definite integral $\int_a^b f(x)\,\mathrm{d}x$ is a *number*; i.e., it is *not* a function of x. The symbol "x" that appears in the notation $\int_a^b f(x)\,\mathrm{d}x$ is the *integration variable*. Once a definite integral is evaluated, the result *cannot* contain any "x".

The specific antiderivative $F(x) = \int_a^x f(t)\mathrm{d}t$ is a *function* of x. This antiderivative is *specific* in the sense that it satisfies the specific condition $F(a) = 0$; no other antiderivative of f satisfies this condition. The derivative of $F(x)$ with respect to x is the function $f(x)$ by the FTC. In contrast, the derivative of a definite integral $\int_a^b f(x)\,\mathrm{d}x$ with respect to x is 0, because a definite integral is a *number*.

The indefinite integral $\int f(x)\,\mathrm{d}x$ is a *family of functions* of x. Each function's derivative is $f(x)$, and the difference between any two functions in this family is a constant. In contrast, the specific antiderivative $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a *single* function, not a family.

* * *

Definite Integral	Specific Antiderivative	Indefinite Integral
$\displaystyle\int_a^b f(x)\,\mathrm{d}x$	$\displaystyle F(x) = \int_a^x f(t)\,\mathrm{d}t$	$\displaystyle\int f(x)\,\mathrm{d}x$
a number	a function of x	family of functions of x

* * *

The connection between the definite and indefinite integrals is established in the FTC2: if f is continuous, then

$$\int_a^b f(x)\,\mathrm{d}x = \left[\int f(x)\,\mathrm{d}x\right]_a^b.$$

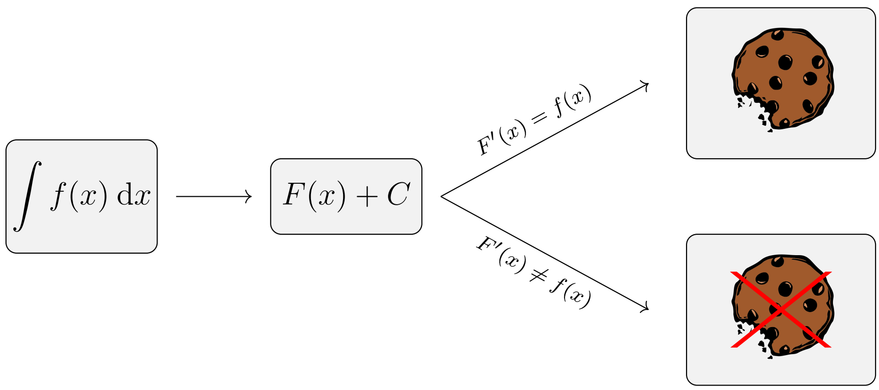
Notice that even though the indefinite integral $\int f(x)\,\mathrm{d}x$ appearing on the right-hand side is a family of functions, the quantity $\left[\int f(x)\,\mathrm{d}x\right]_a^b$ is just a number, because any two functions in this family differ by a constant. Think this point through and make sure it is obvious to you.

Keeping this important distinction in mind, we will from now on use the word *integral* liberally; the context will always make it clear whether we mean *definite integral*, *indefinite integral*, or antiderivative.

The reason the Fundamental Theorem of Calculus is so effective is because we already have a wealth of antiderivatives of functions, such as the table on page 168. You can find many more similar tables in your textbook, or online.

Remark 3.14 (To Check Your Answer)

It cannot be overstated that when you integrate, you can always easily check your answer by simply differentiating your result. Suppose you have just done a very long and messy computation of an integral $\int f(x)\mathrm{d}x$ and you have obtained $F(x)+C$. To check whether your result is correct, differentiate $F(x)$ and check that $F'(x) = f(x)$. If this isn't true, you have made an error, as depicted in the following biscuit diagram:



Example 3.15 (integral)

Find $\displaystyle\int \left(5\sqrt{x} - \frac{2}{x^2+1}\right)\mathrm{d}x$.

Solution.
Using the linearity of the integral and the table on page 168, we have

$$\begin{aligned}\int \left(5\sqrt{x} - \frac{2}{x^2+1}\right)\mathrm{d}x &= 5\int \sqrt{x}\,\mathrm{d}x - 2\int \frac{1}{x^2+1}\mathrm{d}x \\ &= 5\frac{x^{3/2}}{3/2} + C_1 - 2\tan^{-1} x + C_2 \\ &= \tfrac{10}{3}x^{3/2} - 2\tan^{-1} x + C.\end{aligned}$$

(We wrote $C = C_1 + C_2$.) □

Example 3.16 (integral | trigonometric identities)

Find $\displaystyle\int \frac{\sin x}{\cos^2 x}\mathrm{d}x$.

Solution.
This indefinite integral isn't immediate from the table on page 168. We use trigonometric identities to rewrite the integrand:

$$\int \frac{\sin x}{\cos^2 x}\mathrm{d}x = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)\mathrm{d}x = \int \sec x \tan x\,\mathrm{d}x = \sec x + C.$$ □

Example 3.17 (integral)

Find $\int \frac{x+1}{\sqrt{x}} \mathrm{d}x$.

Solution.
Again, this indefinite integral doesn't appear in the table, but we can rewrite the integrand in terms of simpler functions:

$$\int \frac{x+1}{\sqrt{x}} \mathrm{d}x = \int \left(x^{1/2} + x^{-1/2}\right) \mathrm{d}x = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C = \tfrac{2}{3}x^{3/2} + 2x^{1/2} + C. \qquad \square$$

Example 3.18 (integral)

Evaluate $\int_1^4 \frac{x+1}{\sqrt{x}} \mathrm{d}x$.

Solution.
In the previous exercise, we found the indefinite integral:

$$\int \frac{x+1}{\sqrt{x}} \mathrm{d}x = \tfrac{2}{3}x^{3/2} + 2x^{1/2} + C.$$

Thus, using the Fundamental Theorem of Calculus 2, we get

$$\int_1^4 \frac{x+1}{\sqrt{x}} \mathrm{d}x = \left[\int \frac{x+1}{\sqrt{x}} \mathrm{d}x\right]_1^4 = \left[\tfrac{2}{3}x^{3/2} + 2x^{1/2} + C\right]_1^4 = \tfrac{2}{3}4^{3/2} + 2\sqrt{4} - \tfrac{2}{3} - 2 = \tfrac{20}{3}. \qquad \square$$

Remark 3.19 (The Constant of Integration is Vital! Never Forget it!)

Here is why it is crucial to never forget to write " $+C$ " when solving indefinite integrals. Consider the following indefinite integral:

$$\int 2\sin x \cos x \, \mathrm{d}x.$$

On one hand, $(\sin^2 x)' = 2\sin x \cos x$, so $\sin^2 x$ is an anti-derivative of $2\sin x\cos x$, and you would want to write

$$\int 2\sin x \cos x \, \mathrm{d}x = \sin^2 x. \tag{6}$$

On the other hand, $(-\cos^2 x)' = -2\cos x(-\sin x) = 2\sin x\cos x$, so $-\cos^2 x$ is *also* an anti-derivative of $2\sin x\cos x$; so you would *also* want to write

$$\int 2\sin x \cos x \, \mathrm{d}x = -\cos^2 x. \tag{7}$$

But this is a contradiction: (6) and (7) together imply $\sin^2 x = -\cos^2 x$, which is totally *totally* false! (For example, $\sin^2(0) = 0$, but $-\cos^2(0) = -1$.)

So what has gone wrong? Recall the trigonometric identity $\sin^2 x + \cos^2 x = 1$. Using it, we can rewrite the right-hand side of (6) as $-\cos^2 x + 1$. This means that the right-hand sides of (6) and (7) *differ by a constant*, in this case 1. The point here is that

neither (6) nor (7) is correct!

The left hand-side is a family of functions, whilst the right-hand side is a single function. To correct (6) and (7), *we must include the constants of integration* on the right-hand sides like so:

$$\int 2\sin x \cos x \, \mathrm{d}x = \sin^2 x + C_1, \qquad \text{where } C_1 \text{ is an arbitrary constant} \tag{8}$$

$$\int 2\sin x \cos x \, \mathrm{d}x = -\cos^2 x + C_2, \qquad \text{where } C_2 \text{ is an arbitrary constant.} \tag{9}$$

(Notice that we called the constants of integration by different names to emphasise that C_1 and C_2 may not be the same.) Why does this eliminate the contradiction we encountered earlier? This is because (8) and (9) together imply

$$\sin^2 x + C_1 = -\cos^2 x + C_2 \qquad \text{i.e.,} \qquad \sin^2 x + \cos^2 x = C_2 - C_1,$$

which is a true statement whenever $C_2 - C_1 = 1$.

§3.3 Applications

The Fundamental Theorem of Calculus 2 says that if f is a continuous function on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b \qquad \text{where } F'(x) = f(x).$$

This can be written as

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

We know that

- $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x;
- $F(b) - F(a)$ represents the net change in y when x changes from a to b.

In this language, the Fundamental Theorem of Calculus 2 can be formulated as follows.

Theorem 3.20 (The Net Change Theorem)
The integral of a rate of change is a net change:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

Example 3.21 (displacement as definite integral)

If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_0}^{t_1} v(t)\,\mathrm{d}t = s(t_1) - s(t_0)$$

is the net change or displacement.

Example 3.22 (finding total displacement)

A particle moves along a straight line with velocity $v(t) = t^2 - 2 - 2$. Find the displacement of the particle during the time period $0 \leqslant t \leqslant 3$.

Solution.
From example 3.21, the displacement is

$$s(3) - s(0) = \int_0^3 v(t)\,\mathrm{d}t = \int_0^3 (t^2 - t - 2)\,\mathrm{d}t = \left[\frac{t^3}{3} - \frac{t^2}{2} - 2t\right]_0^3 = -\frac{3}{2}.$$

This means that the particle moved 1.5 units to the left. □

§4 Techniques of Integration

§4.1 Substitution Rule

Example 4.1 (integral | substitution)

Find $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\mathrm{d}x$.

Solution.
This integral looks difficult! The simple anti-differentiation formulas we have do not tell us how to compute this integral. We need a trick!

What is the scariest, ugliest looking thing in this integrand? I think it's the square root $\sqrt{x^2+1}$; let's call it a name, like u (stands for 'ugly'):

$$u := \sqrt{x^2+1}.$$

Using chain rule, notice the following:

$$\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \frac{x}{\sqrt{x^2+1}}\mathrm{d}x.$$

A-ha! This is exactly the integrand! Therefore,

$$\int \frac{x}{\sqrt{x^2+1}}\mathrm{d}x = \int \mathrm{d}u = u + C = \sqrt{x^2+1} + C.$$

To make sure we've done this right, we check the answer by differentiating:

$$\left(\sqrt{x^2+1}+C\right)' = \frac{1}{2\sqrt{x^2+1}} \cdot (x^2+1)' + 0 = \frac{x}{\sqrt{x^2+1}},$$

which agrees with the integrand, so we've integrated correctly. □

Example 4.2 (integral | substitution)

Find $\displaystyle\int x\cos(3x^2+2)\mathrm{d}x$.

Solution.
Again, this integrand doesn't appear in any of our anti-derivative tables. However, we know the integral $\int \cos u \mathrm{d}u$, and the main difference between it and our case is the argument $(3x^2+2)$. This is the 'ugly term', so we call it a name:

$$u = 3x^2 + 2 \quad \rightsquigarrow \quad \mathrm{d}u = 6x\,\mathrm{d}x \quad \Longrightarrow \quad x\,\mathrm{d}x = \tfrac{1}{6}\mathrm{d}u.$$

Now, write the integrand entirely in terms of u:

$$x\cos(3x^2+2)\,\mathrm{d}x = \cos(3x^2+2)(x\,\mathrm{d}x) = \cos(u)\tfrac{1}{6}\,\mathrm{d}u.$$

So

$$\int x\cos(3x^2+2)\,\mathrm{d}x = \tfrac{1}{6}\int \cos(u)\,\mathrm{d}u = \tfrac{1}{6}\sin(u) + C = \tfrac{1}{6}\sin(3x^2+2) + C.$$

Again, check the result by differentiation:

$$\left(\tfrac{1}{6}\sin(3x^2+2)+C\right)' = \tfrac{1}{6}\cos(3x^2+2)\cdot 6x = x\cos(3x^2+2).$$ □

These examples make the strategy more or less clear. We summarise it as follows.

Theorem 4.3 (Substitution Rule)
If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\big(g(x)\big)\, g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u.$$

Proof.

The main ingredient in the proof is Chain Rule. Notice that if $F' = f$, then by Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(F\big(g(x)\big)\Big) = F'\big(g(x)\big)\, g'(x) = f\big(g(x)\big)\, g'(x). \tag{10}$$

Thus, the result of a change of variables $u = g(x)$ is as follows:

$$\begin{aligned} \int f\big(g(x)\big)\, g'(x)\, \mathrm{d}x &= F\big(g(x)\big) + C && \text{(by (10))} \\ &= F(u) + C && (u = g(x)) \\ &= \int f(u) + C. && (F' = f) \end{aligned}$$

□

The general guideline for using the Substitution Rule is as follows.

Substitution Rule Strategy:

(1) **Observe and try.**
Try to find an occurrence of a function $g(x)$ and its differential $g'(x)\,\mathrm{d}x$ in the integrand. Often, it is the 'ugly' term in the integrand; try and see what happens.

(2) **Make the substitution $u = g(x)$.**
Let $u = g(x)$ be the new integration variable, then compute $\mathrm{d}u = g'(x)\,\mathrm{d}x$.

(3) **Write everything in terms of u and $\mathrm{d}u$.**
The variable x must not appear anywhere in the integral!

(4) **Integrate** with respect to u.

(5) **Back substitute** $u = g(x)$ in the final answer.
(The variable u must not appear anywhere in the final answer!)

Example 4.4 (integral | substitution)

Use a different substitution to find the integral $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\mathrm{d}x$ of example 4.1.

Solution.
We follow our general guideline. Notice that if $g(x) = x^2 + 1$ then $g'(x) = 2x$, and x appears in the numerator. Thus, we can take $u = x^2 + 1$, so $\mathrm{d}u = 2x\,\mathrm{d}x$, from where it follows that $x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$. Now we write the integral entirely in terms of u:

$$\int \frac{x}{\sqrt{x^2+1}}\mathrm{d}x = \int \frac{1}{\sqrt{u}}\tfrac{1}{2}\mathrm{d}u = \frac{1}{2}\int u^{-1/2}\mathrm{d}u = \tfrac{1}{2}(2u^{1/2}) + C = u^{1/2} + C = \sqrt{x^2+1} + C.$$

□

The moral of the Substitution Rule is that often we can replace a relatively complicated integral with a simpler one.

Example 4.5 (integral | substitution)

Compute $\displaystyle\int x\sqrt{x+1}\,\mathrm{d}x$.

Solution.
Let $u = x+1$, so $\mathrm{d}u = \mathrm{d}x$. Then $x\sqrt{x+1} = (u-1)\sqrt{u}$. Therefore,

$$\int x\sqrt{x+1}\mathrm{d}x = \int (u-1)\sqrt{u}\mathrm{d}u = \int \left(u^{3/2}-u^{1/2}\right)\mathrm{d}u = \tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}+C = \tfrac{2}{5}(x+1)^{5/2}-\tfrac{2}{3}(x+1)^{3/2}+C.$$

□

§4.2 Definite Integrals and Substitution

When evaluating a *definite* integral by substitution, there are two methods.

Method 1. First, find the indefinite integral; then, apply the Fundamental Theorem of Calculus.

Example 4.6 (integral | substitution)

Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x\,\mathrm{d}x$.

Solution.
First, we find the corresponding *indefinite* integral $\displaystyle\int \sin^2 x\cos x\,\mathrm{d}x$.

Notice that $(\sin x)' = \cos x$, so we take $u = \sin x$, whence $\mathrm{d}u = \cos x\,\mathrm{d}x$. Then

$$\int \sin^2 x\cos x\,\mathrm{d}x = \int u^2\,\mathrm{d}u = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3 x + C.$$

(Don't forget to check that this is correct by differentiation.)

Next, we use the Fundamental Theorem of Calculus:

$$\int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x = \tfrac{1}{3}\sin^3 x\Big|_0^{\pi/2} = \tfrac{1}{3}\sin^3(\pi/2) - \tfrac{1}{3}\sin^3(0) = \tfrac{1}{3}.$$

□

Method 2. Change the limits of integration when the variable is changed, as followings.

Theorem 4.7 (Substitution Rule for Definite Integrals)
If $y = g(x)$ is a differentiable function whose range is an interval I *and f is a continuous function* I, *then*

$$\int_a^b f\big(g(x)\big)\, g'(x)\, \mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\, \mathrm{d}u. \tag{11}$$

Proof.
Let F be an anti-derivative of f. Then via Chain Rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} F\big(g(x)\big) = f\big(g(x)\big)\, g'(x),$$

so the Fundamental Theorem of Calculus implies

$$\int_a^b f\big(g(x)\big)\, g'(x)\, \mathrm{d}x = F\big(g(x)\big)\Big|_a^b = F\big(g(b)\big) - F\big(g(a)\big).$$

On the other hand, applying the Fundamental Theorem of Calculus to the right-hand side, we get

$$\int_{g(a)}^{g(b)} f(u)\, \mathrm{d}u = F(u)\Big|_{g(a)}^{g(b)} = F\big(g(b)\big) - F\big(g(a)\big).$$

This completes the proof. □

The rule says that when using a substitution in a definite integral, we must write *everything* in terms the new variable u — this includes the limits of integration.

Example 4.8 (integral | substitution)

Evaluate $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x\, \mathrm{d}x$.

Solution.
We can use the same substitution as before:

$$u = \sin x \qquad \rightsquigarrow \mathrm{d}u = \cos x\, \mathrm{d}x.$$

Next, we find the new limits of integration:

$$\text{when } x = 0, \qquad u = \sin(0) = 0;$$
$$\text{when } x = \pi/2, \qquad u = \sin(\pi/2) = 1.$$

So

$$\int_0^{\pi/2} \sin^2 x \cos x \, dx = \int_0^1 u^2 \, du = \tfrac{1}{3}u^3\Big|_0^1 = \tfrac{1}{3}. \qquad \square$$

Theorem 4.9 (Integrals of Symmetric Functions)
Let f be a continuous function on the interval $[-a, a]$. Then:

(1) *If f is an odd function (i.e., $f(-x) = -f(x)$)* $\implies \displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

(2) *If f is an even function (i.e., $f(-x) = f(x)$)* $\implies \displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

Proof.
First, regardless of whether f is even or odd, we rewrite the integral as follows. Use the additivity to split the integral into two parts:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$
$$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

let $u = -x$, so $du = -dx$; change the limits of integration: when $x = -a$, we have $u = a$; when $x = 0$, get $u = 0$. Thus:

$$= -\int_0^a f(-u)\,(-du) + \int_0^a f(x)\,dx$$
$$= \int_0^a f(-u)\,du + \int_0^a f(x)\,dx.$$

Now, to prove (1), assume f is odd. Then $f(-u) = -f(u)$, so

$$\int_0^a f(-u)\,du + \int_0^a f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0.$$

(Remember: both u and x are just integration variables (i.e., 'dummy' variables); that's why the two integrals cancel.) This proves (1).

To prove (2), assume f is an even function. Then $f(-u) = f(u)$, so

$$\int_0^a f(-u)\,du + \int_0^a f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

This proves (2). □

Example 4.10 (integral of odd function)

Compute the integral $\displaystyle\int_{-100}^{100} \frac{\sin x + x^3}{x^4 + \cos x}\,\mathrm{d}x$.

Solution.
First, observe that the integrand $f(x) := \dfrac{\sin x + x^3}{x^4 + \cos x}$ is an odd function (check it!). Thus,

$$\int_{-100}^{100} \frac{\sin x + x^3}{x^4 + \cos x} = 0.$$

□

§4.3 Integration by Parts

You may have already recognised that, although differentiation and integration are intimately connected, computing derivatives is very different from computing integrals. Computing integrals is hard work! (See an interesting discussion at math.stackexchange). We have simple rules that allow us to compute derivatives of almost any function. On the other hand, computing integrals even of very simple looking functions is often very hard.

So far, we have a collection of basic integrals in our Antiderivative Table. Beyond these formulas, we have seen one integration technique: the Substitution Rule, which we can use to write integrands in a simpler form. For example, to find

$$\int xe^{x^2}\,\mathrm{d}x\ ,$$

we can make the substitution $u = x^2$. This implies $\mathrm{d}u = 2x\mathrm{d}x$, and the integral becomes

$$\int xe^{x^2}\,\mathrm{d}x = \tfrac{1}{2}\int e^u\,\mathrm{d}u = \tfrac{1}{2}e^{x^2} + C\ .$$

This was very easy to compute. However, consider a slight modification of this integral:

$$\int xe^{x}\,\mathrm{d}x\ .$$

This integral does not appear in the Antiderivatives Table. Can you think of the right substitution to make in order to find this integral? I can't; the substitution method is hopeless.

Since integration and differentiation are so closely related, every differentiation rule has a corresponding method for integration. For example, we have deduced the Substitution Rule from the Chain Rule:

$$\text{Substitution Rule} \quad \leftrightsquigarrow \quad \text{Chain Rule}\ .$$

In this section, we study another method of integration, called **Integration by Parts**. It corresponds to the Product Rule for differentiation:

$$\text{Integration by Parts} \quad \leftrightsquigarrow \quad \text{Product Rule} .$$

Recall the Product Rule: if f, g are differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)g(x)\Big) = f'(x)g(x) + f(x)g'(x) .$$

If we integrate both sides of this equation, we get

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)g(x)\Big)\,\mathrm{d}x = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x .$$

Notice that on the left-hand side we can use the Fundamental Theorem of Calculus to "cancel" integration and differentiation[1]:

$$f(x)g(x) = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x .$$

Now, re-arrange this equation:

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x .$$

This is called the **Integration by Parts Formula**.

To help us remember it easier, introduce some notation: let $u := f(x)$ and $v := g(x)$; then $\mathrm{d}u = f'(x)\mathrm{d}x$ and $\mathrm{d}v = g'(x)\mathrm{d}x$.

Theorem 4.11 (Integration by Parts)

$$\boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u .} \tag{12}$$

The Integration by Parts Formula is a very useful tool. According to equation (12), we can calculate an integral of the form $\int u\,\mathrm{d}v$ by calculating the integral $\int v\,\mathrm{d}u$ instead. The power of this method lies in that it may be the case that the integral $\int v\,\mathrm{d}u$ is much simpler than $\int u\,\mathrm{d}v$. Let us demonstrate this with many examples. To apply Integration by Parts, we need to make a clever choice of u and $\mathrm{d}v$ such that the integral $\int v\,\mathrm{d}u$ is easy to compute. The only way to learn how to make these clever choices is to

practise, practise, practise, . . . , practise, and then practise some more!

[1]Notice: when we use the FTC to "cancel" integration and differentiation on the left-hand side, we are technically supposed to add the constant of integration. But we didn't; why? The reason is this: on the right-hand side of this equation, we still have some indefinite integrals appearing. When finding them, we'll need to add integration constants. In the end, we would combine all the integration constants into one. This is confusing! Think about this very thoroughly until it makes sense, and make sure it makes sense to you!

Example 4.12 (integration by parts)

Find $\int xe^x \, dx$.

Solution.
To use Integration by Parts, we need to choose:

$$\begin{aligned} u &\quad \text{(something to differentiate)} \\ dv &\quad \text{(something to integrate)} \ . \end{aligned}$$

Let $u = x$ and $dv = e^x dx$. Then $du = dx$. Integrating dv, we find[a]

$$v = \int e^x dx = e^x \ . \tag{13}$$

When solving an integral via integration by parts, it's often convenient to write this information as a block like this:

$$\begin{aligned} u = x, &\quad dv = e^x \, dx \\ du = dx, &\quad v = e^x \ . \end{aligned}$$

Thus, we get[b]

$$\int \underbrace{x}_{u} \underbrace{e^x \, dx}_{dv} = uv - \int v \, dv = xe^x - \int e^x \, dx = xe^x - e^x + C \ .$$

Don't forget to check the answer by differentiating the result. □

[a] think about why it makes sense to drop the constant of integration here.

[b] Notice: we've been dropping the constants of integration at all the intermediate steps. It isn't because we were "sloppy". For example, in (13), the constant of integration was dropped, but not because we were "lazy". Admittedly, this thing about integration constants — when to drop them and when not to — can be very confusing. You must invest time into understanding why it makes sense to drop it in some steps, but not the others. It's an important point, so don't lay down your sword until it becomes crystal clear to you. Come to our office hours and ask us!

Our aim in using Integration by Parts is to obtain a simpler integral than the original one. The choice of u and dv is critical! Observe what happens if make a poor choice.

Example 4.13 (integration by parts | poor choice of u and dv)

Consider again the integral $\int xe^x \, dx$.

Bad Solution.
If we let $u = e^x$ and $dv = x \, dx$, then

$$\begin{aligned} u = e^x, &\quad dv = x \, dx \\ du = e^x dx, &\quad v = \tfrac{x^2}{2} \ . \end{aligned}$$

Then

$$\int xe^x \, dx = \int \underbrace{e^x}_{u} \underbrace{x \, dx}_{dv} = \tfrac{x^2}{2} e^x - \int \tfrac{x^2}{2} e^x \, dx \; .$$

The integral on the right-hand side is even harder than the original one! □

Example 4.14 (integration by parts | integrand with single term)

Find $\displaystyle\int \ln x \, dx$.

Solution.
This integral doesn't appear in our Antiderivatives Table, and substitution doesn't look promising. We use integration by parts. Remember that we must choose

$$\begin{array}{cl} u & \text{(something to differentiate)} \\ dv & \text{(something to integrate)} \end{array} \; .$$

Of course, taking $dv = \ln x \, dx$ would be stupid, because we would then need to find v, and that amounts to integrating dv, which is what we are trying to find in the first place! Thus, we make the following choices:

$$\begin{aligned} u &= \ln x, & dv &= dx \\ du &= \tfrac{1}{x} \, dx, & v &= x \; . \end{aligned}$$

Therefore,

$$\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = uv - \int v \, dv = x \ln x - \int \underbrace{x \tfrac{1}{x}}_{1} \, dx = x \ln x - x + C \; .$$

Check this by differentiating the result. □

The Integration by Parts worked in this example because the derivative of the function $f(x) = \ln x$ is simpler than f. When computing integrals, you should look for instances of this: *if one of the functions appearing in the integrand has a "simpler" derivative, the Integration by Parts method may work.*

Frequently, an Integration by Parts results in an integral that we cannot compute directly or immediately, but that can still be evaluated by applying Integration by Parts. In other words, some integrals can be solved via a *chain of Integrations by Parts*.

Example 4.15 (integration by parts | repeated application)

Compute $\displaystyle\int x^2 \sin x \, dx$.

Solution.
Notice that x^2 becomes simpler when differentiated. This motivates our choice:

$$\begin{aligned} u &= x^2, \quad \mathrm{d}v = \sin x \mathrm{d}x \\ \mathrm{d}u &= 2x\,\mathrm{d}x, \quad v = -\cos x \end{aligned}$$

Then, Integration by Parts gives:

$$\int \underbrace{x^2}_{u} \underbrace{\sin x\,\mathrm{d}x}_{\mathrm{d}v} = uv - \int v\,\mathrm{d}u = -x^2\cos x - \int (-\cos x)2x\,\mathrm{d}x = -x^2\cos x + 2\int x\cos x\,\mathrm{d}x \quad (14)$$

The integral on the right-hand side is simpler, but it still doesn't appear in our Antiderivatives Table, nor is there any obvious substitution to make. However, we again notice that the derivative of x is 1, so we choose:

$$\begin{aligned} u &= x, \quad \mathrm{d}v = \cos x \mathrm{d}x \\ \mathrm{d}u &= \mathrm{d}x, \quad v = \sin x \end{aligned}$$

Then

$$\int \underbrace{x}_{u} \underbrace{\cos x\,\mathrm{d}x}_{\mathrm{d}v} = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + C .$$

Putting this back into equation (14), we get

$$\int x^2 \sin x\,\mathrm{d}x = -x^2\cos x + 2(x\sin x + \cos x) + C . \qquad \square$$

Based on this example, how many Integrations by Parts do you think would be required to compute the integral

$$\int x^n \sin x\,\mathrm{d}x$$

for a positive integer n? Put these notes away to think about the answer before reading any further.

Welcome back! Each time we apply Integration by Parts to $\int x^n \overbrace{\;—\;}^{\sin x \text{ or } \cos x} \mathrm{d}x$, we end up with the integral $\int x^{n-1} \underbrace{\;—\;}_{\sin x \text{ or } \cos x} \mathrm{d}x$. Thus, each time, we drop the degree of x^n by 1, until it disappears from the integrand. This needs to be done n times. So the answer is: n Integrations by Parts are required.

Sometimes, Integration by Parts can produce the original integral back to you. This can sometimes be used to our advantage.

Example 4.16 (integration by parts | repeated application with a twist)

Compute $\displaystyle\int e^x \cos x\,\mathrm{d}x$.

Solution.
Notice that neither e^x nor $\cos x$ become simpler when differentiated. Try the following:

$$\begin{aligned} u &= e^x, \quad \mathrm{d}v = \cos x \mathrm{d}x \\ \mathrm{d}u &= e^x \,\mathrm{d}x, \quad v = \sin x \ . \end{aligned}$$

(The opposite choice also works — try it!) Integration by Parts gives:

$$\int \underbrace{e^x}_{u} \underbrace{\cos x \,\mathrm{d}x}_{\mathrm{d}v} = e^x \sin x - \int e^x \sin x \,\mathrm{d}x \ . \tag{15}$$

The integral on the right-hand side is not any easier. But let's try applying Integration by Parts to it:

$$\begin{aligned} u &= e^x, \quad \mathrm{d}v = \sin x \mathrm{d}x \\ \mathrm{d}u &= e^x \,\mathrm{d}x, \quad v = -\cos x \ . \end{aligned}$$

Then

$$\int \underbrace{e^x}_{u} \underbrace{\sin x \,\mathrm{d}x}_{\mathrm{d}v} = -e^x \cos x + \int e^x \cos x \,\mathrm{d}x \ . \tag{16}$$

The integral on the right-hand side is exactly the original integral. You might think we're doomed, but stay with me: combine equation (16) with equation (15) to write

$$\int e^x \sin x \,\mathrm{d}x = e^x \sin x + e^x \cos x - \int e^x \cos x \,\mathrm{d}x \ .$$

Here's the **trick**: *bring the integral on the right-hand side over to the left-hand side*:

$$2 \int e^x \sin x \,\mathrm{d}x = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x) \ .$$

Dividing both sides by 2 and adding the constant of integration, and we obtain the final answer:

$$\int e^x \sin x \,\mathrm{d}x = \tfrac{1}{2} e^x(\sin x + \cos x) + C \ .$$

Isn't this a clever trick? □

§4.4 Integration by Parts for Definite Integrals

If we combine the formula for Integration by Parts with the Fundamental Theorem of Calculus, we can easily evaluate definite integrals via Integration by Parts.

Proposition 4.17 (Integration by Parts for Definite Integrals)

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du .$$

Example 4.18 (integration by parts | definite integral)

Evaluate $\displaystyle\int_1^2 x^2 \ln x \, dx$.

Solution.
We use Integration by Parts by choosing

$$u = \ln x, \quad dv = x^2 dx$$
$$du = \tfrac{1}{x}\, dx, \quad v = \tfrac{x^3}{3} .$$

Therefore,

$$\begin{aligned}
\int_1^2 x^2 \ln x \, dx &= \int_1^2 \underbrace{\ln x}_{u}\, \underbrace{x^2 \, dx}_{dv} \\
&= \tfrac{1}{3} x^3 \ln x\Big|_1^2 + \int_1^2 \tfrac{x^3}{3} \cdot \tfrac{1}{x} \, dx \\
&= \tfrac{1}{3}(8 \ln 2 - \ln 1) - \int_1^2 \tfrac{x^2}{3} \, dx \\
&= \tfrac{8}{3} \ln 2 - 0 - \frac{x^3}{9}\Big|_1^2 \\
&= \tfrac{8}{3} \ln 2 - \tfrac{7}{9} .
\end{aligned}$$

□

§4.5 Integration of Trigonometric Functions

Integrating trigonometric functions very often comes down to a combination of an imaginative use of the basic trigonometric identities and making a clever substitution. Here are a few identities which are very useful; it's worth the investment to memorise them.

PYTHAGOREAN IDENTITIES
$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

DOUBLE-ANGLE FORMULAS
$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$
$\sin(2x) = 2\sin x \cos x$

HALF-ANGLE FORMULAS
$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$
$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS
$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$

These identities will be instrumental for the development of this section.

Example 4.19 (integral | trigonometric functions)

Find $\displaystyle\int \cos^8 x \sin x \, \mathrm{d}x$.

Solution.
There is nothing particularly new or tricky about this integral. We can use substitution: look for

terms that are derivatives of other terms. You quickly realise that you should take:

$$u = \cos x, \quad \rightsquigarrow \quad \mathrm{d}u = -\sin x \, \mathrm{d}x \ .$$

So

$$\int \underbrace{\cos^8 x}_{u^8} \ \underbrace{\sin x \, \mathrm{d}x}_{-\mathrm{d}u} = -\int u^8 \, \mathrm{d}u = -\frac{u^9}{9} + C = -\frac{1}{9}\cos^9 x + C \ .$$

□

Example 4.20 (integral | trigonometric functions | odd power of cosine)

Calculate $\int \cos^3 x \sin^4 x \, \mathrm{d}x$.

Solution.
First, let's try

$$u = \cos x, \quad \rightsquigarrow \quad \mathrm{d}u = -\sin x \, \mathrm{d}x \ .$$

Then

$$\int \cos^3 x \sin^4 x \, \mathrm{d}x = -\int \underbrace{\cos^3 x}_{u^3} \ \underbrace{\sin^3 x}_{\substack{\text{write this} \\ \text{term in} \\ \text{terms of} \\ \text{cosine}}} \ \underbrace{(-\sin x) \, \mathrm{d}x}_{\mathrm{d}u} \ .$$

Using the Trigonometric Identity $\sin^2 x = 1 - \cos^2 x$, we find

$$\sin^3 x = \left(\sqrt{1 - \cos^2 x}\right)^3 = (1 - u^2)^{3/2} \ .$$

The square root makes the integrand expression more cumbersome, so this doesn't seem too promising.

Let's try instead

$$u = \sin x, \quad \rightsquigarrow \quad \mathrm{d}u = \cos x \, \mathrm{d}x \ .$$

Then

$$\begin{aligned}
\int \cos^3 x \ \sin^4 x \, \mathrm{d}x &= -\int \underbrace{\cos^2 x}_{\substack{\text{write this} \\ \text{term in} \\ \text{terms of} \\ \text{sine}}} \ \underbrace{\sin^4 x}_{u^4} \ \underbrace{\cos x \, \mathrm{d}x}_{\mathrm{d}u} \\
&= (1 - \sin^2 x) \sin^4 x \ \cos x \, \mathrm{d}x \\
&= \int (1 - u^2) u^4 \, \mathrm{d}u \\
&= \int (u^4 - u^6) \, \mathrm{d}u \\
&= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C \ .
\end{aligned}$$

□

Example 4.21 (integral | trigonometric functions | odd power of sine)

Find $\displaystyle\int \sin^5 x\, \cos^2 x\, \mathrm{d}x$.

Solution.
Consider

$$u = \cos x, \quad \rightsquigarrow \quad \mathrm{d}u = -\sin x\, \mathrm{d}x\ .$$

Then

$$\begin{aligned}\int \sin^5 x\, \cos^2 x\, \mathrm{d}x &= \int \underbrace{\sin^4 x}_{(1-u^2)^2}\ \underbrace{\cos^2 x}_{u^2}\ \underbrace{\sin x\, \mathrm{d}x}_{-\mathrm{d}u} \\ &= -\int (1-u^2)^2 u^2\, \mathrm{d}u \\ &= -\int (u^2 - 2u^4 + u^6)\, \mathrm{d}x \\ &= -\tfrac{u^3}{3} + \tfrac{2}{5}u^5 - \tfrac{u^7}{7} + C \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C\ .\end{aligned}$$

□

Look at how complicated the expression in this answer is! There is no way would could've guessed this anti-derivative, even though the integrand looks rather innocent.

On a more serious note, pay close attention to how we solved this. We took the odd power (in this case, $\sin^5 x$) and factored out a single power (of $\sin x$) to use it for $\mathrm{d}u$. The remaining even powers can be written in terms of $\cos x$ using the Pythagorean identity $\sin^2 x + \cos^2 x + 1$. This is a useful technique: whenever you see an odd power of a trigonometric function in the integrand, try and see if it works.

Example 4.22 (integral | trigonometric functions)

Find $\displaystyle\int \sin^3 x\, \mathrm{d}x$.

Solution.
What should we try here? Taking $u = \sin x$ isn't helpful, because $\mathrm{d}u = \cos x\, \mathrm{d}x$, but the integrand has no $\cos x$. Instead, we see an odd power of a trigonometric function, so let's try the technique we've just discussed. Write $\sin^3 x = \sin^2 x \cdot \sin x$, use the Pythagorean trigonometric identity to write $\sin^2 x = 1 - \cos^2 x$. Take $u = \cos x$, so that $\mathrm{d}u = -\sin x\, \mathrm{d}x$. Then

$$\begin{aligned}\sin^3 x\, \mathrm{d}x &= \int \underbrace{\sin^2 x}_{1-u^2}\ \underbrace{\sin x\, \mathrm{d}x}_{-\mathrm{d}u} \\ &= -\int (1-u^2)\, \mathrm{d}u \\ &= -u + \tfrac{u^3}{3} + C \\ &= -\cos x + \tfrac{1}{3}\cos^3 x + C\ .\end{aligned}$$

□

Method: To compute $\displaystyle\int \sin^n x \, \cos^m x \, \mathrm{d}x$:

- if m is odd, then try $u = \sin x$;
- if n is odd, then try $u = \cos x$.

If both m and n are odd, you can try either substitution.

Example 4.23 (integral | trigonometric functions)

Compute $\displaystyle\int \cos^2 x \, \mathrm{d}x$.

Solution.
Here we can use the half-angle formula for $\cos^2 x$ to get

$$\int \cos^2 x \, \mathrm{d}x = \int \left(\frac{1+\cos(2x)}{2}\right) \mathrm{d}x = \tfrac{1}{2}x + \tfrac{1}{2}\int \cos(2x)\, \mathrm{d}x = \tfrac{1}{2}x + \tfrac{1}{4}\sin(2x) + C \, .$$

(Here, we can integrate $\cos(2x)$ using substitution; which one?) □

Example 4.24 (integral | trigonometric functions)

Find $\displaystyle\int \sin^4 x \, \mathrm{d}x$.

Solution.
This integral does fit into the general method stated above, because the power of sine is even. Instead, notice that $\sin^4 x = (\sin^2 x)^2$ and then use the half-angle formula. So

$$\begin{aligned}
\int \sin^4 x \, \mathrm{d}x &= \int (\sin^2 x)^2 \, \mathrm{d}x \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \mathrm{d}x \\
&= \tfrac{1}{4}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right) \mathrm{d}x \\
&= \tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{4}\int \cos^2(2x)\, \mathrm{d}x \\
&= \tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{4}\int \frac{1+\cos(4x)}{2}\, \mathrm{d}x \\
&= \tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{8}x + \tfrac{1}{32}\sin(4x) + C \, .
\end{aligned}$$

□

We can use similar strategies to compute integrals involving powers of tangent and secant.

Example 4.25 (integral | trigonometric functions)

Find $\int \sec^4 x \ \tan^5 x \, \mathrm{d}x$.

Solution.
Recall some relevant derivatives and trigonometric identities:

$$\tfrac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x, \qquad \tfrac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x, \qquad \sec^2 x = 1 + \tan^2 x \ .$$

Notice that if we separate one $\sec^2 x$ factor, then we can express the remaining factor $\sec^2 x$ factor in terms of $\tan x$. We take $u = \tan x$, so that $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$, and therefore

$$\begin{aligned} \int \sec^4 x \ \tan^5 x \, \mathrm{d}x &= \int \underbrace{\sec^2 x}_{u^2+1} \ \underbrace{\tan^5 x}_{u^5} \ \underbrace{\sec^2 x \, \mathrm{d}x}_{\mathrm{d}u} \\ &= \int (u^2+1) u^5 \, \mathrm{d}u \\ &= \int (u^7 + u^5) \, \mathrm{d}u \\ &= \tfrac{1}{8} \tan^8 x + \tfrac{1}{6} \tan^6 x + C \ . \end{aligned}$$

□

Example 4.26 (integral | trigonometric functions)

Compute $\int \sec^3 x \ \tan^3 x \, \mathrm{d}x$.

Solution.
Try $u = \tan x$, so that $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$. So the integral becomes

$$\int \sec^3 x \ \tan^3 x \, \mathrm{d}x = \int \underbrace{\sec x}_{\substack{\text{isn't easy} \\ \text{to express} \\ \text{in terms} \\ \text{of tangent}}} \ \underbrace{\tan^3}_{u^3} \underbrace{\sec^2 x \, \mathrm{d}x}_{\mathrm{d}u} \ .$$

Instead, try $u = \sec x$, so that $\mathrm{d}u = \sec x \ \tan x \, \mathrm{d}x$. Then

$$\begin{aligned} \int \sec^3 x \ \tan^3 x \, \mathrm{d}x &= \int \underbrace{\sec^2 x}_{u^2} \underbrace{\tan^2 x}_{u^2-1} \underbrace{\sec x \ \tan x \, \mathrm{d}x}_{\mathrm{d}u} \\ &= \int u^2 (u^2 - 1) \, \mathrm{d}u \\ &= \int (u^4 - u^2) \, \mathrm{d}u \\ &= \tfrac{u^5}{5} - \tfrac{u^3}{3} + C \\ &= \tfrac{1}{5} \sec^5 x - \tfrac{1}{3} \sec^3 x + C \ . \end{aligned}$$

□

As usual, once we get the integrand to look like a polynomial, we win.

Method: To compute $\displaystyle\int \sec^n x \, \tan^m x \, \mathrm{d}x$:

- if m is odd, then try $u = \sec x$;
  - $\rightarrow$ save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$.
- if n is even, then try $u = \tan x$.
  - $\rightarrow$ save a factor of $\sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to express the remaining factors in terms of $\tan x$.

A similar method applies to integrals involving $\csc x$ and $\cot x$.

Powers of secant may require Integration by Parts, as shown in the following example.

Example 4.27 (integral | $\sec^3 x$ | integration by parts)

Find $\displaystyle\int \sec^3 x \, \mathrm{d}x$.

Solution.
We could try applying the trigonometric identity $\sec^2 x = \tan^2 x + 1$, which would give

$$\int \sec^3 x \, \mathrm{d}x = \int \sec x(\tan^2 x + 1) \, \mathrm{d}x = \int \sec x \tan^2 x \, \mathrm{d}x + \int \sec x \, \mathrm{d}x \ .$$

The second integral is known from the table of standard integrals. But the first integral is more difficult. You could try another application of the relation $\sec^2 x + \tan^2 x + 1$, but it doesn't help, because it'll only take you right back to where you started:

$$\int \sec x \tan^2 x \mathrm{d}x + \int \sec x \mathrm{d}x = \int \sec x(\sec^2 x - 1)\mathrm{d}x + \int \sec x \mathrm{d}x = \int \sec^3 x \mathrm{d}x - \cancel{\int \sec x \mathrm{d}x} + \cancel{\int \sec x \mathrm{d}x} \ .$$

Instead, let's try Integration by Parts:

$$\begin{aligned} u &= \sec x, & \mathrm{d}v &= \sec^2 x \mathrm{d}x \\ \mathrm{d}u &= \sec x \tan x \, \mathrm{d}x, & v &= \tan x \ . \end{aligned}$$

So

$$\begin{aligned} \int \sec^3 x \, \mathrm{d}x &= \int \underbrace{\sec x}_{u} \underbrace{\sec^2 x \, \mathrm{d}x}_{\mathrm{d}v} \\ &= \sec x \tan x - \int \sec x \tan^2 x \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1) \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x \, \mathrm{d}x + \int \sec x \, \mathrm{d}x \ . \end{aligned}$$

We've already seen this trick: bring the integral of $\sec^3 x$ over to the left-hand side to find

$$2\int \sec^3 x\,\mathrm{d}x = \sec x\tan x + \int \sec x\mathrm{d}x = \sec x\tan x + \ln\big|\sec x + \tan x\big| + C\ ,$$

and so

$$\int \sec^3 x\,\mathrm{d}x = \tfrac{1}{2}\Big(\sec x\tan x + \ln\big|\sec x + \tan x\big|\Big) + C\ .$$ □

§4.6 Trigonometric Substitution | Integrals Featuring $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $\sqrt{x^2+a^2}$

When trying to compute integrals containing terms of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $\sqrt{x^2+a^2}$ for some a, we need to make use of some clever substitutions. For this type of integrals, it is very often the case that making a substitution involving a trigonometric function turns out to be quite helpful.

We start by recalling some very useful trigonometric identities:

$$1-\sin^2\theta = \cos^2\theta \qquad \text{(I)}$$
$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{(II)}$$
$$\sec^2\theta - 1 = \tan^2\theta \qquad \text{(III)}$$

Notice that (III) is exactly the same identity as (II) (we just transferred 1 to the other side); these two ways of writing the same identity will be helpful to us shortly. These three identities are obtained from the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

Example 4.28 (integral | trig substitution | $1-x^2$)

Compute $\int x\sqrt{1-x^2}\,\mathrm{d}x$.

Solution.
Use substitution

$$u := 1-x^2, \quad \rightsquigarrow \quad \mathrm{d}u = -2x\,\mathrm{d}x\ ,$$

so

$$\int x\sqrt{1-x^2}\,\mathrm{d}x = \int \sqrt{u}\,\left(-\tfrac{1}{2}\,\mathrm{d}u\right) = -\tfrac{1}{2}u^{3/2} + C = -\tfrac{1}{3}(1-x^2)^{3/2} + C\ .$$ □

Example 4.29 (integral | trig substitution | $\sqrt{a^2-x^2}$)

Find the area of a half-disc of radius a.

Solution.
The equation of the circle centred at the origin with radius a is

$$x^2 + y^2 = a^2 \qquad \Rightarrow \qquad y = \pm\sqrt{a^2 - x^2} \ .$$

To find the area A of the half-disc we need to integrate $y = \sqrt{a^2 - x^2}$ from $-a$ to a:

$$A = \int_{-a}^{a} \sqrt{a^2 - x^2}\,\mathrm{d}x \ .$$

First, let's try the substitution

$$u = a^2 - x^2, \quad \rightsquigarrow \quad \mathrm{d}u = -2x\,\mathrm{d}x \ .$$

Do you see a problem with this? This is isn't very helpful, because when we try to write $2x$ in terms of u, we get obtain a square root again:

$$2x = 2\sqrt{a^2 - u} \ .$$

We need a different method. The trouble here is that the square root $\sqrt{\ }$ keeps appearing; it would be nice if we could eliminate it. Now, recall back the identity (I): $1 - \sin^2\theta = \cos^2\theta$. The left-hand side of this identity looks very similar to $a^2 - x^2$. On other hand, if we take the square root of both sides of this identity, the right-hand side becomes simply $\cos\theta$ — no square root! Multiply both sides of (I) by a^2:

$$a^2 - (a\sin\theta)^2 = (a\cos\theta)^2 \ .$$

So try the substitution

$$x = a\sin\theta \qquad \rightsquigarrow \qquad \mathrm{d}x = a\cos\theta\,\mathrm{d}\theta \ .$$

Notice the difference between the two substitutions we've tried in this example: when letting $u = a^2 - x^2$, the new variable is a function of the old one; when $x = a\sin\theta$, the old variable is a function of the new one.

Now, we change the limits of integration:

$$\begin{aligned} x &= -a \qquad \rightsquigarrow \qquad -a = a\sin\theta \ \Rightarrow \ \theta = -\tfrac{\pi}{2} \ , \\ x &= a \qquad \rightsquigarrow \qquad a = a\sin\theta \ \Rightarrow \ \theta = \tfrac{\pi}{2} \ . \end{aligned}$$

Since $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$, we know that $\cos\theta \geqslant 0$. Therefore,

$$\begin{aligned}
A = \int_{-a}^{a} \sqrt{a^2 - x^2}\,\mathrm{d}x &= \int_{-\pi/2}^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}(a\cos\theta)\,\mathrm{d}\theta && \text{(sub in } x = a\sin\theta\text{)} \\
&= \int_{-\pi/2}^{\pi/2} a\sqrt{\cos^2\theta}(a\cos\theta)\,\mathrm{d}\theta && \text{(use (I))} \\
&= \int_{-\pi/2}^{\pi/2} a\big|\cos\theta\big|(a\cos\theta)\,\mathrm{d}\theta \\
&= \int_{-\pi/2}^{\pi/2} a^2\cos^2\theta\,\mathrm{d}\theta && (\cos\theta \geqslant 0 \text{ on } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]) \\
&= \dots && \text{(exercise 4.30)} \\
&= \frac{\pi a^2}{2}
\end{aligned}$$

Notice that based on our knowledge of areas of basic shapes, this is exactly what we expected, because we know that the area of the entire disc of radius a is πa^2. □

Exercise 4.30: Fill in the "..." in the previous example's calculation (example 4.29).

Example 4.31 (integral | trig substitution | $\sqrt{x^2 - a^2}$)

Compute $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,\mathrm{d}x$.

Solution.
The expression $x^2 - 4$ in the integrand is similar to the left-hand side of (III): $\sec^2\theta - 1 = \tan^2\theta$. Multiplying this identity by 4, we get $4\sec^2\theta - 4 = (2\tan\theta)^2$. Thus, we try the following substitution:

$$x = 2\sec\theta \quad \rightsquigarrow \quad \mathrm{d}x = 2\sec\theta\tan\theta\,\mathrm{d}\theta\,.$$

Therefore,

$$\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x}\,\mathrm{d}x &= \int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta}(2\sec\theta\tan\theta\,\mathrm{d}\theta) \\
&= \int \sqrt{(2\tan^2\theta)^2}\tan\theta\,\mathrm{d}\theta \\
&= \int \big|\tan\theta\big|\tan\theta\,\mathrm{d}\theta\,.
\end{aligned}$$

Now, in order to take care of the absolute value, we need to understand whether $\tan\theta$ is positive or negative. Notice that the integrand is only defined for x in the interval $(-\infty, -2] \cup [2, +\infty)$, because

$x^2 - 4$ is negative for any x in the interval $(-2, 2)$. Thus, we have two cases: $x \leqslant -2$ or $x \geqslant 2$.

- Consider first the case $x \geqslant 2$. Since $x = 2\sec\theta$, it follows that $2\sec\theta \geqslant 2$, which means $\sec\theta \geqslant 1$. Moreover, for the final answer, we will need to write θ back in terms of x; since θ is defined by the equation $x = 2\sec\theta$, this will involve the function $\sec^{-1}$. Therefore, $\theta \in [0, \frac{\pi}{2})$, and so $\tan\theta \geqslant 0$.
- Similarly, if $x \leqslant -2$, then $\sec\theta \leqslant -1$, so $\theta \in [\pi, \frac{3\pi}{2})$, and so $\tan\theta \geqslant 0$.

In both cases, $\tan\theta \geqslant 0$, so $\left|\tan\theta\right| = \tan\theta$. We can therefore complete the integral

$$\begin{aligned} \int \left|\tan\theta\right| \tan\theta \, \mathrm{d}\theta &= \int \tan^2\theta \, \mathrm{d}\theta \\ &= 2\int (\sec^2\theta - 1) \, \mathrm{d}\theta && \text{(use (III))} \\ &= 2\tan\theta - 2\theta + C \, . \end{aligned}$$

We need to write the final answer in terms of x. For this we use right triangles: since $x = 2\sec\theta$ implies $\sec\theta = \frac{x}{2}$,

$$\sec\theta = \frac{\text{hypothenuse}}{\text{adjacent}} \quad \text{and} \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} \quad \Longrightarrow \quad \tan\theta = \frac{\sqrt{x^2-4}}{2} \, .$$

Finally, $\sec\theta = \frac{x}{2}$ implies $\theta = \sec^{-1}\left(\frac{x}{2}\right)$, so

$$\int \frac{\sqrt{x^2-4}}{x} \, \mathrm{d}x = \sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C \, .$$ □

Example 4.32 (integral | trig substitution | $\sqrt{x^2 + a^2}$)

Compute $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2 - 2x + 5}}$.

Solution.
Notice that the substitution $u = x^2 - 2x + 5$ is not helpful, because you get $\mathrm{d}u = (2x - 2)\,\mathrm{d}x$.

Instead, try completing the square: $x^2 - 2x + 5 = (x-1)^2 + 4$. So

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 - 2x + 5}} = \int \frac{\mathrm{d}x}{\sqrt{(x-1)^2 + 4}} \, .$$

Use the substitution $u := x - 1$, so $\mathrm{d}u = \mathrm{d}x$, to get

$$\int \frac{\mathrm{d}x}{\sqrt{(x-1)^2 + 4}} = \int \frac{\mathrm{d}u}{\sqrt{u^2 + 4}} \, .$$

Try $u = 2\tan\theta$, so $\mathrm{d}u = 2\sec^2\theta \, \mathrm{d}\theta$. The domain of the integrand is $\mathbb{R}$, so $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\sec\theta > 0$.

Therefore $\sqrt{u^2+4} = \sqrt{4(\tan^2\theta+1)} = 2|\sec\theta| = 2\sec\theta$. Thus,

$$\int \frac{\mathrm{d}u}{u^2+1} = \int \frac{2\sec^2\theta\,\mathrm{d}\theta}{2\sec\theta} = \int \sec\theta\,\mathrm{d}\theta = \ln\Big|\sec\theta+\tan\theta\Big| + C\ .$$

Writing this in terms of x, we find

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-2x+5}} = \ln\left|\frac{\sqrt{x^2-2x+5}}{2} + \frac{x-1}{2}\right| + C\ . \qquad \square$$

Exercise 4.33: Verify the last equality by doing the back-substitution in detail.

In the following table, we summarise the substitutions we've encountered in this section.

EXPRESSION	SUBSTITUTION	INTERVAL	IDENTITY
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \leqslant \theta \leqslant \frac{\pi}{2}$ or $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$
$\sqrt{x^2+a^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$\tan^2\theta + 1 = \sec^2\theta$

Exercise 4.34 (challenge question): Compute $\displaystyle\int \frac{x^5}{(4x^2+1)^{3/2}}\,\mathrm{d}x$.

Answer.

$$\int \frac{x^5}{(4x^2+1)^{3/2}}\,\mathrm{d}x = \frac{2x^4-2x^2-1}{24\sqrt{4x^2+1}} + C. \qquad \square$$

§4.7 Partial Fractions

In this section, we will learn a general method that allows us to integrate *rational functions*. Recall that a **rational function** is a function that is a quotient of two polynomials:

$$\text{rational function } R(x) = \frac{P(x)}{Q(x)}, \qquad \text{where } P(x), Q(x) \text{ are polynomials.}$$

The method discussed in this section is called **partial fractions**: it allows us to decompose the quotient $\frac{P}{Q}$ into a sum of simpler fractions, whose integrals are easy to find. To illustrate the idea, we begin with a simple observation. To add two simple fractions, we take their common denominator:

take common denominator

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2x-6+3x+3}{(x+1)(x-3)} = \frac{5x-3}{x^2-2x-3} .$$

This equality is extremely useful if we want to integrate the rational function on the right-hand side:

$$\int \frac{5x-3}{x^2-2x-3} \mathrm{d}x = \int \left(\frac{2}{x+1} + \frac{3}{x-3} \right) \mathrm{d}x = 2 \ln |x+1| + 3 \ln |x-3| + C .$$

But suppose we began with the rational function

$$R(x) = \frac{5x-3}{x^2-2x-3} ,$$

how could we find its decomposition into *partial fractions*?

take common denominator

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2x-6+3x+3}{(x+1)(x-3)} = \frac{5x-3}{x^2-2x-3} .$$

partial fractions

Example 4.35 (integral | partial fractions | distinct linear factors)

Compute $\displaystyle\int \frac{4x-1}{x^2+x-2} \, \mathrm{d}x$.

Solution.
First, factor the denominator: $x^2+x-2 = (x+2)(x-1)$. Thus

$$\frac{4x-1}{x^2+x-2} = \frac{4x-1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} .$$

Multiplying by $(x+2)(x-1)$, we get

$$4x-1 = A(x-1)+B(x+2) \ . \tag{17}$$

Next, we solve for A and B.

Method 1. Recall that *two polynomials are equal if and only if their coefficients are equal*. Rearranging (17), we get an equality of two polynomials:

$$4x-1 = (A+B)x + 2B - A \ .$$

Equating the coefficients, we find

$$4 = A+B \qquad \text{and} \qquad -1 = 2B - A \ .$$

Solving this system, we get $A = 3$ and $B = 1$.

Method 2. This is a shortcut. The idea here is that equation (17) must be true for all x. So, substituting some convenient *test* values, like $x = 1$ and $x = -2$ to find

$$\begin{aligned} x = 1 \quad &\implies \quad 4-1 = 3B \quad \implies \quad B = 1 \ , \\ x = -2 \quad &\implies \quad -8-1 = -3A \quad \implies \quad A = 3 \ . \end{aligned}$$

Therefore,

$$\int \frac{4x-1}{x^2+x-2}\,\mathrm{d}x = \int \left(\frac{3}{x+2} + \frac{1}{x-1} \right) \mathrm{d}x = 3\ln|x+2| + \ln|x-1| + C \ . \qquad \square$$

In general, when $Q(x)$ has no repeated factors, we use this method as follows.

Partial Fractions for $Q(x)$ with no distinct linear factors.
If $\deg P(x) < \deg Q(x)$ and $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdot\ldots\cdot(a_nx+b_n)$ with no repeated factors, then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_n}{a_nx+b_n} \ .$$

Example 4.36 (integral | partial fractions | repeated linear factor)

Compute $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,\mathrm{d}x$.

Solution.
Here $P(x) = 2x^2+3$ and $Q(x) = x(x-1)^2$, so $\deg P(x) < \deg Q(x)$, but $Q(x)$ has a repeated linear factor $(x-1)$, so the above method is not applicable immediately. We break this rational function into simpler fractions as follows:

$$\frac{2x^2+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \ .$$

Multiply by the common denominator (which is $x(x-1)^2$) to get

$$2x^2+3 = A(x-1)^2 + Bx(x-1) + Cx \ .$$

Now, we solve for A, B, C as before: we either expand the right-hand side, equate the coefficients, and solve the resulting linear system; or we substitute a few text values like so:

$$\begin{aligned}
x=0 \quad &\Longrightarrow \quad 3 = A \quad \Longrightarrow \quad A = 3 \ ; \\
x=1 \quad &\Longrightarrow \quad 2+3 = C \quad \Longrightarrow \quad C = 5 \ ; \\
x=2 \quad &\Longrightarrow \quad 2\cdot 4+3 = A+2B+2C = 3+2B+10 \quad \Longrightarrow \quad B = -1 \ .
\end{aligned}$$

Thus, we have

$$\int \frac{2x^2+3}{x(x-1)^2}\,\mathrm{d}x = \int \left(\frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2}\right)\mathrm{d}x = 3\ln|x| - \ln|x-1| - \frac{5}{x-1} + C \ .$$

□

In summary, we have the following.

Partial Fractions for $Q(x)$ with repeated linear factors.
Suppose $\deg P(x) < \deg Q(x)$. Each factor of $Q(x)$ of the form $(ax+b)^k$ gives rise to

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k} \ .$$

Example 4.37 (integral | partial fractions | $Q(x)$ has irreducible quadratic factor)

Compute $\displaystyle\int \frac{3x^2-5x+4}{(x-1)(x^2+1)}\,\mathrm{d}x$.

Solution.
Notice that the denominator has already been factored. We cannot factor x^2+1, because x^2+1 is an irreducible polynomial (it has no real roots). We break our rational function into simpler fractions as follows:

$$\frac{3x^2-5x+4}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \ .$$

Notice that on the right-hand side, the second fraction's numerator is a *linear function* rather than a constant. Multiplying by the common denominator, we get

$$3x^2-5x+4 = (A+B)x^2 + (C-B)x + (A-C) \ .$$

Solving for A, B, C, we find $A=1$, $B=2$, $C=-3$. Thus,

$$\int \frac{3x^2-5x+4}{(x-1)(x^2+1)}\,\mathrm{d}x = \int \left(\frac{1}{x-1} + \frac{2x-3}{x^2+1}\right)\mathrm{d}x \ .$$

In order to integrate the second term, we split it into two parts:

$$\int \frac{2x-3}{x^2+1}\mathrm{d}x = \int \frac{2x}{x^2+1}\mathrm{d}x - 3\int \frac{1}{x^2+1}\mathrm{d}x = \ln\left|x^2+1\right| - 3\tan^{-1}x + C \ .$$

Therefore,

$$\int \frac{3x^2-5x+4}{(x-1)(x^2+1)}\mathrm{d}x = \int \frac{1}{x-1}\mathrm{d}x + \int \frac{2x}{x^2+1}\mathrm{d}x - 3\int \frac{1}{x^2+1}\mathrm{d}x = \ln|x-1| + \ln\left|x^2+1\right| - 3\tan^{-1}(x) + C \ .$$

□

Partial Fractions for $Q(x)$ contains distinct irreducible quadratic factors.
Suppose $\deg P(x) < \deg Q(x)$, and suppose $Q(x)$ has a factor ax^2+bx+c, where $b^2-4ac<0$. Then we get a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c} \ .$$

Example 4.38 (integral | partial fractions | long division)

Compute $\displaystyle\int \frac{x^3-x+2}{x^2-4}\,\mathrm{d}x$.

Solution.
The degree of the numerator is larger than of the denominator. This means we first need to use long division of polynomials. You can verify via long division that

$$x^3-x+2 = (3x+2)(x^2-4) \ .$$

Therefore,

$$\frac{x^3-x+2}{x^2-4} = x + \frac{3x+2}{x^2-4} \ ,$$

so the integral becomes

$$\int \frac{x^3-x+2}{x^2-4}\,\mathrm{d}x = \int x\,\mathrm{d}x + \int \frac{3x+2}{x^2-4}\,\mathrm{d}x \ .$$

We use the method of Partial Fractions to find the second integral.

$$\frac{3x+2}{x^2-4} = \frac{3x+2}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \ ,$$

whence $3x+2 = A(x+2)+B(x-2)$. Solving for A, B, we find $A=2$ and $B=1$. Thus,

$$\int \frac{x^3-x+2}{x^2-4}\,\mathrm{d}x = \int \left(x + \frac{2}{x-2} + \frac{1}{x+2}\right)\,\mathrm{d}x = \frac{x^2}{2} + 2\ln|x-2| + \ln|x+2| + C \ .$$

□

§4.8

Strategies for Integration

(1) **Simplify the integrand** (if possible): sometimes a simple algebraic manipulation or the use of some trigonometric identity simplifies the integrand enough to make it clear how to proceed

Example 4.39 (integral | simplify the integrand)

- $\displaystyle\int \frac{\sin\theta}{\sec^2\theta}\,\mathrm{d}\theta = \int \sin\theta\cos^2\theta\,\mathrm{d}\theta$, so now we can use the substitution $u = \cos\theta$.
- $\displaystyle\int \frac{\sqrt{x}+1}{x}\,\mathrm{d}x = \int \left(x^{-1/2} + \frac{1}{x}\right)\mathrm{d}x$, so now we can integrate directly: $2x^{1/2} + \ln|x| + C$.

(2) **Look for an obvious substitution**: try to find a function $u = g(x)$ whose differential $\mathrm{d}u = g'(x)\,\mathrm{d}x$ also occurs in the integrand.

Example 4.40 (integral | obvious substitution)

$\displaystyle\int \frac{2x}{\sqrt{1-x^2}}\,\mathrm{d}x$; notice that if $u = 1-x^2$, then $\mathrm{d}u = -2x\,\mathrm{d}x$. Using the substitution $u = 1-x^2$ (instead of a trigonometric substitution), we can quickly find the answer:

$$\int \frac{2x}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int \frac{\mathrm{d}u}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1-x^2} + C\ .$$

If (1) and (2) are fruitless, we must work a little harder to understand what type of integral we're dealing with.

(3) **Classify the integrand** according to its form:

(a) Trigonometric function: if $f(x) = \sin^n x\cos^m x$ or $f(x) = \sec^n x\tan^m x$, use appropriate substitution depending on whether n, m are even or odd.

(b) Rational function: if $f(x) = \frac{P(x)}{Q(x)}$ for polynomials $P(x), Q(x)$, use partial fractions. If $\deg P(x) > \deg Q(x)$, do long division first.

(c) Polynomial × transcendental function: if $f(x) = (\text{poly})\times(e^x/\ln x/\cos x/\sin x/\tan x/\ldots)$, use integration by parts.

(d) Radicals: if integrand contains $\sqrt{\pm x^2 + \pm a^2}$, use appropriate trigonometric substitution.

(4) **Try Again**: we basically only have two methods of integration: substitution and integration by parts. Sometimes, they need to be used in clever ways. Cracking an integral can takes several attempts.

(a) Try another substitution: Even if there isn't an *obvious* substitution, try calling various things in your integrand by u and computing $\mathrm{d}u$. This could inspire you to the right path.

(b) Try integration by parts: sometimes this needs to be used several times; sometimes integration by parts works in surprising situations (recall $\int \ln x \, \mathrm{d}x$).

(c) Try manipulating the integrand further: multiply and divide the integrand by something; use trigonometric identities.

Example 4.41 (integral | multiply and divide)

Compute $\displaystyle\int \sec x \, \mathrm{d}x$.

Solution.
We can multiply the integrand $\sec x$ by $\frac{\sec x + \tan x}{\sec x + \tan x}$:

$$\int \sec x \, \mathrm{d}x = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, \mathrm{d}x .$$

Let $u = \sec x + \tan x$, so $\mathrm{d}u = (\sec x \tan x + \sec^2 x) \, \mathrm{d}x$. Then:

$$= \int \frac{\mathrm{d}u}{u} = \ln |u| + C = \ln \big| \sec x + \tan x \big| + C .$$ □

Example 4.42 (integral | multiply and divide)

Compute $\displaystyle\int \frac{1}{1 - \sin x} \, \mathrm{d}x$.

Solution.
Multiply the integrand by $\frac{1+\sin x}{1+\sin x}$:

$$\int \frac{1}{1-\sin x} \, \mathrm{d}x = \int \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} \, \mathrm{d}x = \int \frac{1+\sin x}{1-\sin^2 x} \, \mathrm{d}x = \int \frac{1+\sin x}{\cos^2 x} \, \mathrm{d}x$$
$$= \int \left(\sec^2 x + \frac{\sin x}{\cos^2 x} \right) \, \mathrm{d}x .$$ □

(d) Use several methods: Sometimes you need to use a combination of different methods; sometimes you need to apply the same method several times.

The following message cannot be emphasised enough:

THE ONLY WAY TO MASTER INTEGRATION
IS BY PRACTISING SOLVING MANY EXERCISES!

In the following examples, we describe the main ideas involved in finding the integral; you should finish the computation in each example as an exercise.

Example 4.43 (integral | complete the square)

Compute $\displaystyle\int (x+1)^2\sqrt{3-2x-x^2}\,\mathrm{d}x$.

Solution.
We see in the integrand the square root of a quadratic polynomial. The first thing to try is to complete the square: $3-2x-x^2 = 4-(x+1)^2$. Now, you can make the substitution $u = x+1$ (so $\mathrm{d}u = \mathrm{d}x$) to obtain

$$\int u^2\sqrt{1-u^2}\,\mathrm{d}u\ .$$

Next, use the trigonometric substitution $u = \sin\theta$. □

Exercise 4.44: Finish the integration.

Example 4.45 (integral | long division, partial fractions)

Compute $\displaystyle\int \frac{x^4-x^3-x^2-6x-2}{x^3-2x^2+x-2}\,\mathrm{d}x$.

Solution.
This is a rational function $\frac{P(x)}{Q(x)}$ with $\deg P(x) > \deg Q(x)$, so to use partial fractions method we first need to do long division. We find

$$\int \frac{x^4-x^3-x^2-6x-2}{x^3-2x^2+x-2}\,\mathrm{d}x = \int\left(x+1-\frac{5x}{(x-2)(x^2+1)}\right)\mathrm{d}x\ .$$

Using partial fractions, we can then find:

$$= \int\left(x+1-\frac{2}{x-2}+\frac{2x-1}{x^2+1}\right)\mathrm{d}x\ .$$

□

Exercise 4.46: Finish the integration.

Example 4.47 (integral | trigonometric functions)

Compute $\displaystyle\int \cos^3 x\sin^{3/2} x\,\mathrm{d}x$.

Solution.
Note that we have an odd power of $\cos x$ appearing in the integrand. We can separate one factor of $\cos x$ and write the remaining $\cos^2 x$ factor in terms of $\sin x$ using trigonometric identities. So,

$$\int \cos^3 x\sin^{3/2} x\,\mathrm{d}x = \int \cos^2 x\sin^{3/2} x(\cos x\,\mathrm{d}x) = \int (1-\sin^2 x)\sin^{3/2} x(\cos x\,\mathrm{d}x)\ .$$

Substition: $u = \sin x$, so $\mathrm{d}u = \cos x \,\mathrm{d}x$ and

$$= \int (1-u^2)u^{3/2}\,\mathrm{d}u = \int (u^{3/2} - u^{5/2})\,\mathrm{d}u \ . \qquad \square$$

Exercise 4.48: Finish the integration.

Example 4.49 (integral | $\cos(\sqrt{x})$)

Compute $\displaystyle\int \cos(\sqrt{x})\,\mathrm{d}x$.

Solution.
There is not much we can do here. Let's try substitution and see what happens. Try $u = \sqrt{x}$, so $\mathrm{d}u = \frac{1}{2\sqrt{x}}\mathrm{d}x = \frac{1}{2u}\mathrm{d}x \Rightarrow \mathrm{d}x = 2u\mathrm{d}u$. Therefore, the integral becomes $\int 2u\cos u\,\mathrm{d}u$. Now, you can integrate by parts, with

$$\begin{aligned} \widetilde{u} &= u, \quad \mathrm{d}\widetilde{v} = \cos u\,\mathrm{d}u \\ \mathrm{d}\widetilde{u} &= \mathrm{d}u, \quad \widetilde{v} = \sin u \ . \end{aligned}$$

So,

$$\int 2u\cos u\,\mathrm{d}u = 2\left(\widetilde{u}\widetilde{v} - \int \widetilde{v}\mathrm{d}\widetilde{u}\right) = 2\left(u\sin u - \int \sin u\,\mathrm{d}u\right) \ . \qquad \square$$

Exercise 4.50: Finish the integration.

Exercise 4.51 (Challenge Question): Compute $\displaystyle\int \frac{\ln(\tan x)}{\sin x \cos x}\,\mathrm{d}x$.

§4.9 Improper Integrals

So far, when working with definite integrals $\int_a^b f(x)\,\mathrm{d}x$, we have always assumed that

(1) limits of integration are finite numbers (the interval $[a, b]$ is *bounded*);

(2) f does not have an infinite discontinuity on the interval $[a, b]$ (either f is continuous on $[a, b]$ or discontinuities are of jump-type).

In this section, we generalise the concept of a definite integral in two ways:

(1) we will allow *unbounded* intervals of integration;

(2) we will allow f to develop *infinite discontinuities* on the integration interval.

An integral with either or both of these features will be called an **improper integral**.

Improper integrals have many important applications in many parts of mathematics and physics, such probability theory and quantum mechanics.

§4.9.1 Type 1 | Unbounded Intervals

To motivate the right way of generalising the notion of the definite integral to the setting of unbounded intervals of integration, consider the infinite region S that lies under the curve $y = \frac{1}{x^2}$, above the x-axis, and to the right of the vertical line $x = 1$. What is the area $A(\mathsf{S})$ of the region S?

We are not immediately able to answer this question with the methods we have seen so far. This is because it sounds like to find the area of S what we should do is find the area under the graph of $y = \frac{1}{x^2}$ over the *unbounded* interval $[1, +\infty)$. That is, it sounds like what we want to evaluate is the integral

$$\text{“} \int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x \text{”} .$$

I've written quotation marks because we cannot interpret these symbols using our current notion of the definite integral. (For example, into how many subintervals do we need to divide the interval $[1, +\infty)$, or what is the right notion of a Riemann sum here? It's not clear.) In fact, you may even suspect that the area of the region S cannot possibly be finite: after all, it extends unboundedly to the right along the x-axis. But it would be premature to come to this conclusion, because we can outwit these complications as follows.

First, let us consider a simpler problem, one whose solution we already know. Truncate the region S by cutting off the unbounded tail: consider the region S_t below the curve $y = \frac{1}{x^2}$, above the x-axis, and between $x = 1$ and $x = t$ for some fixed point $t > 1$. We know how to find the area of S_t:

$$A(\mathsf{S}_t) = \int_1^t \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{x}\bigg|_1^t = 1 - \frac{1}{t} .$$

Notice that $A(t) < 1$ for all $t > 1$.

Observe that as we increase t, the region S_t occupies a larger and larger share of the region S; in fact, in the limit as $t \to +\infty$, the region S_t becomes S. Therefore,

$$\lim_{t\to+\infty} A(\mathsf{S}_t) = \lim_{t\to+\infty} \left(1 - \frac{1}{t}\right) = 1 \ .$$

Thus, the area of the unbounded region S is 1, and we write

$$A = \int_1^{+\infty} \frac{1}{x^2}\, \mathrm{d}x = \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\, \mathrm{d}x \ .$$

Definition 4.52 (Improper Integral of Type 1 | unbounded interval)

(a) If $\int_a^t f(x)\,\mathrm{d}x$ exists for all $t \geqslant a$, then $\int_a^{+\infty} f(x)\,\mathrm{d}x := \lim_{t\to+\infty} \int_a^t f(x)\,\mathrm{d}x \ .$

(b) If $\int_t^b f(x)\,\mathrm{d}x$ exists for all $t \leqslant b$, then $\int_{-\infty}^{b} f(x)\,\mathrm{d}x := \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x \ .$

The improper integrals in parts (a) and (b) are called **convergent** if the corresponding limit exists, and **divergnet** if the limit does not exist.

(c) For any $c \in \mathbb{R}$, $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x := \int_{-\infty}^{c} f(x)\,\mathrm{d}x + \int_{c}^{+\infty} f(x)\,\mathrm{d}x \ ,$

where $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ converges if and only if both $\int_{-\infty}^{c} f(x)\,\mathrm{d}x$ and $\int_{c}^{+\infty} f(x)\,\mathrm{d}x$ are convergent. If either one diverges, then the improper integral $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ also diverges.

Example 4.53 (improper integral | unbounded interval, upper limit)

Determine whether the integral $\int_1^{\infty} \frac{1}{x}\,\mathrm{d}x$ is convergent or divergent.

Solution.
The first thing we do is write the improper integral according to its definition as a limit:

$$\int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x \stackrel{\text{def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,\mathrm{d}x \ .$$

Now we calculate the definite integral appearing in the limit:

$$\int_1^t \frac{1}{x}\,\mathrm{d}x = \ln(t) - \ln(1) = \ln(t) \ .$$

Finally, we plug this result back into the limit and compute it:

$$\lim_{t\to+\infty} \int_1^t \frac{1}{x}\,\mathrm{d}x = \lim_{t\to+\infty} \ln(t) \ ,$$

which does not exist ($\ln(t)$ explodes to $+\infty$ as $t \to +\infty$). Since this limit is infinite, the integral is divergent. □

Going back to our discussion of the area under the curve over an unbounded domain, we can also interpret the result of the previous example as saying that the area under the curve $y = \frac{1}{x}$ on $[1, +\infty]$ is infinite. This is in stark contrast to the area under the curve $y = \frac{1}{x^2}$, which was finite (we found it to be 1), even though the graphs if $y = \frac{1}{x^2}$ and $y = \frac{1}{x}$ look rather similar for $x > 1$.

Why is this happening? The main difference between them is that, as $x \to \infty$, the function $\frac{1}{x^2}$ decays to 0 faster than $\frac{1}{x}$. The function $\frac{1}{x^2}$ decays *fast enough*, whilst $\frac{1}{x}$ doesn't decay *fast enough*. So, how *fast* is *fast enough*?

Example 4.54 (improper integral | when is $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ convergent?)

For what values of p is the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ convergent?

Solution.
We already know that for $p = 1$ the integral is divergent. So, assume that $p \neq 1$. Then

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^t = \frac{1}{1-p} \lim_{t\to\infty} \left(t^{1-p} - 1\right) \ .$$

The convergence properties of this limit depend on the value of p:

$$\lim_{t\to\infty} t^{1-p} = \begin{cases} 0 & \text{if } p > 1, \\ \infty & p < 1. \end{cases}$$

Thus,

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1, \\ \text{diverges} & p \leqslant 1. \end{cases}$$ □

Example 4.55 (improper integral | unbounded interval, lower limit)

Determine whether $\displaystyle\int_{-\infty}^{0} \cos x \, \mathrm{d}x$ converges or diverges.

Solution.
First, convert the improper integral into a limit according its definition:

$$\int_{-\infty}^{0} \cos x \, \mathrm{d}x = \lim_{t \to -\infty} \int_{t}^{0} \cos x \, \mathrm{d}x = \lim_{t \to -\infty} \sin x \Big|_{t}^{0} = \lim_{t \to -\infty} (-\sin t) \ .$$

This limit does not exist, because $\sin(t)$ oscillates between -1 and 1. Therefore, the given improper integral is divergent. □

Example 4.56 (improper integral | unbounded interval, both limits)

Determine whether $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2} \, \mathrm{d}x$ converges or diverges.

Solution.
It's convenient to choose c in definition 4.52 to be 0:

$$\int_{-\infty}^{\infty} x e^{-x^2} \, \mathrm{d}x = \int_{-\infty}^{0} x e^{-x^2} \, \mathrm{d}x + \int_{0}^{\infty} x e^{-x^2} \, \mathrm{d}x \ .$$

We investigate the two improper integrals on the right-hand side separately. First,

$$\int_{-\infty}^{0} x e^{-x^2} \, \mathrm{d}x = \lim_{t \to -\infty} \int_{t}^{0} x e^{-x^2} \, \mathrm{d}x = \lim_{t \to -\infty} \left(-\frac{1}{2} + \frac{1}{2} e^{-t^2} \right) = -\frac{1}{2} \ .$$

Similarly,

$$\int_{0}^{\infty} x e^{-x^2} \, \mathrm{d}x = \lim_{t \to \infty} \int_{0}^{t} x e^{-x^2} \, \mathrm{d}x = \frac{1}{2} \ .$$

Thus, we find

$$\int_{-\infty}^{\infty} x e^{-x^2} \, \mathrm{d}x = -\frac{1}{2} + \frac{1}{2} = 0 \ .$$ □

Remark 4.57 (Quantum Field Theory)

Integrals like the one we have just encountered are very commonly encountered in subjects such as probability theory as well as physics. Their prominence is especially significant in Quantum Field

Theory. Understanding their deep geometric meaning is currently a very active area of research.

Example 4.58 (improper integral | unbounded interval, both limits)

Determine whether $\displaystyle\int_{-\infty}^{\infty} e^{-x}\,\mathrm{d}x$ converges or diverges.

Solution.
From the definition 4.52, we have

$$\int_{-\infty}^{\infty} e^{-x}\,\mathrm{d}x = \int_{-\infty}^{0} e^{-x}\,\mathrm{d}x + \int_{0}^{\infty} e^{-x}\,\mathrm{d}x \ .$$

Again, we solve the two improper integrals separately. First, it's easy to see that $\int_0^\infty e^{-x}\,\mathrm{d}x$ converges (exercise). However,

$$\int_{-\infty}^{0} e^{-x}\,\mathrm{d}x = \lim_{t\to-\infty}\int_{t}^{0} e^{-x}\,\mathrm{d}x = \lim_{t\to-\infty}(-1+e^{-t}) = \infty \ .$$

This means that the integral $\int_{-\infty}^{0} e^{-x}\,\mathrm{d}x$ is divergent. Consequently, the integral $\int_{-\infty}^{\infty} e^{-x}\,\mathrm{d}x$ is divergent, too. □

§4.9.2 Type 2 | Infinite Discontinuities

Consider the unbounded region S under the curve $y = \frac{1}{\sqrt{x}}$, above the x-axis, and between 0 and 1. First, we find the area of the part of S between t and 1, for some $0 < t < 1$:

$$A(t) = \int_t^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}\Big|_t^1 = 2 - 2\sqrt{t} \ .$$

Then we find the limit of this area as $t \to 0^+$:

$$\lim_{t\to 0^+} A(t) = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to 0^+} (2 - \sqrt{t}) = 2 \ .$$

The area of S is finite and we write:

$$A = \int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2 \ .$$

This motivates the definition.

Definition 4.59 (Improper Integral of Type 2 | infinite discontinuity)

(a) If f is continuous on $[a, b)$ and discontinuous at b, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x := \lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x$.

(b) If f is continuous on $(a, b]$ and discontinuous at a, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x := \lim_{t\to a^+} \int_t^b f(x)\,\mathrm{d}x$.

The improper integral $\int_a^b f(x)\,\mathrm{d}x$ is called **convergent** if the corresponding limit exists, and **divergent** if the limit doesn't exist.

(c) If f has a discontinuity at c, where $a < c < b$, and f is continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x := \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \ .$$

This integral is **convergent** if and only if both $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ are convergent; if at least one of them diverges, then $\int_a^b f(x)\,\mathrm{d}x$ is **divergent**.

Example 4.60

Evaluate $\displaystyle\int_0^5 \frac{1}{\sqrt{5-x}}\,\mathrm{d}x$ if it exists.

Solution.
We first notice that this is an improper integral, because $f(x) = \dfrac{1}{\sqrt{5-x}}$ has the vertical asymptote $x = 5$. Since the infinite discontinuity occurs at the right end-point of the interval $[0, 5]$, we use part (1) of definition 4.59:

$$\int_0^5 \frac{1}{\sqrt{5-x}}\,\mathrm{d}x = \lim_{t\to 5^-} \int_0^t \frac{1}{\sqrt{5-x}}\,\mathrm{d}x = \lim_{t\to 5^-} -2\sqrt{5-x}\Big|_0^t = \lim_{t\to 5^-} \left(-2\sqrt{5-t} + 2\sqrt{5}\right) = 2\sqrt{5}\,.$$

□

Example 4.61

Evaluate $\displaystyle\int_{-1}^1 \frac{\mathrm{d}x}{x^2}$ if possible.

Solution.
Observe that $x = 0$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval of integration, we must use part (3) in the definition 4.59 with $c = 0$:

$$\int_{-1}^1 \frac{\mathrm{d}x}{x^2} = \int_{-1}^0 \frac{\mathrm{d}x}{x^2} + \int_0^1 \frac{\mathrm{d}x}{x^2}\,.$$

Observe that

$$\int_0^1 \frac{\mathrm{d}x}{x^2} = \lim_{t\to 0^+} \int_t^1 \frac{\mathrm{d}x}{x^2} = \lim_{t\to 0^+} -\frac{1}{x}\Big|_t^1 = \lim_{t\to 0^+} \left(-1 + \frac{1}{t}\right) = \infty\,.$$

Thus, $\int_0^1 \frac{\mathrm{d}x}{x^2}$ is divergent; hence $\int_{-1}^1 \frac{\mathrm{d}x}{x^2}$ is divergent, too. □

Notice that we didn't even need to evaluate the integral $\int_{-1}^0 \frac{\mathrm{d}x}{x^2}$; as long as one of the summands is divergent, the whole sum is divergent.

Had we not noticed the vertical asymptote $x = 0$ in the previous example, we would have made the following **erroneous calculation**:

$$\int_{-1}^1 \frac{\mathrm{d}x}{x^2} = -\frac{1}{x}\Big|_{-1}^1 = -2\,.$$

But this is **wrong!** Indeed, the function $\frac{1}{x^2}$ is positive everywhere on $[-1, 1]$, but we've got a negative value for the integral. This cannot be right.

From now on, whenever we encounter the integral

$$\int_a^b f(x)\,\mathrm{d}x\ ,$$

we must always examine f on the interval $[a,b]$ and decide whether or not the integral is improper, and proceed accordingly.

Example 4.62

Determine if the integral $\displaystyle\int_0^\infty \frac{1}{x^2}\,\mathrm{d}x$ is convergent or not.

Solution.
This integral exhibits improper behaviour into two different ways: the integration domain is an unbounded interval $[0,\infty)$, and the integrand $\frac{1}{x^2}$ has a vertical asymptote at $x=0$. To analyse this integral, we must split the integration domain $[0,\infty)$ into two subintervals $[0,c]\cup[c,\infty]$, such that each of the two resulting integrals $\int_0^c \frac{1}{x^2}\,\mathrm{d}x$, $\int_c^\infty \frac{1}{x^2}\,\mathrm{d}x$ exhibits only one type of improper behaviour. In this case, we can choose the point c to be anywhere in the open interval $(0,\infty)$; we choose $c=1$ for convenience of evaluation:

$$\int_0^\infty \frac{1}{x^2}\,\mathrm{d}x = \int_0^1 \frac{1}{x^2}\,\mathrm{d}x + \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x\ .$$

Remember that for $\int_0^\infty \frac{1}{x^2}\,\mathrm{d}x$ to be convergent, BOTH integrals on the right-hand side must be convergent. However, in example 4.61, we discovered that $\int_0^1 \frac{1}{x^2}\,\mathrm{d}x$ is divergent. Therefore, $\int_0^\infty \frac{1}{x^2}\,\mathrm{d}x$ is divergent. □

§4.9.3 Comparison Test for Improper Integrals

Often it is not necessary to calculate the actual value of an improper integral, and it is sufficient to determine only its convergence properties.

Theorem 4.63 (Comparison Test for Improper Integrals)
Let f and g be continuous functions with $f(x) \geqslant g(x)$ for $x \geqslant a$.

(a) *If* $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ *is convergent, then* $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ *is convergent.*

(b) *If* $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ *is divergent, then* $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ *is divergent.*

We omit the proof of this theorem, but it is simply a generalisation to improper integrals of the domination Comparison Property of the Integral.

It is important to note that the implications in this theorem are one direction only: if $\int_0^\infty g(x)\,\mathrm{d}x$ is convergent, the theorem does NOT say that $\int_0^\infty f(x)\,\mathrm{d}x$ is convergent; the Comparison Test simply provides no information. Similarly, if $\int_0^\infty f(x)\,\mathrm{d}x$ is divergent, we CANNOT conclude that $\int_0^\infty g(x)\,\mathrm{d}x$ is divergent.

Example 4.64 (Improper Integral | Comparison Test)

Show that $\displaystyle\int_1^\infty \frac{2+\sin x}{x}\,\mathrm{d}x$ diverges.

Solution.
Notice that we can't evaluate this integral directly, because the antiderivative of $\frac{\sin x}{x}$ cannot be written in terms of elementary functions. But we can use the Comparison Test. Notice that $-1 \leqslant \sin x \leqslant 1$; i.e., $1 \leqslant 2+\sin x \leqslant 3$. Therefore,

$$\frac{2+\sin x}{x} \geqslant \frac{1}{x} \, .$$

Since $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ diverges, the integral $\int_1^\infty \frac{2+\sin x}{x}\,\mathrm{d}x$ diverges by the Comparison Test. □

Example 4.65

Determine whether $\displaystyle\int_1^\infty \frac{x}{\sqrt{x^6+1}}\,\mathrm{d}x$ converges or diverges.

Solution.
Notice that $x^6+1 \geqslant x^6$, whence $\frac{1}{x^6+1} \leqslant \frac{1}{x^6}$. Thus,

$$\frac{x}{\sqrt{x^6+1}} \leqslant \frac{x}{\sqrt{x^6}} = \frac{1}{x^2} \, .$$

The integral $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges, then so does $\int_1^\infty \frac{x}{\sqrt{x^6+1}}\,\mathrm{d}x$ by the Comparison Test. ☐

§5 Applications of the Integral

§5.1 Area Between Two Curves

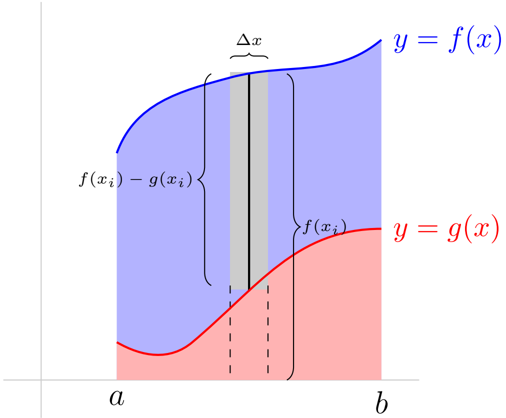
We have already learnt how to compute the area under a curve. We will now extend this method to finding area of regions that lie between the graphs of two curves.

The area A of a region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f, g are continuous functions such that $f(x) \geqslant g(x)$ for all $x \in [a, b]$ is:

$$A = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x \,. \qquad \text{(where } f(x) \geqslant g(x) \text{ on } [a,b]) \quad (18)$$

(It is crucial to note that this formula is *only valid* when $f(x) \geqslant g(x)$. If, for instance, $g(x) \geqslant f(x)$, the formula returns a negative number. We'll discuss this further after a few examples.)

To understand this formula, assume $g(x) \geqslant 0$ and consult the following picture.



The area below f is $\int_a^b f(x)\,\mathrm{d}x$.

The area below g is $\int_a^b g(x)\,\mathrm{d}x$.

Therefore, the area A between f and g is

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x.$$

If $g(x) \leqslant 0$, then the area below f is $\int_a^b f(x)\,\mathrm{d}x$, whilst the area below g is $-\int_a^b g(x)\,\mathrm{d}x$. Therefore, the area between f and g is

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x \,,$$

so the formula is still true.

We can approximate the area using rectangles as before:

$$A \approx \sum_{i=1}^{n} \big(f(x_i) - g(x_i)\big)\Delta x \,.$$

Taking the limit as $n \to \infty$, we get

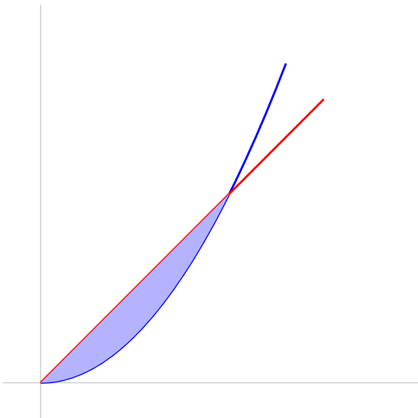
$$A = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x\ .$$

When using integrals to find the area between two curves, it is very helpful to graphs the curves. The graph reveals which curve is the "upper" curve, and which is the "lower" one. It also helps to find the limits of integration when they are not explicitly indicated.

Example 5.1 (area between curves)

Find the area bounded by the graphs of $f(x) = x$ and $g(x) = x^2$.

Solution.
Notice that we were not given an interval over which to integrate explicitly. We have to deduce the interval of integration from the context. If we sketch the graphs of f and g, there is only one region enclosed by these two curves; this region is bounded.



To find intersection points, we need to solve the equation $f(x) = g(x)$. It gives:

$$x = x^2 \quad \Leftrightarrow \quad x^2 - x = 0 \quad \Leftrightarrow \quad x = 0 \quad \text{or} \quad x = 1.$$

Notice that the graph of f is "above" the graph of g, so the area is

$$A = \int_0^1 (x - x^2)\mathrm{d}x = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{6}\ .$$

□

Example 5.2 (area between curves)

Find the area under the region bounded by $y = \sin x, y = \cos x, x = 0, x = \frac{\pi}{2}$.

Solution.
First, find the points of intersection:

$$\sin x = \cos x \quad \Longrightarrow \quad x = \tfrac{\pi}{4} \qquad \text{(because } 0 \leqslant x \leqslant \tfrac{\pi}{2}\text{)}$$

To understand which curve is "above" and which is "below", you have to graph these functions. We find:

$$\begin{aligned} \cos x \geqslant \sin x \quad & \text{for} \quad 0 \leqslant x \leqslant \tfrac{\pi}{4} \\ \sin x \geqslant \cos x \quad & \text{for} \quad \tfrac{\pi}{4} \leqslant x \leqslant \tfrac{\pi}{2}. \end{aligned}$$

To use the formula (18), we need to use the additivity of the integral: we split the integration interval into $[0, \frac{\pi}{4}]$ and $[\frac{\pi}{4}, \frac{\pi}{2}]$; on these intervals, one of the functions is always "above" the other. Therefore, the calculation of the total area between the curves splits into two calculations:

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx \\ &= \Big[\sin x - \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\ &= 2\sqrt{2} - 2. \end{aligned}$$

□

In general, we have the following formula.

The area between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b \Big|f(x) - g(x)\Big|\, dx \tag{19}$$

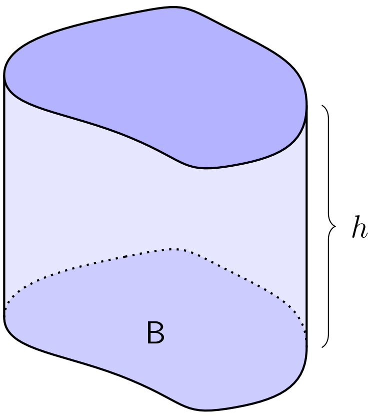
The absolute value appears in this formula because

$$\Big|f(x) - g(x)\Big| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geqslant g(x), \\ g(x) - f(x) & \text{if } f(x) \leqslant g(x). \end{cases}$$

Therefore, whenever we evaluate an integral like (19), we must first split the integration interval into the corresponding subintervals upon which we can definitely say that one function is "above" the other.

§5.2 Volumes

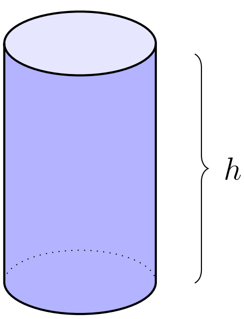
The problem of finding volumes is very similar to finding areas. To find the volume of a solid, we will approximate it by solids whose volume we can easily calculate, like cubes, cylinders, spheres, etc..



A **right cylinder** is obtained by translating a region B (the *base* of the cylinder) along a line perpendicular to the plane where B is contained (as pictured on the left). If A is the area of the base B, then the volume of the right cylinder with base B and height h is

$$V = Ah \ .$$

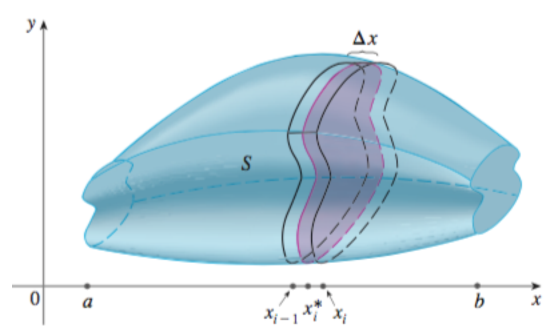
Example 5.3 (area of circular cylinder)



The circular cylinder of height h and radius r has volume

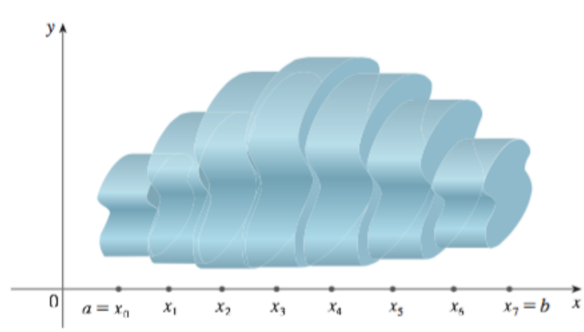
$$V = \pi r^2 h \ .$$

For a general solid S, we can first "cut" it into several slices (called *cross-sections* and approximate each one by a cylinder. Think of slicing S with a knife through x_i^* and computing the area $A(x_i^*)$ of the slice.



Then approximate the volume of each slice by the volume of a cylinder with base area $A(x_i^*)$ and height Δx. So

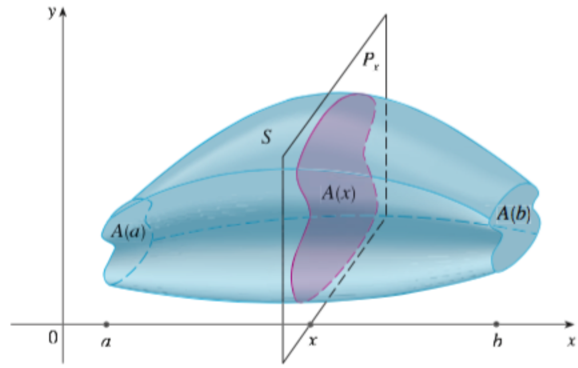
$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x \ .$$



Just as in the Area Problem (section 2.1), we find the exact volume of S by taking the limit as $n \to \infty$:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,\mathrm{d}x,$$

where $A(x)$ is the cross-sectioned area as a function of x.



In the area problem, we integrated height to find area. To find volumes, we integrate *areas*. Notice that for a cylinder, the cross-sectioned area is constant $A(x) = A$, so

$$V = \int_a^b A\,\mathrm{d}x = A(b-a),$$

which agrees with our formula $V = Ah$ for the volume of the right cylinder.

In the area problem, we integrated height to find area. To find volumes, we integrate *areas*. Notice that for a cylinder, the cross-sectioned area is constant $A(x) = A$, so

$$V = \int_a^b A\,\mathrm{d}x = A(b-a),$$

which agrees with our formula $V = Ah$ for the volume of the right cylinder.

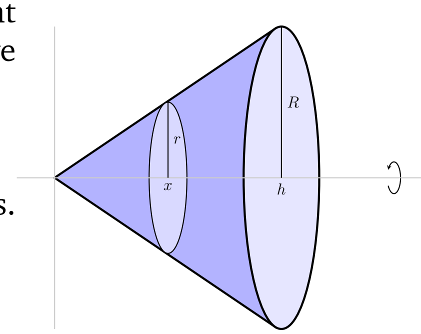
Example 5.4 (volume of circular cone)

Find the volume of the right circular cone with radius R and height h.

Solution.
It is convenient to centre the cone around the x-axis with its vertex at the origin, as you can see on the right. By similar triangles, we have that

$$\frac{r}{x} = \frac{R}{h} \implies r = \frac{R}{h}x,$$

so we can obtain the cone by rotating the line $y = \frac{R}{h}x$ about the x-axis. The cross-sectioned area is



$$A(x) = \pi r^2 = \pi\left(\tfrac{R}{h}x\right)^2 = \pi\frac{R^2}{h^2}x^2.$$

Using the definition of volumes with $a = 0$ and $b = h$, we have (integrating from 0 to h):

$$V = \int_0^h \pi\frac{R^2}{h^2}x^2\,\mathrm{d}x = \pi\frac{R^2}{h^2}\int_0^h x^2\,\mathrm{d}x = \pi\frac{R^2}{h^2}\left.\frac{x}{3}\right|_0^h = \pi\frac{R^2}{h^2}\frac{h^3}{3} = \tfrac{1}{3}\pi R^2 h.$$

This can be a useful formula to remember:

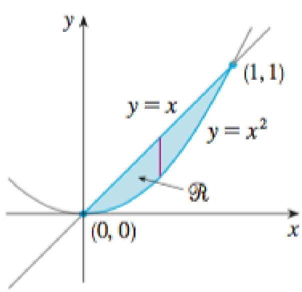
$$V = \tfrac{1}{3}\pi R^2 h$$

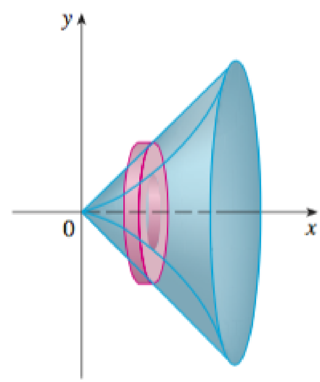
□

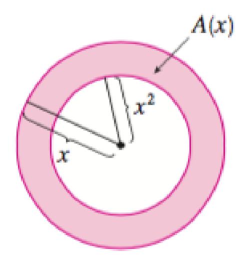
Example 5.5 (volume of conic cup)

The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Solution.
We've already seen that the curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$ (see example 5.1); see a detailed picture of this below on the left. What are the cross-sections? If you make the right drawing (like you can see below in the middle), you see immediately that a typical cross-section is a washer (i.e., an annulus) with inner radius x^2 and outer radius x, as pictured below on the right.







The cross-sectional area is

$$A = (\text{area of outer disc}) - (\text{area of inner disc}) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4).$$

So

$$V = \int_0^1 \pi(x^2 - x^4)\,\mathrm{d}x = \tfrac{2}{15}\pi.$$

□

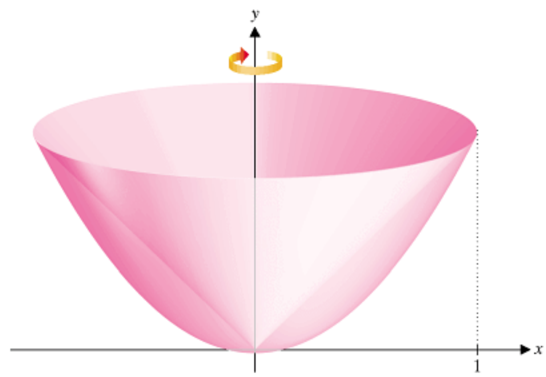
Here's an interesting question: what if we *rotate* the region from last example about the y-axis? Will we get the same volume? How can we handle the integral resulting from this?

Example 5.6 (volume of conic cup | again)

The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the y-axis. Find the volume of the resulting solid.

Solution.
This time, taking cross-sections about a point x gives cross-sections whose areas are hard to compute. Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the

y-axis and therefore integrate with respect to y.



So, let's try taking cross-sections in the y-direction. Notice that the cross-sections look again like a washer. Writing the equations for our curves as a function of y, we have

$$\text{inner radius:} \quad y$$
$$\text{outer radius:} \quad \sqrt{y}$$

So the cross-sectional area is $A(y) = \pi(y - y^2)$. Again, our interval is from 0 to 1, so the volume is given by

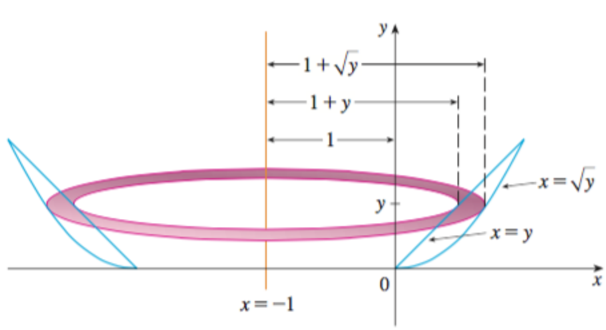
$$V = \int_0^1 \pi(y - y^2)\,\mathrm{d}y = \tfrac{\pi}{6}. \qquad \square$$

We can also rotate about any other axis.

Example 5.7 (volume of conic cup | yet again)

The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $x = -1$. Find the volume of the resulting solid.

Solution.
The horizontal cross-section is a washer with inner radius $1 + y$ and outer radius $1 + \sqrt{y}$.



So the cross-sectional area is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi(1+\sqrt{y})^2 - \pi(1+y)^2.$$

So the volume is

$$V = \int_0^1 A(y)\,\mathrm{d}y = \int_0^1 \pi\Big((1+\sqrt{y})^2 - (1+y)^2\Big)\,\mathrm{d}y = \pi\int_0^1 \big(2\sqrt{y} - y - y^2\big)\mathrm{d}y = \frac{\pi}{2}. \qquad \square$$

The solids in the examples we've seen so far are called **solids of revolution**, because they are obtained by revolving a region about a line. In general, we calculate the volumes using the formulas

$$V = \int_a^b A(x)\,\mathrm{d}x \qquad \text{or} \qquad V = \int_c^d A(y)\,\mathrm{d}y$$

We find the cross-sectional area in one of the following ways:

- **Disc**: if the cross-section is a disc, we find the radius and use

$$A = \pi(\text{radius})^2.$$

- **Washer**: if the cross-section is a washer, find inner and outer radii and use

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

§5.3 Volumes | Cylindrical Shells Method

Some volumes are too difficult to handle using the methods that we've seen so far. Imagine that we want to find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.

Notice that $y = 2x^2 - x^3 = x^2(2 - x)$, so we have that the graph intersects the y-axis at $x = 0$ and $x = 2$. This is the graph of the region. We could slice it horizontally and notice that we get a washer. But this leads to several problems.

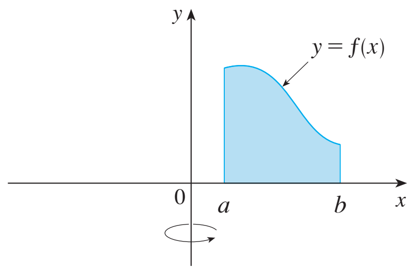
First, both inner and outer radii are defined by the same function, so to find them we need to put this function into the form $x = f(y)$. This means that we need to solve the equation

$$y = 2x^2 - x^3 \qquad \text{for } x \text{ in terms of } y$$

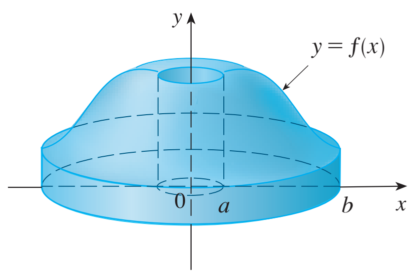
Often this isn't easy, and usually it's extremely difficult to write down a solution, especially by hand. Even when it is possible, the resulting equation can turn out to be messier than the original. This can be very problematic when we attempt to integrate.

Secondly, to use the washers we need to find the limits of integration. In this case, we need to know how high the graph goes, which means that we have to find a relative maximum which as you may remember from `MAT135` is hard work.

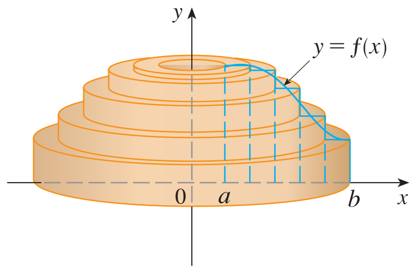
Luckily, there is a method called **method of cylindrical shells**. The idea is simple: we approximate a given solid using cylinders instead of slices. We can find volumes in the following way.



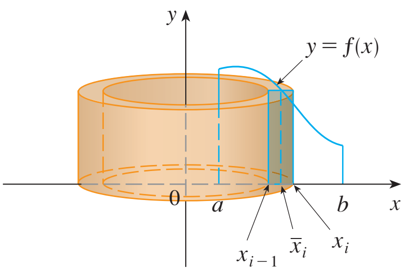
Consider a region bounded by the graph $y = f(x)$ (assume $f(x) \geqslant 0$) and the lines $y = 0$, $x = a$, and $x = b$, as depicted on the left.



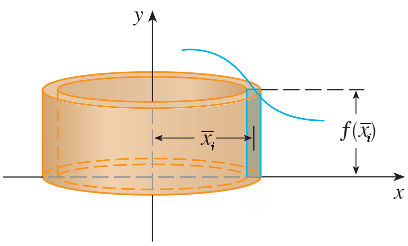
Let S be the solid obtained by rotating the region R about the y-axis. This solid is pictured on the left.

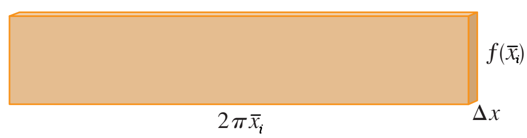


Instead of using a knife to cut the solid S into slices, we are going to use cylindrical "cookie cutters" to cut our solid into cylindrical shells, like you see on the left.



Consider an individual typical such cylindrical shell, as you see on the left. We divide the interval $[a, b]$ into n subintervals of width Δx. On each subinterval, pick a point $\overline{x}_i$ and construct a rectangle with base Δx and height $f(\overline{x}_i)$.
Revolving this rectangle about the y-axis forms a thin cylindrical shell (i.e., a hollow cylinder, like a can). To find the volume of this cylindrical shell, imagine cutting it along the height, rolling it out and flattening:





So $V_i = 2\pi\overline{x}_i f(\overline{x}_i)\Delta x$.

The total volume can be approximated by the sum of the volumes of n cylindrical shells:

$$V \approx \sum_{i=1}^{n} 2\pi\overline{x}_i f(\overline{x}_i)\Delta x \ .$$

Taking the limit as $n \to \infty$, we obtain the true volume:

$$V = \int_a^b 2\pi x f(x)\,\mathrm{d}x \ .$$

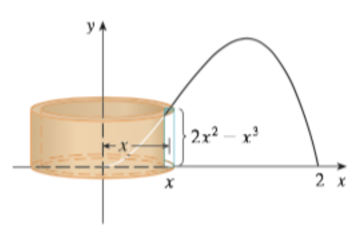
Notice that we are again integrating an area to compute the volume.

Example 5.8 (computing volume | solid of revolution | cylindrical shells method)

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Solution.
From the picture on the right, we can see that a typical shell has radius x (hence circumference $2\pi x$) and height $2x^2 - x^3$. So, by the shell method, the volume is

$$V = \int_0^2 2\pi x(2x^2 - x^3)\,\mathrm{d}x = \tfrac{16}{5}\pi.$$



□

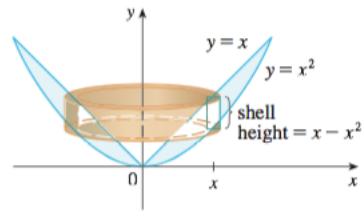
Notice that in this case the cylindrical shells method was much easier than the washer method. We didn't have to solve the equation for x in terms of y, and we didn't have to find the maximum to fund the limits of integration. However, in some other cases, it is easier to use the previous methods.

Example 5.9 (computing volume | solid of revolution | cylindrical shells method)

Find the volume of the solid obtained by rotating about the y-axis then region between $y = x$ and $y = x^2$, using cylindrical shells (integrating wrt x).

Solution.
Notice that we have already computed the volume of this solid, integrating wrt y. A typical shell has radius x and height $x - x^2$. So the volume is

$$V = \int_0^1 2\pi x(x - x^2)\,\mathrm{d}x = \tfrac{\pi}{6}.$$



□

The shell method also works if we rotate about the x-axis. We simply need to draw a diagram to identify the radius and height of the shells.

Example 5.10 (computing volume | solid of revolution | cylindrical shells method)

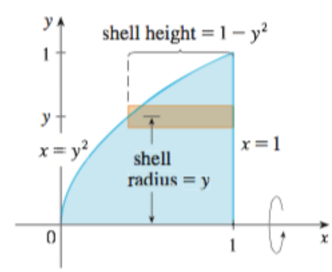
Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1 (integrating wrt y).

Solution.
Write in terms of y:

$$y = \sqrt{x} \implies x = y^2.$$

If we were integrating wrt x, we would use discs. But integrating wrt y forces us to use cylindrical shells. Each shell has radius y and height $1 - y^2$. So the volume is

$$V = \int_0^1 2\pi y(1 - y^2)\,\mathrm{d}y = \tfrac{\pi}{2}.$$



□

§5.4

Solids of Revolution | Summary

We close this section with a summary of the strategies for computing volumes of solids of revolution. To compute the volume of a solid of revolution, follow these steps:

(1) **Sketch the region to be revolved**.

(2) **Determine the variable of integration**.

- if the region has well-defined top and bottom (i.e., you have explicit equations for them), it's usually easier to integrate wrt x.
- if the region has well-defined left and right boundaries (i.e., you have explicit equations for them), it's usually easier to integrate wrt y.

(3) **Determine the method** based on the variable of integration and the axis of revolution:

- **discs or washers**: for integration wrt x about a horizontal axis
  or for integration wrt y about a vertical axis.
- **cylindrical shells**: for integration wrt x about a vertical axis
  or for integration wrt y about a horizontal axis.

(4) **Label your picture** with: • inner and outer radii for discs/washers
• radius and height for cylindrical shells.

(5) **Set up the integral and evaluate**.

§5.5 The Average Value of a Function

You are no doubt familiar with averages. If you write four exams in a year-long course, and your grades are 70%, 81%, 67%, 74%. Then your average grade is

$$\text{average} = \frac{70+81+67+74}{4} = 73 .$$

In general, the average of n values $y_1, \ldots, y_n$ is

$$y_{\text{ave}} = \frac{y_1 + \ldots + y_n}{n} .$$

What about *continuous functions*? Is there a way to calculate the *average value of a continuous function*? For example, to find the average temperature T_{ave} during a 24-hour day, we could measure the temperature every hour and take the average of the 24 measurements: this gives the approximation

$$T_{\text{ave}} \approx \frac{T(1) + \ldots + T(24)}{24} .$$

This approximation is good only as long as the temperature variation over the span of an hour is small, but increasing the number of measurements gives better and better approximations. This is beginning to sound like the set-up for integration.

Let's try to compute the average value of a continuous function $y = f(x)$ over the interval $[a, b]$. We start by dividing the interval $[a, b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$, choose points x_i in each subinterval, and calculate the average of the values $f(x_1), \ldots, f(x_n)$:

$$f_{\text{ave}} \approx \frac{f(x_1) + \ldots + f(x_n)}{n} .$$

Here's the key observation: since $\Delta x = \frac{b-a}{n}$, it follows that $n = \frac{b-a}{\Delta x}$, and therefore

$$f_{\text{ave}} \approx \frac{f(x_1) + \ldots + f(x_n)}{n} = \frac{f(x_1) + \ldots + f(x_n)}{\frac{b-a}{\Delta x}} = \frac{f(x_1) + \ldots + f(x_n)}{b-a} \Delta x = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x .$$

Taking the limit as $n \to \infty$, we get

$$f_{\text{ave}} = \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x .$$

Therefore, we have the following result.

Theorem 5.11 (The Average Value of a Function)
The average value of a continuous function f on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x .$$

For nonnegative functions f (i.e., $f \geqslant 0$), we have simple geometric interpretation of this formula:

$$\frac{\text{area}}{\text{width}} = \text{average height} .$$

Example 5.12 (average value of function)

Find the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution.
In this case, $a = 0$ and $b = \pi$, so we apply the formula for the average value of a function to find:

$$f_{\text{ave}} = \frac{1}{\pi - 0} \int_0^\pi \sin x \, dx = \frac{1}{\pi} (-\cos x)\Big|_0^\pi = \frac{2}{\pi} .$$ □

We might wonder if there is a number c at which the value of a function f is exactly equal to its average value. That is

Is there some $c \in [a, b]$ such that $f(c) = f_{\text{ave}}$?

In the case of the temperature function, this would mean that there is a specific time at which the temperature is the same as the average temperature. This is the case in the previous example. The following is a beautiful classical theorem which states that this is true for continuous functions in general.

Theorem 5.13 (Mean Value Theorem for Integrals (MVT))
If f is a continuous function on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b - a} \int_a^b f(x) \, dx .$$

In other words,

$$f(c)(b - a) = \int_a^b f(x) \, dx .$$

The geometric interpretation of the MVT for integrals is that for a nonnegative function f, there is some number c such that the *rectangle with height $f(c)$ and width $(b - a)$ has the same area as the region under the graph of f from a to b.*

Example 5.14 (MVT | finding c)

Find the number c that satisfies the Mean Value Theorem for integrals for the function $f(x) = (x-3)^3$ on the interval $[2, 5]$.

Solution.
We have:

$$f_{\text{ave}} = \frac{1}{5 - 2} \int_2^5 (x - 3)^2 \, dx = \frac{1}{3} \int_{-1}^2 u^2 \, du = \frac{1}{3} \frac{u^3}{3}\Big|_{-1}^2 = 1.$$

Since f is continuous, the MVT for integrals says that there exists some number $c \in [2, 5]$ such that $f(c) = f_{\text{ave}}$. Thus, $f(c) = f_{\text{ave}} = 1$, so $(c - 3)^2 = 1$, so $c - 3 = \pm 1$. Therefore, $c = 2$ or $c = 4$. □

Notice that this in this example, there are *two* numbers $c = 2$ and $c = 4$ that satisfy the MVT for integrals. That's okay: the theorem says there exists *at least one* such point c; in general, there could be many.

Example 5.15 (MVT | finding c)

Consider the function $f(x) = x^2$. Find the average value of f on the interval $[0, 3]$ and the number c that satisfies the MVT for integrals on $[0, 3]$.

Solution.
We have:

$$f_{\text{ave}} = \frac{1}{3-0}\int_0^3 x^2 \, dx = 3.$$

By the MVT for integrals, there exists some $c \in [0, 3]$ such that $f(c) = f_{\text{ave}}$. So $c^2 = f(c) = 3$, whence $c = \pm\sqrt{3}$. The only solution of this equation in $[0, 3]$ is $c = \sqrt{3}$. □

In the previous examples, we were able to solve easily for the point c. In general, this is very difficult, or not possible. The power of MVT for integrals lies not in the ability to find an appropriate point c, but instead in the theoretical value of the existence of such c. In other words, the fact that such a point c is guaranteed to exist (no matter what it is) is used to deduce other fact. The following example demonstrates this.

Example 5.16 (MVT | usage)

Show that if f is continuous on $[a, b]$ and $\displaystyle\int_a^b f(x) \, dx = 0$, then f has at least one zero in $[a, b]$.

Solution.
We have

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x) \, dx = 0 \ ,$$

so by MVT for integrals, there exists some $c \in [a, b]$ such that $f(c) = 0$. □

§5.6

Arc Length

We know how to measure the length of a straight line segment, or something made up of several straight line segments. From school we even know that the length of the circumference of a circle is $2\pi r$ where r is the radius of the circle. But what if we wanted to know the length of a more complicated curve? Of course, one way to do it is to fit a string to the shape of the curve, then measure the needed amount of string against a ruler. This method has significant disadvantages in that you need to draw the curve very accurately and fit the string very precisely; and even in that case your answer would be accurate only to a certain extent. (Never mind the fact that you'd probably also need to invest in a string!)

It turns out that Integration yet again provides with us with a much better tool for solving this problem. The main idea is that we can use *polygons* to approximate a curve. Since we have a simple formula for the length of a polygon, such approximations are very easy to calculate. Then allowing for polygons smaller in size but larger in number gives better and better approximations, recovering the true length in the limit.

Definition 5.17 (The Arc Length Formula)

LET: f be a differentiable function on $[a,b]$ whose derivative f' is continuous on $[a,b]$.

THEN: the **arc length** of the curve $y = f(x)$, $a \leqslant x \leqslant b$ is given by

$$L := \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, \mathrm{d}x .$$

Using Leibniz notation, this can be written as

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x .$$

We will not derive the formula for the arc length in these notes.

Example 5.18 (arc length of a curve | $x = g(y)$)

Find the arc length of the curve $y = \frac{2}{3}(x-1)^{3/2}$, $1 \leqslant x \leqslant 4$.

Solution.
Since $y = \frac{2}{3}(x-1)^{3/2}$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x-1}$. Therefore, by the Arc Length Formula,

$$L = \int_1^4 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x = \int_1^4 \sqrt{1 + \left(\sqrt{x-1}\right)^2}\, \mathrm{d}x = \int_1^4 \sqrt{1 + x - 1}\, \mathrm{d}x = \frac{14}{3} .$$

□

If a curve has the equation $x = g(y)$, $c \leqslant y \leqslant d$, and $g'(y)$ is continuous on $[c,d]$, then interchanging the roles of x and y we obtain the following formula for its arc length:

$$L = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\, \mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\, \mathrm{d}y \ .$$

Example 5.19 (arc length of a curve | $x = g(y)$)

Determine the arc length of the curve $x = \ln(\sec y)$, $0 \leqslant y \leqslant \frac{\pi}{4}$.

Notice that it's not a very good idea to write this as a function of x, because solving for y we would get an expression that is much more complicated to differentiate and then integrate.

Since $x = \ln(\sec y)$, we have $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\sec y}(\sec y \tan y) = \tan y$, which is continuous on $[0, \frac{\pi}{4}]$. Therefore,

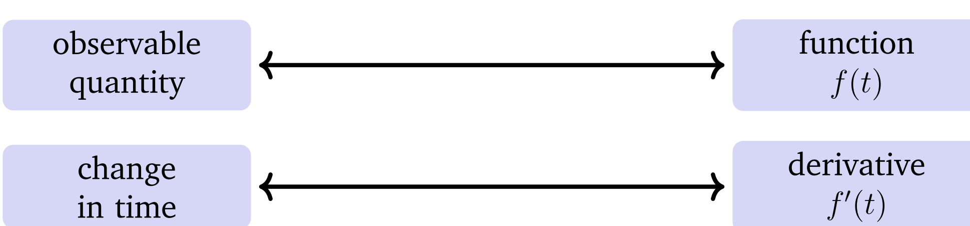
$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 y}\, \mathrm{d}y = \int_0^{\pi/4} \sqrt{\sec^2 y}\, \mathrm{d}y = \int_0^{\pi/4} \sec y\, \mathrm{d}y = \ln\left|\sqrt{2} + 1\right| \ .$$

§6 Differential Equations

Perhaps one of the most important applications of Calculus is to the study of *Differential Equations*. It is not an overstatement to say that almost *all* of mathematics in one way or another has been developed in order to solve and study differential equations. Differential equations are everywhere – our entire everyday life is governed by differential equations in all kinds of different ways.

Commonly, the work of scientists and engineers revolves around answering the following question: How can we translate a physical phenomenon into a set of equations which describes it? By studying these equations and their solutions, the scientist or engineer then uses this information to draw conclusions about the physical phenomenon:

What does this mean? For example, a scientist or an engineer is interested in some observable quantity (like temperature, population size, position, etc.), and how it is varying with respect to time. (For instance, "how does the population of my ecosystem change over the span of one year?"). This he or she encodes mathematically as some function $f(t)$, with t the independent variable representing time. This change of the observable quantity then corresponds to the change of the function $f(t)$; i.e., the derivative $f'(t)$:



Then, in this framework, describing a physical phenomenon amounts to finding an appropriate relationship between the function $f(t)$, its derivative $f'(t)$, and possibly its higher derivatives $f''(t), f'''(t), \ldots$. Such a relationship is called a *differential equation*.

Definition 6.1 (Differential Equation)
A **differential equation** is an equation that contains an unknown function f and any of its derivatives, $f', f'', f''', \ldots$.

Examples 6.2 (first examples of differential equations)

$f' = 3f \qquad f' = f - x \qquad f'' - 2f' = x^2$

$f'' - xf = 0 \qquad f' + f^2 = x$

(Here, x is the independent variable, and $f' := \frac{\mathrm{d}}{\mathrm{d}x} f$)

Definition 6.3 (Solution of Differential Equation)
A **solution** of a differential equation is a function which satisfies the differential equation.

Example 6.4 (a solution of a differential equation)

The function $f(x) = e^{3x}$ is a solution of $f' = 3f$. Indeed,

$$f'(x) = \left(e^{3x}\right)' = 3e^{3x} = 3f(x)$$

Differential equations are hard! Many mathematicians devote their entire careers to studying differential equations. In fact, differential equations are so hard that humans have essentially given up trying to solve differential equations explicitly; instead, they employ a very wide range of tools and techniques to study solutions' properties: remarkably, given a differential equation, one can often deduce a great amount of information[2] about a solution without knowing its formula. As a matter of fact, more often than not, such information is actually more useful than knowing an explicit formula for the solution!

Although it's usually impossible to describe a physical phenomenon exactly or even in great detail, we can find a set of equations that describe a system approximately and adequately for a given purpose.

To illustrate how differential equations can be used to describe a wide range of processes in physics, chemistry, biology, engineering, economics, and even social sciences, we begin by describing ways to model and analyse the long-term behaviour of animal populations.

[2]things like "is the solution continuous?" or "does it grow or decay as $x \to \infty$?"

§6.1

Models for Population Growth (Part I)

Imagine that we leave a group of rabbits loose on a large, unpopulated island, but which has plenty of food and no predators. How does the population of rabbits vary over time?

The general strategy of modelling is as follows: we begin with the simplest model first, and we analyse its predictions. We then modify the model to obtain a better approximation to reality.

We begin with a very simple model first. Under ideal conditions (e.g., unlimited environment, adequate nutrition, no predators, no dying, etc.), we can reasonably assume that the more rabbits there are, the more offspring are produced every month. For instance, if we have twice as many rabbits, then the rate at which new rabbits appear is also doubled. Mathematically, this translates into the following assumption: *the population grows at a rate proportional to the size of the population.*

Thus, let an independent variable t represent time, and let a dependent variable $P(t)$ represent the number of individuals (i.e., rabbits) in the population at any time t. Then the rate of growth of the population is the derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$. Let's summarise the system's variables that we have identified:

- t (the independent variable) $:=$ time
- $P(t)$ (dependent variable) $:=$ number of individuals at time t; then $\dfrac{\mathrm{d}P}{\mathrm{d}t}$ is the rate of growth.

Then the assumption about the rate of growth can be translated into the following differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \,, \tag{20}$$

where $k \in \mathbb{R}$ is some proportionality constant (per capita growth rate / reproduction rate).

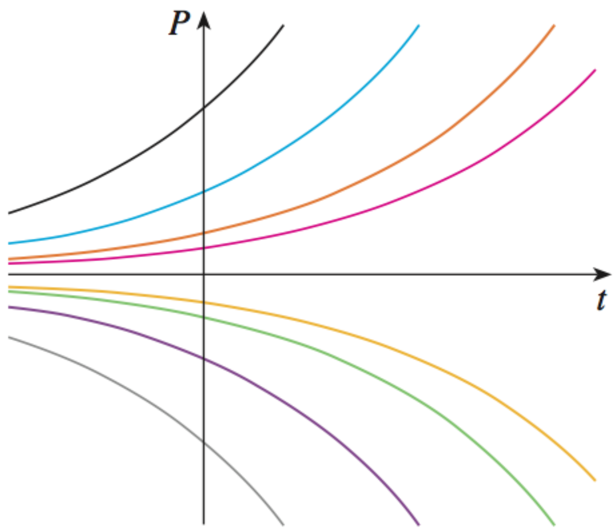
Having formulated a model, let's take a look at its predictions. First, assume $P(t) \neq 0$ for $t > 0$; i.e., we have a non-zero number of rabbits — this is reasonable assumption in our model, because we assume we start with a non-zero number of rabbits and that there is no dying. If $k > 0$, then equation (20) shows that $P'(t) > 0$ for all t; that is, the population always increases. In fact, as $P(t)$ increases, the growth rate $\frac{\mathrm{d}P}{\mathrm{d}t}$ becomes even larger:

What about a solution to equation (20)? Notice that $P(t) = Ce^{kt}$, for any constant C, satisfies this differential equation:

$$P'(t) = Cke^{kt} = k(Ce^{kt}) = kP(t) \,.$$

Thus, $P(t) = Ce^{kt}$ is a solution of equation (20). (Later we will see that there is no other solution.)

Allowing C vary over $\mathbb{R}$, we get a **family of solutions**:



But $P(t) > 0$ for all $t > 0$, so we are only interested in $C > 0$ and $t > 0$. For $t = 0$, we get $P(0) = Ce^0 = C$. Thus, C is the initial population $P(0)$.

This model is too simple to describe well what happens to a rabbit population. An obvious problem with this model is that it predicts that the population keeps growing forever — it doesn't take into account limited resources, predators, etc.. Whilst rabbit populations can, under good conditions, in a nearly exponential manner for a surprisingly long time (this actually happened in Australia during the 19th century), our model is ultimately unrealistic.

A more realistic model must reflect the fact that a given environment has limited resources. Therefore, any ecological system can support only some finite number of creatures over the long term. This number M is called the **carrying capacity**. It's reasonable to assume that population growth is proportional to their present size, but there is a maximum sustainable population M determined by the available resources. As the population size approaches M (and so the resources become scarce), the population growth slows down; once the population surpasses the maximum sustainable number M, the growth is reversed and the population decreases. For a model to take into account both of these trends, we make two additional assumptions:

- $\frac{\mathrm{d}P}{\mathrm{d}t} \approx kP$ if P is small (i.e., for small populations, the growth rate is roughly proportional to the current population);
- $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$ if $P > M$ (i.e., P decreases if it exceeds M).

A simple expression that incorporates both of these assumptions is:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right) \quad (21)$$

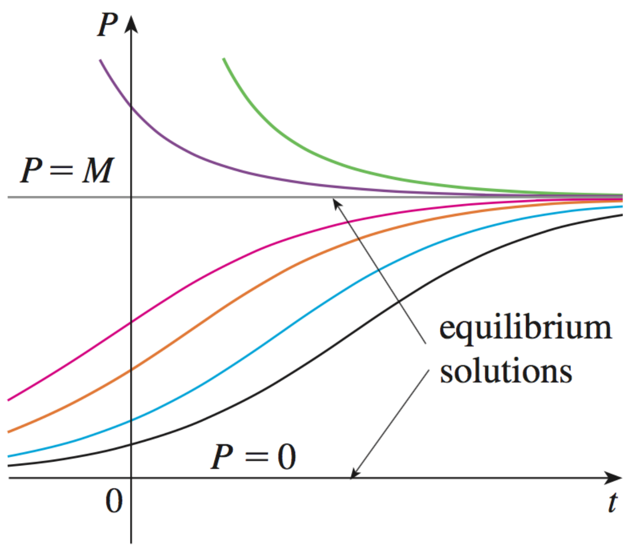
Notice:

- If $P \ll M$, then $\frac{P}{M}$ is close to 0 and so $\frac{\mathrm{d}P}{\mathrm{d}t} \approx kP$.
- If $P \gg M$, then $1 - \frac{P}{M} < 0$, so $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$.

Equation (21) is called the **logistic equation**.

Later we will learn techniques that enable us to solve the logistic equation. Right now we can do some qualitative analysis of its solutions.

- The constant functions $P(t) = 0$ and $P(t) = M$. These are called **equilibrium solutions**.
- If $0 < P(0) < M$, then $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$ (the population increases).
- If $P > M$, then $1 - \frac{P}{M} < 0$, so $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$ (the population decreases).
- If P approaches M, then $\frac{\mathrm{d}P}{\mathrm{d}t}$ approaches 0 (the population approaches the carrying capacity and levels off).



§6.2 Differential Equations

We shall now begin a more formalised study of differential equations, and how they can be solved in some special cases. For completeness, we restate some definitions that have already appeared.

Definition 6.5 (Differential Equation)
A **differential equation** is an equation that contains an unknown function f and any of its derivatives, f', f'', f''', $\ldots$.

The **order** of a differential equation is the order of the highest derivative appearing in the equation.

A **solution** of a differential equation is a function which satisfies the differential equation.

Examples 6.6 (order of a differential equation)

(1) $f' = f$ is a first order differential equation.

(2) $f'' = f$ and $f'' + f' + f = 0$ are examples of second order differential equations.

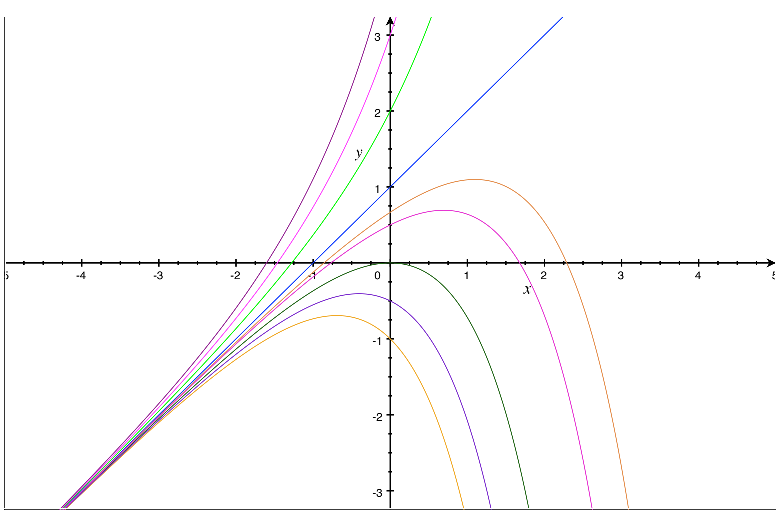
Example 6.7 (checking a family is a solution)

Show that every member of the family of functions

$$f(x) = Ce^x + x + 1$$

is a solution of the differential equation $f' = f - x$.

Solution.
To check whether a given (family of) function(s) is solution to a differential equation, we simply need to check that it satisfies the differential equation. On the one hand, we have $f' = Ce^x + 1$. On the other hand, $f - x = Ce^x + x + 1 - x = Ce^x + 1$. Therefore, for every value of C, the given function is a solution of the given differential equation. It turns out that *every* solution of this differential equation a member of this family. Here's a picture displaying this family of solutions:



Each displayed curve is the graph of $f(x)$ for a particular choice of the constant C. □

Very often, we are interested in a **particular solution**; that is, a solution for the particular choice of the constant C. For example, notice how in the previous example, only one of the curves in the family passes through the point $(0,0)$ — it's the curve coloured green. That is — if we denote $(x_0, y_0) := (0,0)$, — only one solution f satisfies the condition $f(x_0) = y_0$. This is called the **initial condition**, and the problem of finding a solution of a differential equation that satisfies the initial condition is called the **initial-value problem**.

Definition 6.8 (Solution of Differential Equation)
To **solve a differential equation** is to find all possible solutions.

The **particular solution satisfying the initial condition (x_0, y_0)** is the solution f satisfying $f(x_0) = y_0$.

An **initial-value problem (IVP)** is a differential equation whose solution must satisfy an initial condition.

Example 6.9 (IVP | first order)

Find a solution of the IVP

$$f' = f - x, \qquad f(0) = -\frac{1}{2} \;.$$

Solution.
From example 6.7, we know that the general solution is

$$f(x) = Ce^x + x + 1 \;. \tag{22}$$

To find C corresponding to the particular solution we seek, we substitute the initial condition $(x_0, y_0) = \left(0, -\frac{1}{2}\right)$ into equation (22):

$$-\frac{1}{2} = Ce^0 + 0 + 1 = C + 1 \qquad \Longrightarrow \qquad C = -\frac{3}{2} \;.$$

Therefore, the solution of the IVP is

$$f(x) = -\frac{3}{2}e^x + x + 1. \;.$$ □

Example 6.10 (IVP | second order)

Show that the function $y = 3\cos(2x)$ is a solution of the IVP

$$y'' = -4y \;, \qquad y(0) = 3 \;.$$

Solution.
Differentiation the function y twice, we find: $y'' = -12\cos(2x) = -4\big(3\cos(2x)\big)$. Moreover, the given function satisfies the initial condition, because $y(0) = 3\cos(2 \cdot 0) = 3$. Therefore, $y = 3\cos(2x)$ is indeed a solution of the IVP. □

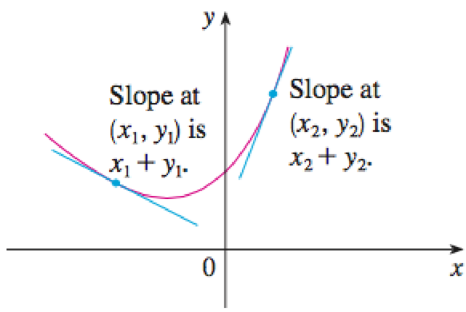
§6.3

Direction Fields

Suppose that we want to sketch the graph of the solution of the IVP

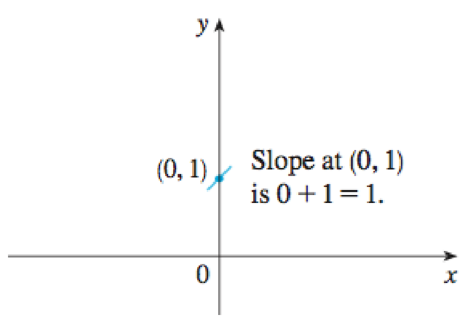
$$y' = x + y \ , \quad y(0) = 1 \ .$$

We don't know how to obtain a formula for the solution, so we need to be creative and find a clever way of graphing the solution without having an explicit formula. Let's think about what the differential equation means. The equation $y' = x + y$ tells that the slope at any point (x, y) on the graph (called the **solution curve** is equal to $x + y$; i.e., the slope is the sum of the x- and y-coordinates of the point (x, y).

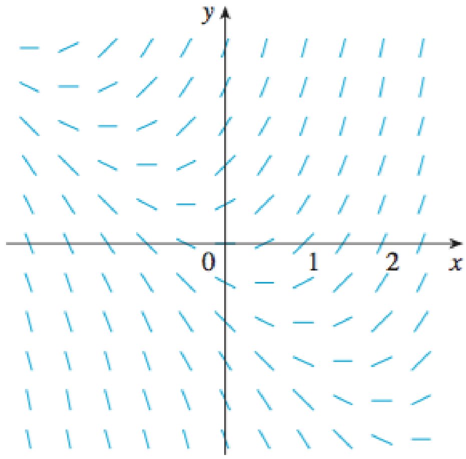


So, in particular, since the curve is required to pass through point $(0, 1)$ (this is the initial condition $y(0) = 1$), its slope at $(0, 1)$ is $0 + 1 = 1$.

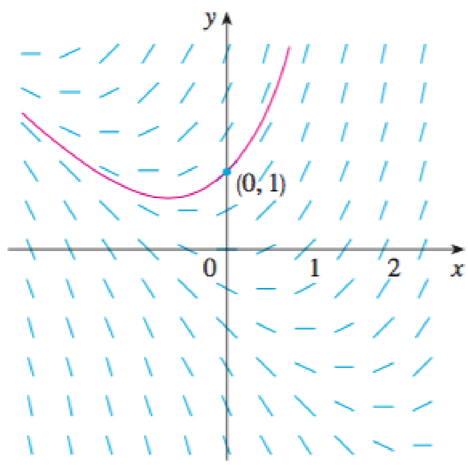
Therefore, near $(0, 1)$, a tiny portion of the solution curve looks like a tiny line segment through $(0, 1)$ with slope 1.



In order to get a handle on what the solution curve looks like, we draw short line segments at several points (x, y) with slope $x + y$. This collection of line segments is called the **direction field**. The direction field allows us to visualise the general shape of the solution curve by indicating the direction in which the curve proceeds at each point.



Now we can sketch the solution curve through the point $(0, 1)$ by following the direction field:



In general, suppose we have a first order differential equation of the form

$$y' = f(x, y) .$$

Even though it may not always be possible to solve such equations, we can always find an approximate solution curve. The differential equation tells us that the slope of a solution curve at (x, y) is $f(x, y)$. In order to know what the solution curve looks like, we draw short line segments with slope $f(x, y)$ at several points (x, y). This collection of line segments is called the **direction field** or **slope field**. Notice that if a particular solution passes through (x, y), then its slope at that point is $f(x, y)$. Each segment has the same slope as the solution curve through (x, y) and so it's tangent to the curve.

Thus, the direction field gives an indication of the behaviour of the family of solutions of a differential equation. It indicates the direction which a solution curve takes at each point it passes through.

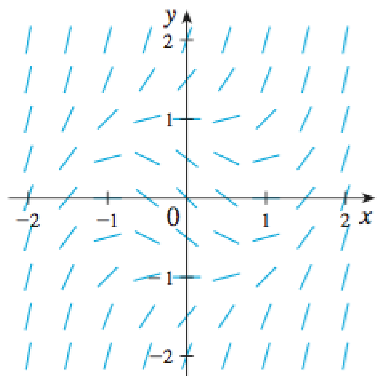
Example 6.11

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.
(b) Use part (a) to sketch the solution curve that passes through $(0, 0)$.

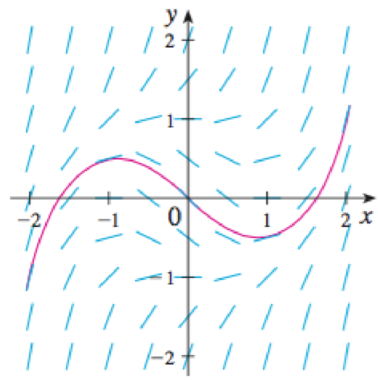
Solution.
(a) We start by computing the slope at several points:

x	-1	0	1	2	-1	0	1	$\cdots$
y	0	0	0	0	1	1	1	$\cdots$
$y' = x^2 + y^2 - 1$	0	-1	0	3	1	0	1	$\cdots$

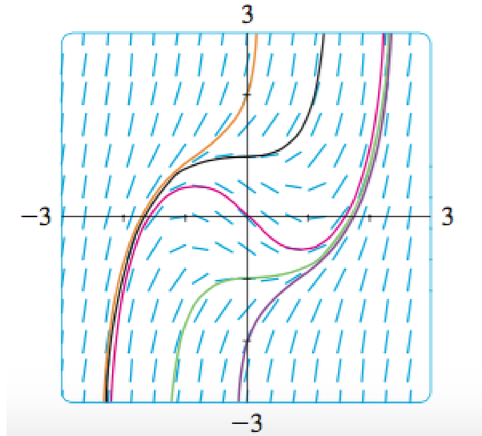
Now we draw short line segments with these slopes at the corresponding points, so we obtain the direction field.



(b) We start at the origin and move to the right in the direction of the line segment (which has slope -1). We continue to draw the solution curve so that it's parallel to the nearby segments. Returning to the origin, we can also draw the curve extending to the left. The picture looks like this:



Here's a picture of a few more solutions curves with y-intercepts $-2, -1, 0, 1, 2$:



□

What is significant about this is that we were able to construct the direction field using only elementary algebra, without first solving the differential equation. That is, by constructing a direction field, we can obtain a reasonably good picture of how the solution curves behave. Since most differential equations are not soluble exactly, this is an extremely useful technique. This is what is meant by qualitative information about the solution: we get a graphical ideal of how solutions behave, but not the details such as the value of a solution at a specific point.

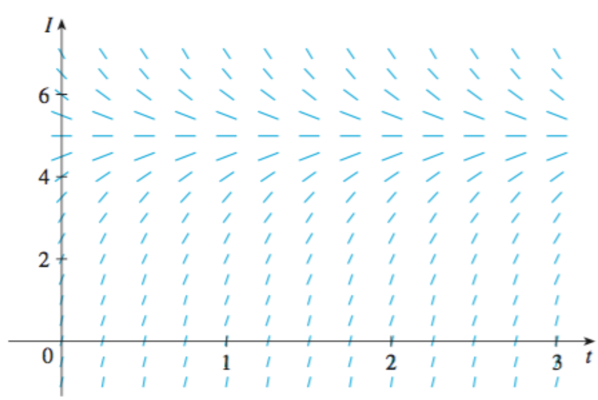
Example 6.12 (using a direction field to visualise the behaviour of solutions)

Draw a direction field for the equation

$$y' = 5 - y \ .$$

What can you deduce about the solutions?

Solution.
Here's the picture of the direction field for this differential equation:



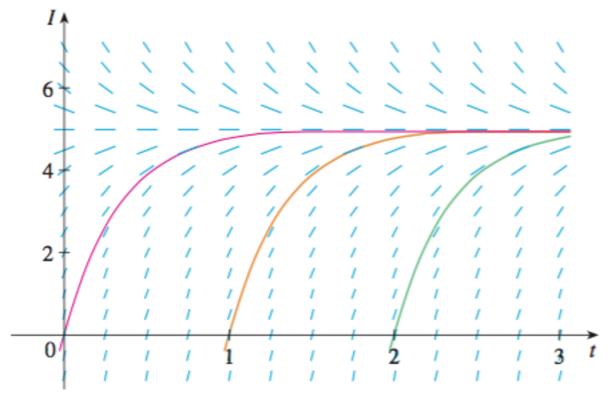
- It appears from the direction field that all solutions approach the value 5; that is,

$$\lim_{x\to\infty} y(x) = 5 \ .$$

- The constant solution $y(x) = 5$ is an equilibrium solution. Indeed, if $y(x) = 5$, then $y' = 0$ and the right-hand side of the differential equation is $5 - 5 = 0$.

- The solutions tend to the equilibrium solution.
- Solution curves that start below $y = 5$ are increasing; those that start below $y = 5$ are decreasing.

Based on these observations, we use the direction field to sketch a few specific solution curves:



Note that the line segments along any horizontal line are parallel. This is because the independent variable x does not appear explicitly on the right-hand side of the differential equation. In general, a differential equation of the form $y' = f(y)$ is called **autonomous**. For such equations, the slopes corresponding to two points with the same y-coordinate must be equal. This means that if we know the solution to an autonomous differential equation, then we can obtain infinitely many other solutions just by shifting the graph of the known solution to right or to the left. □

§6.4 Separable Equations

Definition 6.13 (Separable Equation)
A **separable equation** is a first-order differential equation of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)f(y) \ ,$$

in which the expression for the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written as a product of a function $g(x)$ and a function $f(y)$.

The name "separable" is meant to be suggestive: the right-hand side can be 'separated' into a function of x and a function of y.

Example 6.14

Determine if the differential equation $y' = xy^2 - 2xy$ is separable.

Solution.
Notice that we can rewrite this equation as follows: $y' = x(y^2 - 2y)$. So, in definition 6.13, $g(x) = x$ and $f(y) = y^2 - 2y$. Therefore, the equation is separable. □

Example 6.15

The equation $y' = xy^2 - 2x^2y$ is NOT separable, because there is no separate the right-hand side into a product of the form $g(x)f(y)$.

Separable equations are a particularly nice class of differential equations, because they are very easy to solve.

To solve a separable equation, we do the following. If $f(y) \neq 0$, then let $h(y) := \frac{1}{f(y)}$, and rewrite the differential equation by multiplying both sides by $h(y)$:

$$h(y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \ .$$

Now, we apply an artificial trick[3] of "multiplying both sides by $\mathrm{d}x$" to write

$$h(y)\,\mathrm{d}y = g(x)\,\mathrm{d}x \ .$$

Now, integrate both sides this equation:

$$\int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x \ .$$

Integration then results in the solution y that is defined implicitly as a function of x. In some cases it is possible to solve for y in terms of x, but often it isn't.

[3] there is a neat, deep, and completely rigorous way to make sense of this, but this rigour is not important to us.

Example 6.16

Solve the differential equation

$$y' = \frac{x^2 + 7x + 3}{y^2} \; .$$

Solution.
Separating the variables, we have

$$y^2 \, \mathrm{d}y = (x^2 + 7x + 3) \, \mathrm{d}x \; .$$

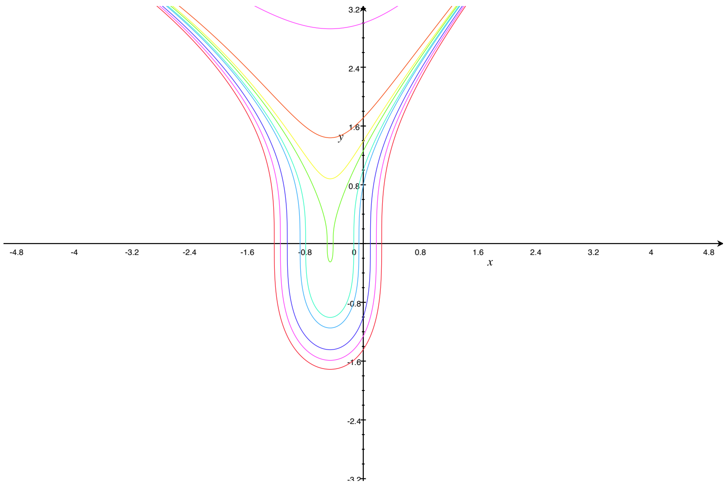
Integrating both sides, we have

$$\int y^2 \, \mathrm{d}y = \int (x^2 + 7x + 3) \, \mathrm{d}x \qquad \text{i.e.} \qquad \tfrac{1}{3} y^3 = \tfrac{1}{3} x^3 + \tfrac{7}{2} x^2 + 3x + C \; ,$$

where C is an arbitrary constant. Solving for y, we get the general solution

$$y(x) = \sqrt[3]{3\left(\tfrac{1}{3}x^3 + \tfrac{7}{2}x^2 + 3x + C\right)} = \sqrt[3]{x^3 + \tfrac{21}{2}x^2 + 9x + K} \; , \tag{23}$$

where $K := 3C$. Notice that for each value of K, we get a different solution of the differential equation. This is called a **one-parameter family of solutions**, and it's depicted here:



□

Example 6.17

Find the solution of the equation in example 6.16 that satisfies the initial condition $y(0) = 3$.

Solution.
Substituting $x = 0$ into the general solution equation (23), we find

$$y(0) = \sqrt[3]{0 + K} = \sqrt[3]{K} \; .$$

Thus, in order to satisfy $y(0) = 3$, the constant K must be 27. So the solution of the given IVP is

$$y(x) = \sqrt[3]{x^3 + \tfrac{21}{2}x^2 + 9x + 27} \; . \qquad \square$$

It is not always possible to obtain an explicit presentation of the solution.

Example 6.18

Find the solution of the IVP:

$$y' = \frac{9x^2 - \sin x}{\cos y + 5e^y} \, , \qquad y(0) = \pi \; .$$

Solution.
First, we rewrite the equation in Leibniz notation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9x^2 - \sin x}{\cos y + 5e^y} \; .$$

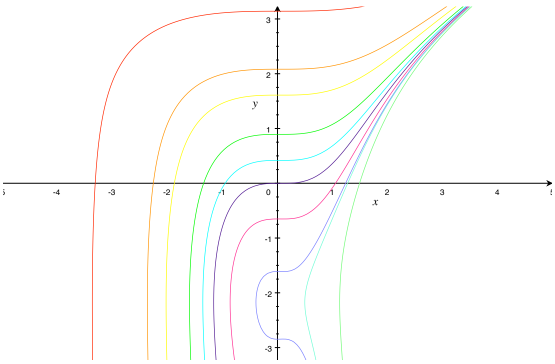
Notice that the equation is separable, because we can rewrite it as:

$$(\cos y + 5e^y)\,\mathrm{d}y = (9x^2 - \sin x)\,\mathrm{d}x \; .$$

Integrating both sides, we get

$$\sin y + 5e^y = 3x^3 + \cos x + C \; . \tag{24}$$

We cannot solve this equation for y explicitly in terms of x. Equation (24) is the general solution, given implicitly. Here's a picture of this family of solutions:



Even though we were unable to solve for y explicitly, we are still able to find the particular solution satisfying the given initial condition. Substitute $x = 0$ and $y = \pi$ into equation (24) to find the right constant C:

$$\sin\pi + 5e^{\pi} = 0 + \cos(0) + C \qquad \Longrightarrow \qquad C = 5e^{\pi} - 1 \; .$$

Hence, $\sin y + 5e^y = 3x^3 + \cos x + 5e^{\pi} - 1$ is an implicit presentation of the solution of the IVP. $\square$

Example 6.19

Solve the equation $y' = x^2 y$.

Solution.
Rewrite the equation in Leibniz notation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y \ .$$

If $y \neq 0$, we separate the variables and integrate:

$$\frac{\mathrm{d}y}{y} = x^2\,\mathrm{d}x \qquad \rightsquigarrow \qquad \int \frac{\mathrm{d}y}{y} = \int x^2\,\mathrm{d}x \qquad \rightsquigarrow \qquad \ln|y| = \tfrac{1}{3}x^3 + C \ .$$

This equation defines y implicitly as a function of x, but this case is special because we *can* solve for y explicitly:

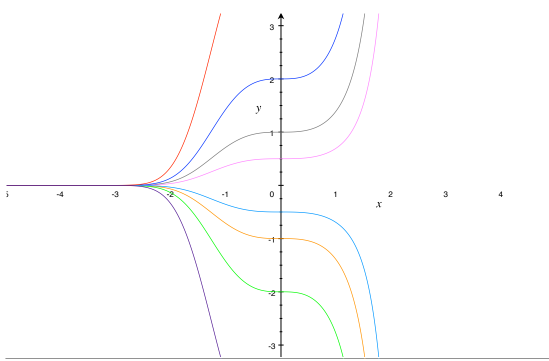
$$|y| = e^{\ln|y|} = e^{x^3/3+C} = e^C e^{x^3/3} \ .$$

So $y = \pm e^C e^{x^3/3}$.

Notice that we can also easily verify that $y = 0$ is also a solution of the given differential equation, so we can write the general solution as

$$y = Ae^{x^3/3} \ ,$$

where A is an arbitrary constant. Here's a picture displaying this family of solutions:



□

§6.5 Models for Population Growth (Part II)

In section 6.1 we discussed how differential equations can be used to model population growth. Now that we have more tools to study differential equations, we are going to further study two differential equations used to model population growth: the Law of Natural Growth and the Logistic Differential Equation.

§6.5.1 The Law of Natural Growth

In this model we assume that the population, P, grows at a rate proportional to the size of the population. In terms of differential equations this can be written as

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \quad \text{where } k \text{ is a constant} \tag{25}$$

Equation (25) is sometimes called the **Law of Natural Growth**. If $k > 0$, the population increases. If $k < 0$, the population decreases.

Notice that equation (25) is separable, so we can solve it explicitly. Separating variables we have

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \Rightarrow \frac{\mathrm{d}P}{P} = k\mathrm{d}t$$

Integrating both sides we obtain

$$\begin{aligned} \int \frac{\mathrm{d}P}{P} &= \int k\mathrm{d}t \\ \ln|P| &= kt + C \\ |P| &= e^{kt+C} = e^C e^{kt} \\ P &= \pm e^C e^{kt} \end{aligned}$$

Notice that $P = 0$ is also a solution. So the general solution is

$$P = Ae^{kt} \quad \text{where } A = 0 \text{ or } A = \pm e^C$$

Now notice that $P(0) = Ae^0 = A$. So A is the initial value of the function.

THE SOLUTION TO THE INITIAL VALUE PROBLEM

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \quad P(0) = P_0$$

IS

$$P(t) = P_0 e^{kt}$$

Example 6.20

A freshly inoculated streptococcus A (a common group of microorganisms that cause strep throat) contains 60 cells. When the culture is checked 20 minutes later, it is determined that there are 120 cells present. Assuming Law of Natural Growth, determine the number of cells present at any time, t, (measured in hours).

Solution.
Law of Natural Growth means that the number of cells satisfies the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

So we know that $P(t) = Ae^{kt}$ where $A = P(0)$. Since $P(0) = 60$, we have

$$P(t) = 60e^{kt}$$

We can use the second observation to determine the value of the constant k. We know that at time $t = 20$ minutes $= \frac{1}{3}$ hours, $P\left(\frac{1}{3}\right) = 120$. So

$$\begin{aligned} 120 = P\left(\frac{1}{3}\right) = 60e^{\frac{k}{3}} &\Rightarrow 2 = e^{\frac{k}{3}} \\ &\Rightarrow \ln 2 = \frac{k}{3} \\ &\Rightarrow k = 3\ln 2 \\ &\Rightarrow k = \ln 8 \end{aligned}$$

Therefore,

$$P(t) = 60e^{(\ln 8)t} = 60 \cdot 8^t$$

This tells us that after 8 hours,

$$P(8) = 60 \cdot 8^8 = 1,006,632,960 \quad \text{(WOW!)}$$

□

§6.5.2 The Logistic Model

As we saw in section 6.1, a more realistic model for population growth is given by the **Logistic Differential Equation**:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$$

Notice that this equation is separable so we can solve it explicitly. Separating variables we have

$$\frac{\mathrm{d}P}{P\left(1 - \frac{P}{M}\right)} = k\mathrm{d}t$$

Integrating both sides

$$\int \frac{\mathrm{d}P}{P\left(1 - \frac{P}{M}\right)} = \int k\mathrm{d}t \tag{26}$$

To evaluate the integral of the left hand side, we first notice that

$$\frac{1}{P\left(1 - \frac{P}{M}\right)} = \frac{M}{P(M-P)}$$

and using partial fraction, we obtain

$$\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$$

So we can rewrite (26) as

$$\begin{aligned}
\int \left(\frac{1}{P} + \frac{1}{M-P}\right) &= \int k\mathrm{d}t \\
\ln|P| - \ln|M-P| &= kt + C \\
\ln\left|\frac{P}{M-P}\right| &= kt + C \\
-\ln\left|\frac{P}{M-P}\right| &= -kt - C \\
\ln\left|\frac{M-P}{P}\right| &= -kt - C \\
\left|\frac{M-P}{P}\right| &= e^{-kt-C} = e^{-C}e^{-kt}
\end{aligned}$$

So, $\dfrac{M-P}{P} = \pm e^{-C}e^{-kt}$. Hence,

$$\frac{M-P}{P} = Ae^{-kt} \quad \text{where } A = \pm e^{-C} \tag{27}$$

Solving for P we obtain,

$$\frac{M}{P} - 1 = Ae^{-kt} \Rightarrow \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

so

$$P = \frac{M}{1 + Ae^{-kt}}$$

If $P(0) = P_0$, then putting $t = 0$ into equation (27), we can find the value of A,

$$\frac{M - P_0}{P_0} = Ae^0 = A$$

THE SOLUTION TO THE LOGISTIC EQUATION

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right) \qquad \text{WITH } P(0) = P_0$$

IS

$$P = \frac{M}{1 + Ae^{-kt}} = \qquad \text{WHERE } A = \frac{M - P_0}{P_0}$$

Example 6.21 (Modelling Bear Population)

A national park is known to be capable of supporting 1000 brown bears, but no more. One hundred bears are in the park at the present. We model the population with a logistic differential equation with $k = 0.08$.

(1) How many bears will there be after 40 years?
(2) When will the population reach 900?

Solution.
The carrying capacity is 1000, so $M = 1000$ and the initial condition is $P(0) = 100$. So we first need to solve the Initial-value theorem,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

so the population at time t is

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}} \text{ where } A = \frac{1000 - 100}{100} = 9$$

So,

$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

(1) So the population when $t = 40$ is

$$P(40) = \frac{1000}{1 + 9e^{-0.08(40)}} \approx 731.6$$

(2) The population is 900 when

$$\begin{aligned}\frac{1000}{1+9e^{-0.08t}} = 900 &\Rightarrow \frac{10}{9} = 1 + 9e^{-0.08t} \\ &\Rightarrow \frac{1}{9} = 9e^{-0.08t} \\ &\Rightarrow e^{-0.08t} = \frac{1}{81} \\ &\Rightarrow t = \frac{\ln(81)}{0.08} \approx 54.9\end{aligned}$$

So the population reaches 900 after approximately 55 years.

□

§7 Sequences and Series

§7.1 Sequences

The mathematical notion of a sequence is not much different from the way we use it in everyday English language. For example, if we are asked to tell the sequence of events that led up to a traffic accident, we would not only need to list the events, but we would tell them in a specific order (hopefully, the order in which they actually happened).

In mathematics, we use the term *sequence* to mean an infinite list of real numbers written in a specific order:

$$a_1, a_2, a_3, \ldots, a_n \ldots \quad .$$

The number a_1 is called **first term**, a_2 is the **second term**, and in general a_n is called the **n^{th} term of the sequence**. We use the integers to give the sequence the order, but it doesn't matter where we start, so the following are also perfectly acceptable ways to enumerate a sequence:

$$a_0, a_1, a_2, \ldots, a_n \ldots \quad ,$$
$$a_{17}, a_{18}, a_{19}, \ldots, a_{17+n} \ldots \quad ,$$
$$a_{-3}, a_{-2}, a_{-1}, \ldots, a_{-3+n} \ldots \quad ,$$

and, more generally, for any integer $N_0 \in \mathbb{Z}$,

$$a_{N_0}, a_{N_0+1}, a_{N_0+2}, \ldots, a_{N_0+n} \ldots \quad .$$

We still usually refer to a_{N_0} as the **first term** of the sequence (because it appears first in the sequence, even if $N_0 \neq 1$), and so on. Most often, though, N_0 is either 0 or 1.

The mathematically rigorous, concise and useful way to define the notion of a *sequence* is as a *function* that to an integer n assigns a real number a_n.

Definition 7.1 (Sequence)
A **sequence** is a function whose domain is the subset of integers starting with some integer N_0.

For instance, the function $a : \mathbb{N} \to \mathbb{R}$, given by $a(n) := \frac{1}{n}$, defines a sequence:

$$a(1) = 1, \quad a(2) = \tfrac{1}{2}, \quad a(3) = \tfrac{1}{3}, \quad \ldots, \quad a(n) = \tfrac{1}{n}, \quad \ldots .$$

Notation: There is a zoo of notations used to denote sequences. The following list is some common ways to denote the sequence $\{a_1, a_2, a_3, \ldots\}$:

$$\{a_n\}_{n=1}^{\infty}, \quad \{a_n\}, \quad (a_n), \quad (a_n)_{n=1}^{\infty}, \quad \ldots .$$

Often, people even omit brackets altogether, and simply write "a_n".

Also, often the starting index of a sequence is not even mentioned at all. There is one main reason for that: as we will see shortly, the most important thing about a sequence is how it behaves as $n \to \infty$; such properties remain the same no matter where exactly the given sequence begins. Instead, it's the *tail* of the sequence that actually carries useful information. This point will become clear to you very soon.

How do we describe a sequence? There are two main ways to do it: by means of a *formula*, and *recursively*.

Example 7.2 (defining a sequence by giving a general formula)

A very common way to describe/define a sequence is by giving a formula for the general term a_n in terms of n. Here are some examples:

$\{a_n\}_{n=1}^{\infty}$ NOTATION	GENERAL TERM	LISTING TERMS OF THE SEQUENCE
$\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$	$a_n = \frac{1}{n^2}$	$\{1, \frac{1}{4}, \frac{1}{9}, \ldots\}$
$\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=0}^{\infty}$	$a_n = \sin\left(\frac{n\pi}{2}\right)$	$\{0, 1, 0, -1, 0, \ldots\}$
$\left\{(-1)^n \frac{n+1}{3^n}\right\}_{n=0}^{\infty}$	$a_n = (-1)^n \frac{n+1}{3^n}$	$\{1, -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \ldots\}$
$\left\{\frac{1}{\ln n}\right\}_{n=10}^{\infty}$	$a_n = \frac{1}{\ln n}$	$\{\frac{1}{\ln 10}, \frac{1}{\ln 11}, \frac{1}{\ln 12}, \ldots\}$

Often, however, a formula for the general term a_n in terms of n is not available, or hard to obtain. Another common way to describe a sequence is by defining it *recursively*.

Definition 7.3 (Recursive Definition of a Sequence)
A sequence is defined **recursively** by giving:

(1) the value (or values) of the initial term (or terms);

(2) a rule (called the **recursion formula**) for calculating any later term from terms that precede it.

Here are two examples that demonstrate this.

Example 7.4 (recursive definition of a sequence)

(1) The famous **Fibonacci sequence** $\{f_n\}$ is defined recursively by the following initial conditions

$$f_1 := 1 \quad \text{and} \quad f_2 := 1 \ ,$$

and the recursion formula

$$f_n := f_{n-1} + f_{n-2} \ . \qquad (n \geqslant 3)$$

In other words, each term is the sum of the previous two terms. So, the first few terms are

$$1, 1, 3, 5, 8, 13, \ldots \ .$$

(2) A sequence $\{a_n\}$ is defined recursively by

$$a_1 := 1 \quad \text{and} \quad a_n := \sqrt{1 + a_{n-1}} \; .$$

The first few terms of this sequence are:

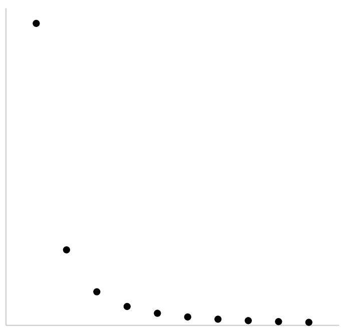
$$1, \sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}, \dots \; .$$

§7.2 Limit of Sequences

How do we think about sequences? Well, since sequences are functions – and a very helpful way to think about a function is to plot its graph, – we plot their graphs.

For instance, let's plot the graph of the sequence $a_n = \frac{1}{n^2}$. By definition, a sequence is a function whose domain is the (subset of the) integers, its graph consists of isolated points with coordinates

$$(1, a_1), \quad (2, a_2), \quad (3, a_3), \dots \; .$$



The graph of $a_n = \frac{1}{n^2}$ is pictured on the right. You can see that as n becomes larger and larger, the points of the graph of a_n approach the horizontal line 0. That is, the terms of the sequence $a_n = \frac{1}{n^2}$ get closer and closer to 0. And by taking n sufficiently large, we can make a_n as close to 0 as we wish.

This is almost literally the intuitive definition of the limit of a sequence.

Definition 7.5 (Limit of a Sequence | Intuitive Definition)
A sequence $\{a_n\}$ **has the limit L** if we can make the terms a_n as close to L as we like by taking n sufficiently large. We denote this by

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty \qquad \text{or} \qquad a_n \xrightarrow[n\to\infty]{} L \; .$$

If the limit $\lim\limits_{n\to\infty} a_n$ exists (and is a finite number), we say that the sequence is **convergent**. Otherwise, we say that the sequence is **divergent**.

Notice that the definition of the limit of a sequence is very similar to the definition of the limit of a function of a real variable x at infinity:

$$\lim_{x\to\infty} f(x) = L \; .$$

This isn't a surprise, because, after all, we have defined a sequence as a function. But the crucial difference is that n can only take integer values, whilst x can take on any real value.

Theorem 7.6
If $\lim\limits_{x\to\infty} f(x) = L$, and $f(n) = a_n$ when n is an integer, then

$$\lim_{n\to\infty} a_n = L \; .$$

If a_n becomes arbitrarily large as n increases, the sequence diverges, but in a special way.

Definition 7.7 (Sequence Diverges to ∞ | Intuitive Definition)
A sequence $\{a_n\}$ **diverges to ∞** if the terms a_n become arbitrarily large as n increases; we denote this by

$$\lim_{n\to\infty} a_n = \infty .$$

Example 7.8 (sequence diverges to ∞)

The sequence $a_n = 2^n$ diverges to ∞, because the terms become larger and larger as n increases:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16\,384, 32\,768, 65\,536, 131\,072 \ldots.$$

(This sequence is no doubt familiar to computer science students. The first chunk of this sequence (up to 2048) is no doubt familiar to anyone who has ever wasted hours playing 2048.)

Finding limits of sequences is a lot like finding limits of functions. Again, this must not come as a surprise; after all, sequences are functions. But there are some key differences, as we'll see in the examples that follow.

A key advantage of our definition of a sequence as a function is that we get many limit laws for sequences for free from the familiar limit laws for functions:

Proposition 7.9 (Limit Laws for Sequences)
If (a_n) and (b_n) are convergent sequences with limits $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$, and c is a constant, then

(a) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L + M.$

(b) $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n = cL.$

(c) $\lim\limits_{n\to\infty} a_nb_n = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right) = LM.$

(d) $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty}}{\lim\limits_{n\to\infty} b_n} = \dfrac{L}{M}$ *if* $\lim_{n\to\infty} b_n = M \neq 0.$

(e) $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ *if* *if $p > 0$ and $a_n > 0$.*

Example 7.10 (finding the limit of a sequence)

Evaluate $\lim\limits_{n\to\infty} \dfrac{2n+3}{5n+1}$.

Solution.
This has an indeterminate form $\frac{\infty}{\infty}$. We divide the numerator and denominator by the highest power

of n in the denominator, and then we use limit laws:

$$\lim_{n\to\infty} \frac{2n+3}{5n+1} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{5+\frac{1}{n}} = \frac{\lim\limits_{n\to\infty}(2+\frac{3}{n})}{\lim\limits_{n\to\infty}(5+\frac{1}{n})} = \frac{2}{5} .$$

□

Note that we cannot apply L'Hôpital's rule in the last example, since the functions in the numerator and denominator are not continuous — they are only defined for integer values. If you (incorrectly) apply L'Hôpital's rule, you get the right answer.

Example 7.11 (a divergent series)

Evaluate the limit $\lim\limits_{n\to\infty} \dfrac{n^2+1}{2n-1}$.

Solution.
Again, this has the intermediate form $\frac{\infty}{\infty}$. Dividing top and bottom by n (which is the highest power of n in the denominator), we have

$$\lim_{n\to\infty} \frac{n^2+1}{2n-1} = \lim_{n\to\infty} \frac{n+\frac{1}{n}}{2-\frac{1}{n}} = \frac{\lim\limits_{n\to\infty}\left(n+\frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(2-\frac{1}{n}\right)} = \infty ,$$

so the sequence $\left(\frac{n^2+1}{2n-1}\right)_{n=1}^{\infty}$ diverges. □

In the following example, we see that a sequence does not need to tend to $\pm\infty$ to diverge.

Example 7.12 (a divergent sequence whose terms do not tend to ∞)

Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution.
If we write out the terms of the sequence, we have

$$-1, \quad 1, \quad -1, \quad 1, \quad -1, \ldots .$$

The terms of this sequence alternate between 1 and -1 infinitely often. Therefore, a_n doesn't approach any number. Thus, $\lim_{n\to\infty}(-1)^n$ does not exist, and so the sequence $\left((-1)^n\right)$ is divergent. □

Example 7.13 (applying L'Hôpital's rule to a related function)

Evaluate $\lim\limits_{n\to\infty} \dfrac{n+1}{e^n}$.

Solution.
This has the indeterminate form $\frac{\infty}{\infty}$. However, there is no obvious way to resolve this, except by L'Hôpital's rule. *But we cannot apply L'Hôpital's rule directly because it applies not to sequences but to functions of a real variable*. So, instead, we consider the limit of the corresponding auxiliary function

of a real variable, to which we can apply L'Hôpital's rule: let

$$f(x) := \frac{x+1}{e^x}$$

and we obtain

$$\lim_{x\to\infty} \frac{x+1}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \ .$$

Thus, by theorem 7.6, we have:

$$\lim_{n\to\infty} \frac{n+1}{e^x} = 0 \ .$$

□

§7.3 Convergence Tests

Theorem 7.14 (Squeeze Theorem for Sequences)
If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example 7.15 (applying Squeeze Theorem to a sequence)

Determine whether $\left(\frac{\cos n}{n^2}\right)_{n=1}^{\infty}$ is convergent or divergent.

Solution.
Recall that $-1 \leqslant \cos n \leqslant 1$ for all n. Dividing by n^2, we get

$$\frac{-1}{n^2} \leqslant \frac{\cos n}{n^2} \leqslant \frac{1}{n^2} \ .$$

But notice that $\lim_{n\to\infty} \frac{-1}{n^2} = 0 = \lim_{n\to\infty} \frac{1}{n^2}$. So by the Squeeze Theorem, we have

$$\lim_{n\to\infty} \frac{\cos n}{n^2} = 0 \ .$$

□

The following result follows immediately from the Squeeze Theorem.

Corollary 7.16
If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 7.17

Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$, if it exists.

Solution.
Notice that since $(-1)^n$ alternates between -1 and 1, we cannot compute the limit directly. But we see that

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0 \ .$$

Thus, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$. □

Example 7.18

Investigate the convergence of $\left(\frac{n!}{n^n}\right)_{n=1}^{\infty}$.

Solution.
Notice that we have an indeterminate form $\frac{\infty}{\infty}$. Here we have no corresponding function to use L'Hôpital's rule with ("$x!$" is not defined when x is not an integer!). Notice that the general term of this sequence satisfies

$$a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{n\cdot n\cdot n\cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2}{n}\cdot\frac{3}{n}\cdot \ldots \cdot \frac{n}{n}\right) \leqslant \frac{1}{n}\cdot 1 = \frac{1}{n}.$$

So $0 \leqslant a_n \leqslant \frac{1}{n}$. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$, so by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0,$$

thus the sequence $\left(\frac{n!}{n^n}\right)_{n=1}^{\infty}$ is convergent. □

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

Theorem 7.19
If $\lim_{n\to\infty} a_n = L$ and f is a continous function at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example 7.20

Because the function $f(x) = 2^x$ is continuous, we have

$$\lim_{n\to\infty} 2^{1/n} = 2^{\lim_{n\to\infty} \frac{1}{n}} = 2^0 = 1.$$

Example 7.21

The sequence (r^n) converges for $-1 < r \leqslant 1$ and

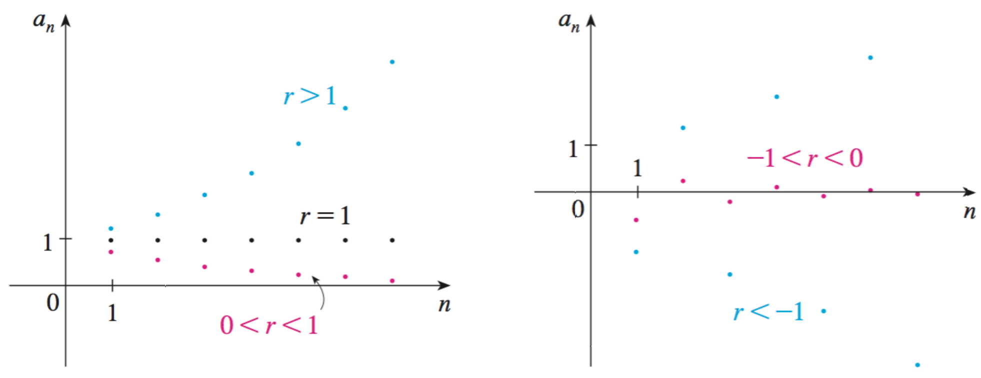
$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1, \\ 1 & \text{if } r = 1. \end{cases}$$

Solution.
It is obvious that $\lim_{n\to\infty} 1^n = 1$ since $\{1^n\} = \{1, 1, \ldots\}$ is the constant sequence.

If $-1 < r < 1$, then $|r| < 1$ so $\lim_{n\to\infty} |r|^n = 0$, because $\{|r|^n\}$ is a sequence of increasing powers of a positive number less than 1.

If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$.

If $r < -1$, the sequence $\{r^n\}$ diverges.



□

Definition 7.22 (Increasing, Decreasing, Monotone Sequences)
A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$.

A sequence $\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$.

A sequence is **monotonic** if it is either decreasing or increasing.

Example 7.23

Show that the sequence $\{a_n\} = \left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ is increasing.

Solution.
We must show that $a_n < a_{n+1}$ for all $n \geqslant 1$. To do this, we look at the ratio

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{(n+1)(n+1)}{n(n+2)} = \frac{n^2+2n+1}{n^2+2n} = 1 + \frac{1}{n^2+2n} > 1 .$$

Since $a_n > 0$, we can multiply both sides by a_n to get $a_{n+1} > a_n$. □

Alternatively, we can consider the function $f(x) = \frac{x}{x+1}$; notice that $\{f(n)\} = \{a_n\}$. Observe that $f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$. So f is increasing on $(1, \infty)$ and so $f(n+1) > f(n)$. Hence, $\{a_n\}$ is increasing.

Definition 7.24 (Bounded Sequences)
A sequence $\{a_n\}$ is

- **bounded above** if there is a number M such that $a_n \leqslant M$ for all n.
- **bounded below** if there is a number m such that $m \leqslant a_n$ for all n.

If $\{a_n\}$ is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Example 7.25

- The sequence $a_n = n$ is bounded below (because $a_n > 0$) but not above.
- The sequence $a_n = \dfrac{n}{n+1}$ is bounded since $0 < a_n \leqslant 1$ for all n.

We know that not every bounded sequence is convergent. For instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leqslant a_n \leqslant 1$, but it's divergent. And, of course, not every monotonic sequence is convergent; for instance, $a_n = n$. Thus, neither boundedness nor monotonicity *alone* force convergence. However, the following very powerful tool says that *together* these properties are sufficient to force convergence.

Theorem 7.26 (Monotone Sequence Theorem)
Every bounded, monotonic sequence is convergent.

Example 7.27 (an indirect proof of convergence)

Investigate the convergence of the sequence $a_n = \frac{2^n}{n!}$.

Solution.
We don't know how to compute the limit $\lim\limits_{n\to\infty} \frac{2^n}{n!}$ directly, and we cannot use L'Hôpital's rule (why not?). We first show that the sequence is monotonic:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{n!2^{n+1}}{(n+1)!2^n} = \frac{n!}{(n+1)!}2 = \frac{n!}{(n+1)n!}2 = \frac{2}{n+1} \leqslant 1$$

for all $n \geqslant 1$. So $\{a_n\}$ is decreasing. At the same time, $a_n < a_1 \quad \Rightarrow \quad 0 < a_n \leqslant 2$ for all $n \geqslant 1$. So $\{a_n\}$ is also bounded. Thus, by Monotone Sequence Theorem, the sequence $\{a_n\}$ is convergent. □

§7.4 Series

We usually say that the decimal expansion of number such as $\frac{1}{3}$ is as a repeated *infinite decimal*

$$\frac{1}{3} = 0.3333\ldots \quad ,$$

where the 3's in this expansion go on forever. But what does this mean? To make sense of this, we can rewrite it as

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots \quad ,$$

where the ellipsis "…" is meant to indicate that the presented pattern repeats indefinitely. Thus, we have presented the number $\frac{1}{3}$ as an *infinite sum*; i.e., as a sum of infinitely many terms. In this section, we formalise this notion.

Definition 7.28 (Infinite Series)
Given a sequence $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n := a_1 + a_2 + \ldots + a_n + \ldots$$

is called an **infinite series** (or simply **series**).

Shortly, we will properly define the notion of the *sum of a series*. The difficulty lies in the fact that — whilst adding together *two* numbers involves a finite number operations — an *infinite sum* involves an infinite number of operations, and therefore cannot be computed directly. In other words, to compute an infinite sum, you cannot simply start adding term after term, because you will never be able to end this process. For now let's continue working with a couple of motivating examples, keeping in mind the 'intuitive' notion of what the sum of a series is.

Many series do not have a finite sum, such as, for instance, the series

$$\sum_{n=1}^{\infty} n = 1 + 2 + \ldots + n + \ldots \quad .$$

Intuitively, it's clear why this is so: the terms in this infinite sum become larger and larger, and we add more and more of them. A more precise way to say this is to observe that if we add only the first N terms,

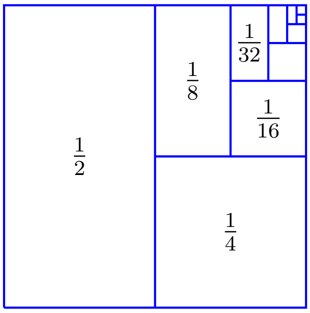
$$\sum_{n=1}^{N} n = 1 + 2 + \ldots + N$$

(this is a sum of *finitely* many terms, which is something we really can calculate), then we get $\frac{N(N+1)}{2}$ (cf. Appendix A), which is increasing as N increases. That is, the *partial sum* $\frac{N(N+1)}{2}$ tends to ∞ in the limit as $N \to \infty$.

The technique of computing *partial sums* of a series, deducing the pattern, and exhibiting their limits is a very powerful one. As a demonstration, consider the following infinite series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots \quad .$$

First, it is easy to see geometrically that this series sums to a finite number. Each term of this infinite series can be represented as the area of rectangles, which are repeatedly halved in size and stacked together, as shown on the right. This pattern of rectangles fill in the entire unit square; as a result the areas of the small rectangles add together to give the area of the unit square, which is 1. We will now make a detailed analysis of this infinite series to deduce the same result, corroborating our geometric intuition.



We start by computing *partial sums*:

$$\begin{aligned} s_1 &= \frac{1}{2} = 1 - \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4} \\ s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8} \\ s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 1 - \frac{1}{16} \\ &\vdots \\ s_n &= \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} \end{aligned}$$

From this pattern, we can see that adding more and more terms of our infinite series, the results gets closer and closer to 1. To make this precise, consider the sequence $\{s_n\} = \{1 - \frac{1}{2^n}\}$, and take its limit:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 .$$

Thus, it is *in this appropriate sense* that we can conclude that our infinite series sums to 1. It is very important to understand very clearly what is going here. The new mathematical object "$\sum \frac{1}{2^n}$" is called a series; it is *not* a sum in the usual sense, because addition is a finitary operation; instead, it is the *limit* of an appropriate sequence — the sequence being the sequence of partial sums.

Definition 7.29 (Sum of a Series)

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$, its ***n*-th partial sum** is the sum

$$s_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n .$$

The sequence $\{s_n\}$ is referred to as the **sequence of partial sums**.

If the sequence $\{s_n\}$ is convergent with limit s, then the series $\sum a_n$ is called **convergent**, and we write

$$\sum_{n=1}^{\infty} a_n = s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k .$$

The number s is called the **sum** of the series.

If the sequence $\{s_n\}$ is divergent, then the series $\sum a_n$ is **divergent**.

Example 7.30

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (we considered it in the introduction). It converges, because its n-th partial sum is $s_n = 1 - \frac{1}{2^n}$, which is a convergent sequence, and so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 \ .$$

Example 7.31

The series $\sum_{k=0}^{\infty} (-1)^k$ diverges.

Solution.
Note that

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 - 1 = 0 \\ s_3 &= 1 - 1 + 1 = 1 \\ s_4 &= 1 - 1 + 1 - 1 = 0 \\ &\vdots \\ s_n &= \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Thus, the sequence of partial sums $\{s_n\}$ is divergent (as we saw in example 7.12), and so the given series is divergent. □

Example 7.32 (geometric series)

One of the most important examples of a series is the **geometric series**, which is the series

$$\boxed{\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \ldots + r^n + \ldots} \ ,$$

where r is a fixed real number (which is often called the **ratio** of the geometric series). (Notice also that the summation starts at $n = 0$, and *not* at $n = 1$.) Actually, we have already examined a geometric series: in example 7.30, in a way, we dealt with the special case $r = \frac{1}{2}$; we will comment on this in more detail after this example.

To analyse the geometric series, we will use a clever trick, which only works when $r \neq 1$; so let's first study the case $r = 1$ separately. If $r = 1$, then the n-th partial sum

$$s_n = \underbrace{1 + \ldots + 1}_{n \text{ times}} = n \xrightarrow[n\to\infty]{} \infty \ ,$$

and so in this case the geometric series diverges.

Now, if $r \neq 1$, we can use the following clever observation. Take the n-th partial sum

$$s_n = 1 + r + r^2 + \ldots + r^{n-1} \ ,$$

and multiply both sides by r:

$$rs_n = r + r^2 + r^3 + \ldots + r^n \ .$$

Now, subtract these two equations from each other:

$$\begin{aligned} s_n - rs_n &= 1 - r^n \\ (1-r)s_n &= 1 - r^n \ . \end{aligned}$$

Finally, dividing both sides by $(1-r)$ (this is why this trick works only when $r \neq 1$), we obtain an explicit formula for the n-th partial sum of the geometric series:

$$s_n = \frac{1-r^n}{1-r} \ .$$

Convergence properties of this sequence of partial sums depends on whether the magnitude of r is less or greater than 1:

- If $|r| < 1$ (i.e., if $-1 < r < 1$), then $r^n \xrightarrow[n\to\infty]{} 0$, so

  $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{(1-r^n)}{1-r} = \frac{1}{1-r} \ .$$

  Thus, in this case, the geometric series is convergent.
- If $|r| > 1$ (i.e., if $r \leqslant -1$ or $r > 1$), then the sequence $\{r^n\}$ is divergent, so the limit $\lim\limits_{n\to\infty} s_n$ does not exist. Therefore, the geometric series in divergent in this case.

More generally, if a is a non-zero real number, then

$$\begin{aligned} \sum_{n=0}^{\infty} ar^n &= a + ar + ar^2 + ar^3 + \ldots \\ &= a(1 + r + r^2 + r^3 + \ldots) \\ &= a\sum_{n=0}^{\infty} r^n \ . \end{aligned}$$

(Being able to factor out a from an infinite series requires justification.)

Theorem 7.33 (Convergence Properties of Geometric Series)
The geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

if $|r| \geqslant 1$*, the geometric series diverges.*

Example 7.34 (geometric series | $r = \frac{1}{2}$ and $a = \frac{1}{2}$)

Example 7.30, in which we studied the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is an example of a geometric series with $r = \frac{1}{2}$ and $a = \frac{1}{2}$. This is because

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2^n} &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots \\ &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{1}{2^n} \quad .\end{aligned}$$

By theorem 7.33,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \quad .$$

This result coincides with our findings in example 7.30.

Often, we might be given a series which is not immediately obviously a geometric series. We need to train our eyes to recognise when we are dealing with a geometric series in disguise. The following three examples demonstrate the thought process that goes into it.

Example 7.35 (geometric series | recognising a geometric series)

Investigate convergence properties of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^n}$.

Solution.
Notice that $\frac{(-1)^n}{\pi^n} = \left(\frac{-1}{\pi}\right)^n$, so this is in fact a geometric series with $a = 1$ and $r = \frac{-1}{\pi}$.
Since $\pi > 1$, it follows that $\left|\frac{-1}{\pi}\right| < 1$. Therefore, by theorem 7.33, this series converges, and moreover

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^n} = \frac{1}{1 - \left(\frac{-1}{\pi}\right)} = \frac{\pi}{\pi + 1} \quad . \qquad \square$$

Example 7.36 (geometric series | recognising a geometric series)

Investigate convergence properties of the series $\sum_{n=0}^{\infty}(-1)^n\, 3^{2n}\, 2^{-n}$.

Solution.
To recognise a geometric series here, we need to write the n^{th} term of the series in the form ar^n. Do so, it's best to try to bring every factor to the same power n:

$$a_n = (-1)^n\, 3^{2n}\, 2^{-n} = (-1)^n\, 9^n\, \frac{1}{2^n} = \left(\frac{-9}{2}\right)^n .$$

Therefore,

$$\sum_{n=0}^{\infty}(-1)^n\, 3^{2n}\, 2^{-n} = \sum_{n=0}^{\infty}\left(\frac{-9}{2}\right)^n ,$$

which is a geometric series with $a = 1$ and $r = \frac{-9}{2}$. Since, $|r| > 1$, the series is divergent. □

Example 7.37 (geometric series | recognising a geometric series)

Calculate the sum of the series $\sum_{n=2}^{\infty} 2\left(\frac{1}{5}\right)^n$

Solution.
Notice that we can rewrite the series as follows

$$\begin{aligned}
\sum_{n=2}^{\infty} 2\left(\frac{1}{5}\right)^n &= \frac{2}{5^2} + \frac{2}{5^3} + \frac{2}{5^4} + \frac{2}{5^5} + \cdots \\
&= \frac{2}{5^2} + \frac{2}{5^2}\cdot\frac{1}{5} + \frac{2}{5^2}\cdot\frac{1}{5^2} + \frac{2}{5^2}\cdot\frac{1}{5^3} + \cdots \\
&= \sum_{n=2}^{\infty}\frac{2}{5^2}\left(\frac{1}{5}\right)^{n-2} \\
&= \sum_{k=0}^{\infty}\frac{2}{5^2}\left(\frac{1}{5}\right)^{k} && (k = n-2)
\end{aligned}$$

Note that this is a geometric series with $a = \frac{2}{5^2}$ and $r = \frac{1}{5}$. Since $|r| < 1$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} 2\left(\frac{1}{5}\right)^n &= \sum_{k=0}^{\infty}\frac{2}{5^2}\left(\frac{1}{5}\right)^{k} \\
&= \frac{\frac{2}{5^2}}{1-\frac{1}{5}} \\
&= \frac{1}{10}
\end{aligned}$$

□

Of course, not all series are geometric series in disguise, and we need to go back to the definition of the sum of a series as a limit of partial sums in order to make the conclusion.

Example 7.38 (Telescoping Series)

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution.
This is not a geometric series so we need to go back to the definition and compute the partial sums. The trick here is to realise that using partial fractions we get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So the nth partial sum is given by

$$\begin{aligned} \mathsf{S}_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \\ &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Notice that almost every term in the sum is cancelled by another term (the next term). Because of all the cancellations, the sum collapses (like a pirate's telescope) into just two terms. For this reason, such sums are called telescoping sums.

Now that we have a nice looking formula for the nth partial sum S_n, we can compute the limit

$$\lim_{n\to\infty} \mathsf{S}_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

□

Since the convergence of a series is defined in terms of limits of the sequence of partial sums, the limit laws for sequences give us the following (unsurprising) result.

Theorem 7.39 (Arithmetic Properties of Convergent Series)
If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ and $\sum(a_n \pm b_n)$ and

- $\sum ca_n = c \sum a_n$
- $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$

Example 7.40

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{1}{2^n} \right)$

Solution.
We already know that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Therefore, by the previous theorem we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} + \frac{1}{2^n} \right) &= 4 \sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= (4) \cdot (1) + 1 \\ &= 5 \end{aligned}$$

□

Remark 7.41

A finite number of terms doesn't affect the convergence of a series. For instance, suppose that you are able to show that

$$\sum_{n=4}^{\infty} \frac{n}{n^3+1} \text{ is convergent}$$

Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3+27} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3+27},$$

it follows that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+27}$ is also convergent.

Theorem 7.42
If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$

Remark 7.43

The converse of this theorem is not true. If $\lim_{n\to\infty} a_n = 0$, we CANNOT conclude that $\sum a_n$ is convergent. Be very clear about this point. This is a very common misconception.

Next section we will see that $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

The following very useful test follows directly from the previous theorem.

Theorem 7.44 (Test for Divergence)

If $\lim_{n\to\infty} a_n$ *does not exist or* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example 7.45 (A Divergent Series)

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+n+1}$ diverges.

Solution.
Notice that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n^2+n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{2} \neq 0$$

So the series diverges by the Test for Divergence.

□

Remark 7.46

If we find that $\lim_{n\to\infty} a_n = 0$, we know nothing about the convergence or divergence of the series $\sum a_n$. The series might converge or it might diverge.

For instance, next section we will see that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

§7.5
The Integral Test

Usually it is very difficult to find the exact sum of a convergent series. We were able to do this for geometric and telescoping series because we were able to find a simple formula for the nth partial sum, S_n. But this is not usually the case, more often than not we cannot determine whether a series is convergent or divergent by simply taking the limit of the sequence of partial sums. For most series we will need to use some indirect method in order to determine whether they are convergent or not.

Example 7.47 (Harmonic Series)

The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

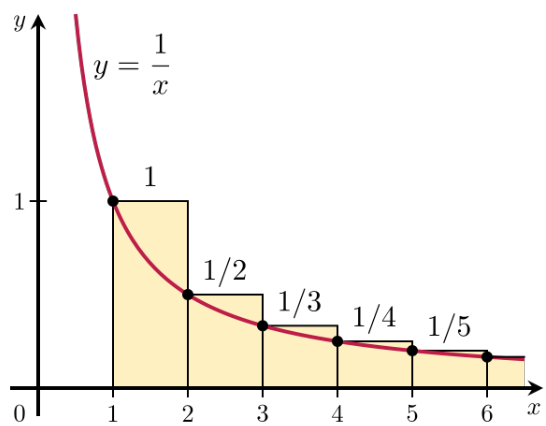
We begin by investigating the convergence or divergence of the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$.

We first notice that $\lim_{n\to\infty} \frac{1}{n} = 0$ so we can only conclude that further study is needed (because the series might converge or it might diverge).

The following geometric argument will help us show that the harmonic series is divergent. Consider the nth partial sum

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

The figure shows the curve $y = \frac{1}{x}$ and rectangles whose tops lie above the curve. Note that the base of each rectangle is an interval of length 1 and the height is equal to the value of the function $f(x) = \frac{1}{x}$ at the left endpoint of the interval.



Note that S_n corresponds to the sum of the areas of these n rectangles. Therefore, since the top of each rectangle lies above the curve $y = \frac{1}{x}$, we have

$$\begin{aligned} \mathsf{S}_n &= \text{sum of area of } n \text{ rectangle} \\ &\geqslant \text{area under the curve} \\ &= \int_1^{n+1} \frac{1}{x}\mathrm{d}x \end{aligned}$$

On the other hand, taking limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \int_1^{n+1} \frac{1}{x}\mathrm{d}x = \int_1^{\infty} \frac{1}{x}\mathrm{d}x$$

and we know that this improper integral is divergent (We know that the area under the curve $y = \frac{1}{x}$, $x \geqslant 1$ is infinite).

Since $\mathsf{S}_n \geqslant \int_1^{n+1} \frac{1}{x}\mathrm{d}x$ for all $n \geqslant 1$, we must also have that

$$\lim_{n\to\infty} \mathsf{S}_n = \infty$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

The following example shows how to use a similar idea to show that a series is convergent.

Example 7.48

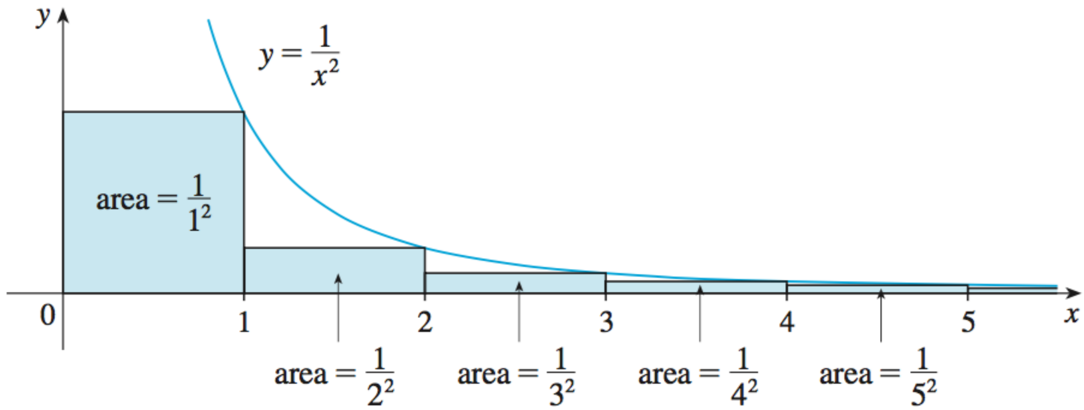
The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

We will use a very similar geometric argument but this time we will consider rectangles that lie below the curve $y = \frac{1}{x^2}$. The base of each has length 1 and the height is equal to the value of the function $f(x) = \frac{1}{x^2}$ at the right endpoint of the interval.

Notice that

$$\mathsf{S}_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

Excluding the first rectangle, the sum of the areas of the remaining $(n-1)$ rectangles is less than the area under the curve $y = \frac{1}{x^2}$ from 1 to n.



So the sum of the areas of the rectangles is

$$
\begin{aligned}
S_n &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \\
&< 1 + \int_1^n \frac{1}{x^2}\mathrm{d}x \\
&< 1 + \int_1^\infty \frac{1}{x^2}\mathrm{d}x && \int_1^n \tfrac{1}{x^2}\mathrm{d}x < \int_1^\infty \tfrac{1}{x^2}\mathrm{d}x \\
&= 1 + 1 && \int_1^\infty \tfrac{1}{x^2}\mathrm{d}x = 1 \\
&= 2
\end{aligned}
$$

Hence, the partial sums are bounded. Also, since all the terms in the series are positive, we have that the sequence of partial sums is increasing. Therefore, the sequence of partial sums converges by the Monotonic Sequence Theorem. So the series converges and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leqslant 2$$

However, we cannot conclude that the exact value of the series is 2. In fact, the exact value of this sum was found by Euler to be $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

The same sort of geometric argument can be used to prove the following theorem.

Theorem 7.49 (The Integral Test)
Suppose that f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ converge} \Leftrightarrow \int_1^\infty f(x)\mathrm{d}x \text{ converges.}$$

In other words, the series $\sum a_n$ and the improper integral $\int_1^\infty f(x)\mathrm{d}x$ are both convergent or both divergent.

Remark 7.50

When we use the Integral Test, it is not necessary to start the series at $n = 1$. For example, testing the series

$$\sum_{n=3}^{\infty} \frac{1}{(n-2)^5} \text{ we use } \int_3^{\infty} \frac{1}{(x-2)^5} \mathrm{d}x$$

Also, it is not necessary for f to be always decreasing, What really matters is that f is decreasing for all $x \geqslant N$ for some N. Then we can prove that $\sum_{n=N}^{\infty} a_n$ is convergent, but adding a finite number of terms does not affect convergence so $\sum_{n=1}^{\infty} a_n$ would also be convergent.

Remark 7.51

The Integral Test does not gives us the exact value of the sum. As we saw before,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{but} \quad \int_1^{\infty} \frac{1}{x^2} \mathrm{d}x = 1$$

Therefore, in general

$$\sum a_n \neq \int f(x) \mathrm{d}x$$

Example 7.52 (p-series)

For what values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Solution.

- If $p < 0$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 1$.

Since $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, the series diverges by the Test for Divergence.

- If $p > 0$, then the function $f(x) = \frac{1}{x^p}$ is continuous, positive and decreasing on $[1, \infty)$ so we can apply the Integral Test.

Recall that

$$\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x \text{ converges if } p > 1 \text{ and diverges if } p \leqslant 1$$

so by the integral test, we have that

$$\text{the } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

□

Remark 7.53

The p-series with $p = 1$ is the harmonic series. The p-series test shows that the harmonic series is just barely divergent; if we increase p just by a little, the series converges. It is quite impressive how slowly the partial sums of the harmonic series approach infinity. For instance, after $2.5 \cdot 10^8$ (250 million!) terms the sum is still less than 20.

Example 7.54

Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

Consider the function $f(x) = \dfrac{1}{x(\ln x)^2}$. Notice that f is not defined at $x = 1$ but this is not a problem because our series starts at $n = 2$. It is easy to check that the function f is continuous and $f(x) > 0$ for all $x \in [2, \infty)$. So we now just need to check that f is decreasing. To do that, we compute its derivative:

$$f'(x) = -\frac{(\ln x)^2 + x\frac{2\ln x}{x}}{x^2(\ln x)^4} = -\frac{\ln x + 2}{x^2(\ln x)^3}$$

Note that if $x > 1$, then $f'(x) < 0$. So we can apply the Integral Test:

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^2}\mathrm{d}x &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^2}\mathrm{d}x \\
&= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2}\mathrm{d}u && \left(\text{Let } u = \ln x,\ \mathrm{d}u = \tfrac{\mathrm{d}x}{x}\right) \\
&= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln t} \\
&= \lim_{t\to\infty} \left[\frac{1}{\ln 2} - \frac{1}{\ln t}\right] \\
&= \frac{1}{\ln 2}
\end{aligned}$$

We again emphasise that this doesn't mean that the sum of the series is $\frac{1}{\ln 2}$. The series converges but we don't know its exact value.

§7.6 The Comparison Tests

In the Comparison Test, the idea is that we can test the convergence of many series by comparing their terms to those of a series which we already know is convergent or divergent. The first of the two comparison test that we will see is very similar to the comparison test we have already seen for integrals.

Theorem 7.55 (Comparison Test)
Suppose that $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$ for some positive number N.

(1) *If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.*

(2) *$\sum a_n$ is divergent, then $\sum b_n$ is divergent.*

In other words, this theorem tells us the following. If the larger series converges, it acts as a wall and it forces the smaller series to converge. Similarly, if the smaller series diverges, it forces the larger one to diverge.

Example 7.56 (Comparison Test | Use)

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3+n}$ converges or diverges.

Solution.
Notice that for large values of n the dominant term in the denominator is $2n^3$ (For instance, when n is one hundred, $2n^3$ is two million which is definitely much larger that n). This suggests we should compare the given series with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3}$. Observe that $2n^3+n \geqslant 2n^3$. So we conclude that

$$\frac{1}{2n^3+n} < \frac{1}{2n^3} \text{ for all } n \geqslant 1$$

Also, we know that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent because it is a constant times a p-series with $p = 3 > 1$.

Hence, by the Comparison Test, the given series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3+n}$ is convergent .

□

Example 7.57 (Comparison Test | Use)

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ converges or diverges

Solution.
Notice that

$$\frac{4^n}{3^n-1} \geqslant \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$$

Since the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$ diverges (geometric series with $|r|>1$), by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{3^n-1}$ also diverges.

□

Remark 7.58

To apply the Comparison Test the terms of the series being tested must be either smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than the terms of a divergent series, then the Comparison Test doesn't apply.

Example 7.59 (Comparison Test | Wrong Inequality)

Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^3-n}$. Notice that this series looks very similar to the one in example 7.56. The only difference is that instead of a $+$ sign, we have a $-$ sign in the denominator.

We obviously have the inequality

$$\frac{1}{2n^3-n} > \frac{1}{2n^3} \text{ for all } n \geqslant 1$$

The problem is that this inequality goes the wrong way. We know that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^3}$ is convergent, so this inequality tells that the terms of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^3-n}$ are larger than those of a convergent series. Hence, we cannot use the Comparison Test to draw any conclusions.

However, it is reasonable to expect that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^3-n}$ converges since for large n its terms are very similarly to those of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^3}$ which we know is convergent.

As in the previous example, there are many series whose terms look very similar to those of a series we already know converges or diverges but we cannot apply the Comparison Test. For those cases, we can use the following test.

Theorem 7.60 (Limit Comparison Test)

Suppose that $a_n, b_n > 0$ for all n and $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = L$ for some $0 < L < \infty$. Then, either $\sum a_n$ and $\sum b_n$ both converge or both diverge.

We can now use the Limit Comparison Test to confirm that the series in example 7.59 is indeed convergent.

Example 7.61 (Limit Comparison Test)

Show that the series $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n}$ is convergent.

Solution.
In example 7.59, we have already observed that for large n, the general term $a_n = \frac{1}{2n^3-n}$ is very similar to $b_n = \frac{1}{2n^3}$. So we use Limit Comparison Test with

$$a_n = \frac{1}{2n^3 - n} \quad \text{and} \quad b_n = \frac{1}{2n^3}$$

and obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2n^3-n}}{\frac{1}{2n^3}} \\ &= \lim_{n\to\infty} \frac{2n^3}{2n^3 - n} \\ &= - \lim_{n\to\infty} \frac{2}{2 - \frac{1}{n^2}} \\ &= 1 > 0 \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n^3}$ is convergent (p-series with $p = 3$), the Limit Comparison Test says that $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n}$ is also convergent.

□

The Limit Comparison Test is very useful as it allows us to compare two series without having to compare them term by term. For instance, in example 7.60, we were able to compare the series $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n}$ with the series $\sum_{n=1}^{\infty} \frac{1}{2n^3}$ even though the inequality $\frac{1}{2n^3 - n} \leqslant \frac{1}{2n^3}$ is false for all $n > 1$.

Example 7.62 (Limit Comparison Test)

Determine whether the series $\sum_{n=1}^{\infty} \frac{3n^3 + n}{\sqrt{n^7 - n^3 + 3}}$ converges or diverges.

Solution.
Note that the dominant term in the numerator is $3n^3$ and the dominant term in the denominator is $\sqrt{n^7} = n^{7/2}$. With this in mind, we take

$$a_n = \frac{3n^3 + n}{\sqrt{n^7 - n^3 + 3}} \quad \text{and} \quad b_n = \frac{3n^3}{n^{7/2}} = \frac{3}{n^{1/2}}$$

So

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{3n^3+n}{\sqrt{n^7-n^3+3}}}{\frac{3}{n^{1/2}}}\\
&= \lim_{n\to\infty}\frac{3n^3+n}{\sqrt{n^7-n^3+3}}\cdot\frac{n^{1/2}}{3}\\
&= \lim_{n\to\infty}\frac{3n^{7/2}+n^{3/2}}{3\sqrt{n^7-n^3+3}}\\
&= \lim_{n\to\infty}\frac{3+\frac{1}{n^2}}{3\sqrt{1-\frac{1}{n^4}+\frac{3}{n^7}}}\\
&= 1>0
\end{aligned}$$

Since the series $\sum\frac{1}{n^{1/2}}$ is divergent (p-series wit $p=1/2<1$), the series $\sum_{n=1}^{\infty}\frac{3n^3+n}{\sqrt{n^7-n^3+3}}$ also diverges by the Limit Comparison Test.

□

§7.7 Alternating Series

The convergence tests we looked at in the previous two sections applied only to series whose terms are all positive. Of course, there are many series that have negative terms in them so we need to start looking at tests for these series. We begin by examining alternating series, that is series whose terms alternate sign (alternate back and forth from positive to negative).

Definition 7.63 (Alternating Series)
An **alternating series** is a series whose terms are alternately positive and negative.

Some examples of alternating series are

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$$-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+2}$$

In general, the nth term of an alternating series is of the form

$$a_n=(-1)^{n-1}b_n \quad \text{or } a_n=(-1)^n b_n$$

where $b_n>0$ ($b_n=|a_n|$)

The following test gives a us a criterion for determining whether an alternating series is convergent.

Theorem 7.64 (Alternating Series Test)

If the alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$, *with* $b_n > 0$ *for all* n, *satisfies*

(i) $\lim_{n\to\infty} b_n = 0$

(ii) $b_{n+1} \leqslant b_n$ *for all* $n \geqslant N$ *for some positive integer* N *that is, the sequence* $\{b_n\}$ *is eventually decreasing.*

Then the alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ *converges.*

Remark 7.65

Unlike the Integral Test and the Comparison/Limit Comparison Tests, this test only tells us when a series is convergent and not if a series is divergent.

Example 7.66 (Alternating Harmonic Series)

The **alternating harmonic series** $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is an alternating series with $b_n = \frac{1}{n}$ and satisfies

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$

(ii) $b_{n+1} \leqslant b_n$ for all $n \geqslant 1$ because $\frac{1}{n+1} < \frac{1}{n}$

Therefore, by the Alternating Series Test, the alternating harmonic series is convergent.

Example 7.67 (Alternating Series | Alternating Series Test doesn't apply)

Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{2n+5}$ converges or diverges.

Solution.
This is an alternating series with $b_n = \frac{n}{2n+5}$. However, we have that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{2n+5} = \lim_{n\to\infty}\frac{1}{2+\frac{5}{n}} = \frac{1}{2} \neq 0.$$

Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n\frac{n}{2n+5} \text{ does not exist.}$$

Hence, the given series diverges by the Test for Divergence. □

Remark 7.68

To apply the Alternating Series Test we only need the sequence $\{b_n\}$ to be eventually decreasing. That is, it is enough to have $b_{n+1} < b_n$ for all $n \geqslant N$ for some positive integer N.

Example 7.69 (Alternating Series Test | Use)

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+4}$ for convergence or divergence.

Solution.

Notice that this is an alternating series with $b_n = \dfrac{\sqrt{n}}{n+4}$. We need to check that the series satisfies the hypotheses of the Alternating Series Test.

(i) It is easy to check that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} + \frac{4}{\sqrt{n}} = 0$$

(ii) The other condition requires some work. Unlike in example 7.66, it is not obvious that the sequence $b_n = \frac{\sqrt{n}}{n+4}$ is decreasing. So we consider the related function $f(x) = \dfrac{\sqrt{x}}{x+4}$ and its derivative

$$f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}$$

Notice that $f'(x) \leqslant 0$ if $x \geqslant 4$. Thus f is decreasing on $[4, \infty)$. Since $b_n = f(n)$, we have that $b_{n+1} < b_n$ for all $n \geqslant 4$. So the sequence $\{b_n\}$ is eventually decreasing.

Since all the conditions of the Alternating Series Test are met, the given series converges.

□

Alternating series can sometimes come in different forms.

Example 7.70

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ converges or diverges.

Solution.

Notice that $\cos(n\pi) = (-1)^n$. Therefore, the terms of the series actually are $a_n = \dfrac{(-1)^n}{n}$. Since the alternating harmonic series is convergent, we have that

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is also convergent.

□

§7.8 Absolute Convergence

So far we have convergence test for series with positive terms and alternating series. But what can we do to study the convergence of a series that has both positive and negative terms but is not alternating? Consider the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

It has both positive and negative terms but is clearly not alternating (the first term is positive, the next three are negative, and the following three are positive: the sign changes irregularly). In many cases we will able to solve this issue by studying the series whose terms are the absolute values of the terms of the original series.

Definition 7.71 (Absolute Convergence)
A series $\sum a_n$ is called **absolutely convergent** if the series of the absolute values $\sum |a_n|$ is convergent.

Notice that since all terms of $\sum |a_n|$ are positive, we can use our earlier tests for series that applied only to series of positive terms.

Example 7.72 (Absolutely Convergent Series)

The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent series (p-series with $p = 2 > 1$).

Can you think of a series that is convergent but not absolutely convergent? Take a minute to think about this question before you read the next example.

Example 7.73 (Convergent but not absolutely convergent)

We know that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent, but it is not absolute convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n-1}^{\infty}\frac{1}{n}$$

which is the harmonic series (p-series with $p = 1$) and it is therefore divergent.

Was this the example that you had in mind?

Definition 7.74 (Conditional Convergence)
A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

As we just saw, the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is conditionally convergent.

The next theorem tells us that absolute convergence implies convergence.

Theorem 7.75 (Absolute convergence ⇒ convergence)
If a series $\sum a_n$ is absolutely convergent, then it is convergent.

This theorem is a very important tool. Whenever we study the convergence a series, we first check if it is absolutely convergent. If so, then we immediately know that the series is convergent. Notice that testing for absolute convergence is very convenient because we have very good tests for convergence of series of positive terms.

Example 7.76 (Absolute Convergence Test)

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ is convergent or divergent.

Solution.
We have already noticed that this series has positive and negative terms, but it is not alternating. To test for absolute convergence, we consider the series of absolute values

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$$

Notice that

$$\left|\frac{\cos n}{n^2}\right| = \frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2} \quad \text{since} \quad |\cos n| \leqslant 1 \text{ for all } n.$$

On the other hand, we know that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (p-series with $p = 2$). Therefore, by the Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$ is convergent. Hence, the series $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ is absolutely convergent and therefore convergent. □

§7.9 Ratio Test

The Ratio Test is a very useful tool that allows us to determine whether a series is absolutely convergent. Also, this test will be very useful when studying power series which we will see very soon.

Theorem 7.77 (The Ratio Test)

Suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ *(including the possibility of* $L = \infty$*).*

(1) *If* $L < 1$*, then the series* $\sum a_n$ *is* **absolutely convergent** *(and therefore, convergent).*

(2) *If* $L > 1$ *(including the possibility of* $L = \infty$*), then the series* $\sum a_n$ *is* **divergent**.

(3) *If* $L = 1$*, then the Ratio Test is* **inconclusive** *(no conclusion can be drawn about the convergence or divergence of the series* $\sum a_n$*).*

Example 7.78 (Ratio Test | inconclusive)

Part (3) of the Ratio Test says that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test gives no information. For example,

- For the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$$

- For the divergent series $\sum_{n=1}^{\infty}\frac{1}{n}$ we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

This shows that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, anything can happen. The series $\sum a_n$ might be convergent or it might be divergent. So in this case, we need to use some other test.

In general, the Ratio Test is inconclusive for any p-series. But this is not a problem since we already how to determine whether p-series are convergent or divergent using the Integral Test.

Example 7.79

Test the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n}$ for absolute convergence.

Solution.
We can use the Ratio Test with $a_n = \frac{(-1)^n n}{5^n}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)}{5^{n+1}}}{\frac{(-1)^n n}{5^n}}\right| = \frac{5^n(n+1)}{5^{n+1}n} = \frac{1}{5}\frac{(n+1)}{n}$$

Taking limits we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{5}\frac{(n+1)}{n} = \frac{1}{5} < 1$$

Therefore, by the Ratio Test, the given series is convergent. □

As the next example shows, the Ratio Test turns out to be very useful when the general term of a series contains factorials or exponential terms.

Example 7.80

Test the series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$ for convergence.

Solution.
We use the Ratio Test

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)!}{3^n}}{\frac{(-1)^n n!}{3^n}}\right| \\ &= \lim_{n\to\infty}\frac{3^n(n+1)!}{3^{n+1}n!} \\ &= \lim_{n\to\infty}\frac{(n+1)\cancel{n!}}{3\cancel{n!}} \\ &= \lim_{n\to\infty}\frac{n+1}{3} = \infty\end{aligned}$$

Therefore, the given series diverges by the Ratio Test.

□

Example 7.81

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(2n+1)!}{5^n(n!)^2}$.

Solution.

Since the terms $a_n = \dfrac{(2n+1)!}{5^n (n!)^2}$ are positive, we don't need to write the absolute value signs.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{(2(n+1)+1)!}{5^{n+1}((n+1)!)^2}}{\frac{(2n+1)!}{5^n (n!)^2}} \\
&= \frac{5^n (n!)^2 (2n+3)!}{5^{n+1}((n+1)!)^2 (2n+1)!} \\
&= \frac{1}{5}\frac{\cancel{(n!)^2}(2n+3)(2n+2)\cancel{(2n+1)!}}{(n+1)^2\cancel{(n!)^2}\cancel{(2n+1)!}} \\
&= \frac{1}{5}\frac{(2n+3)(2n+2)}{(n+1)^2} \\
&= \frac{1}{5}\frac{2(2n+3)\cancel{(n+1)}}{(n+1)^{\cancel{2}}} \\
&= \frac{2}{5}\frac{2n+3}{n+1}
\end{aligned}$$

Taking limits we obtain

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{5}\frac{2n+3}{n+1} = \frac{4}{5} < 1$$

Therefore, the given series is convergent by the Ratio Test.

□

Exercise 7.82: Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{2^n (n!)^2}$ convergent?

§7.10 Root Test

The following test is convenient to apply when nth powers occur.

Theorem 7.83 (The Root Test)
Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ *(including the possibility of* $L = \infty$*).*

(1) *If* $L < 1$*, then the series* $\sum a_n$ *is* **absolutely convergent** *(and therefore, convergent)*

(2) *If* $L > 1$ *(including the possibility of* $L = \infty$*), then the series* $\sum a_n$ *is* **divergent***.*

(3) *If* $L = 1$*, then the Root Test is* **inconclusive** *(no conclusion can be drawn about the convergence or divergence of the series* $\sum a_n$*)*

Similarly to what happens in the Ratio Test, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the root test gives no information. The series $\sum a_n$ might converge or it might diverge.

Example 7.84 (Using the Root Test)
Test the convergence of series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+1}\right)^n$.

Solution.
In this case, we use the Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+1}\right)^n} = \lim_{n\to\infty} \frac{2n+3}{3n+1} = \frac{2}{3} < 1$$

Therefore, the given series converges by the Root Test.

□

§7.11 Summary of Convergence Tests for Series.

We have seen several ways of testing a series for convergence or divergence; the problem is to decide which test to use when we are studying the convergence or divergence of a given series. As in the case of integration, there are no specific rules about which test to apply to a given series. The only suggestion we have is that you work through many problems so you will gain intuition and learn how to recognise which tests works best for certain types of series. In order to help you decide which test to use, we summarise all the test for convergence of series we have seen in the following table.

Summary of convergence tests for series.

Test	When to Use	Conclusions
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geqslant 1$.
Divergence Test	All series	If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
Integral Test	• $a_n = f(n)$ • f is continuous, positive and decreasing. • $\int_1^{\infty} f(x)\,dx$ is easy to compute.	$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges for $p > 1$; diverges for $p \leqslant 1$.
Comparison Test	$0 \leqslant a_n \leqslant b_n$	If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges
Limit Comparison Test	$a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \mathsf{L}$ $(0 < \mathsf{L} < \infty)$	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^n b_n,\ b_n > 0$	If • $b_n > 0,\ \forall n$ • $\{b_n\}$ is decreasing • $\lim_{n\to\infty} b_n = 0$ Then $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.
Absolute Convergence	Series with some positive terms and some negative terms	If $\sum_{n=1}^{\infty} \lvert a_n\rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).
Ratio Test	Any series (especially those involving exponentials or factorials)	For $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = \mathsf{L}$ (including $\mathsf{L} = \infty$), • If $\mathsf{L} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges • If $\mathsf{L} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges • If $\mathsf{L} = 1$, then we can draw no conclusion.
Root Test	Especially for series whose terms a_n are of the form $(b_n)^n$.	For $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = \mathsf{L}$ (including $\mathsf{L} = \infty$), • If $\mathsf{L} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges • If $\mathsf{L} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges • If $\mathsf{L} = 1$, then we can draw no conclusion.

In the following examples we don't work out all the details, in most cases we will simply indicate which test you should use.

Example 7.85

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+2}{n^2-2}$ is convergent or divergent.

Solution.
Note that

$$\lim_{n\to\infty}\frac{n^2+2}{n^2-2}=1$$

Therefore, $\displaystyle\lim_{n\to\infty}(-1)^n\frac{n^2+2}{n^2-2}$ doesn't exist.

Hence, the series is divergent by the Test for Divergence. □

Example 7.86

What test should you use to study the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2}{\sqrt[3]{n^7+n^3}}$?

Solution.
Since a_n is an algebraic function of n, we compare the given series with a p-series. We use the Limit Comparison Test. Notice that the dominant term in the numerator is $2n^2$ and the dominant term in the denominator is $\sqrt[3]{n^7}$. So we use the Limit Comparison Test with

$$b_n=\frac{2n^2}{\sqrt[3]{n^7}}=\frac{2}{n^{1/3}}.$$

□

Example 7.87

Study the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(2n)}{1+2^n}$.

Solution.
Notice that $\sin(2n)$ is sometimes positive and sometimes negative. So we use the Absolute Convergence Test. Notice that

$$\left|\frac{\sin(2n)}{1+2^n}\right|\leqslant\frac{1}{1+2^n}\leqslant\frac{1}{2^n}$$

Since the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ is convergent (geometric series with $|r|<1$), by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin(2n)}{1+2^n}\right|$ is also convergent.

Hence, the given series is absolutely convergent, and therefore convergent. □

Example 7.88

What test should you use to study the convergence or divergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$?

Solution.
We can easily check that the function $f(x) = \frac{1}{x\sqrt{\ln x}}$ is continuous, positive and decreasing on $[2, \infty)$. Since $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} \mathrm{d}x$ is easily evaluated, we use the Integral Test. □

Exercise 7.89: Evaluate $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} \mathrm{d}x$.

Example 7.90

What test would you use to study the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$?

Solution.
Since the series involves $n!$, we use the Ratio Test. □

Example 7.91

Show that the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt{\ln n}}$ is conditionally convergent.

Solution.
We first show that the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt{\ln n}}$ is convergent. Notice that this is an alternating series and satisfies

- $\lim_{n\to\infty} \frac{1}{\sqrt{\ln n}} = 0$
- $\frac{1}{\sqrt{\ln(n+1)}} < \frac{1}{\sqrt{\ln n}}$ because $\ln(n) < \ln(n+1)$

Therefore, by the Alternating series Test, the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt{\ln n}}$ is convergent.

We now need to show that the series is not absolutely convergent. We consider the series whose terms are the absolute value of the terms of the given series. That is,

$$\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{\sqrt{\ln n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{\ln n}}$$

We know that $\ln n < n$ for all $n \geqslant 1$. Therefore

$$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{\ln n}}$$

Since the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (p-series with $p = \frac{1}{2} < 1$), by the Comparison Test, we have that the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{\ln n}}$ is also divergent.

□

Remark 7.92

Notice that unlike in example 7.91, in example 7.88 we cannot use the fact that $\ln n < n$ to use the Comparison Test to determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is convergent or divergent. For the series in example 7.88, we have the inequality

$$\frac{1}{n\sqrt{n}} < \frac{1}{n\sqrt{\ln n}}$$

But we know that the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ is convergent (p-series with $p = \frac{3}{2} < 1$). So the inequality goes the wrong way. All we know is that the terms of the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ are larger than those of a convergent series and therefore, the Comparison Test gives us no information.

§8 Power Series

In the previous sections we learned several tests that allowed us to determine whether an infinite series of real numbers is convergent or divergent. Now we will use what we have learned about series of numbers to study sums that look like "infinite polynomials". We call these infinite sums **power series** because they are defined as a series of powers of x.

Definition 8.1 (Power Series)
A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{28}$$

where x is a variable and the c_n's are constants called the coefficients of the series.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \tag{29}$$

is called a **power series in $(x-a)$** or **power series centred at $x=a$**.

Remark 8.2

In equation (29), we adopt the convention $(x-a)^0 = 1$ even when $x=a$.

Note that for each x fixed, a power series is a just series of numbers that can be tested for convergence or divergence using the methods that we learned before. A power series might be convergent for some values of x and divergent for some other values of x. However, we know that every power series is convergent at least at one value of x. Indeed, note that for $x=a$, in equation (29) all terms are zero for $n \geqslant 1$. Therefore, the power series always converges for $x=a$.

Example 8.3

Taking all coefficients to be $c_n = 1$ in equation (28) gives the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

which converges to $\frac{1}{1-x}$ for $|x| < 1$ and diverges for $|x| \geqslant 1$.

Notice that the sum of a power series defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

whose domain is the set of x for which the series converges.

Our main goal in this section will be finding out for what values of x a given series is convergent. The main tool we will use for studying the convergence of power series is the Ratio Test.

Example 8.4 (Power Series | Converges Everywhere)

For what values of x is the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent?

Solution.
We use the Ratio Test. Let $a_n = \dfrac{x^n}{n!}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| \\
&= \lim_{n\to\infty} \left|\frac{n!x^{n+1}}{(n+1)!x^n}\right| \\
&= \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}}|x| \\
&= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \qquad \text{for all } x
\end{aligned}$$

Therefore, by the Ratio Test, the given series is convergent for all x. Later we will see that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

□

Example 8.5 (Power Series | Converges at Only One Point)

For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} n!(x-5)^n$ converge?

Solution.
We again use the Ratio Test. Let $a_n = n!(x-5)^n$. Then

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!(x-5)^{n+1}}{n!(x-5)^n}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}}|x-5| \\
&= \lim_{n\to\infty} (n+1)|x-5| = \begin{cases} 0 & \text{if } x = 5 \\ \infty & \text{if } x \neq 5 \end{cases}
\end{aligned}$$

Hence, by the Ratio Test, the series converges when $x = 5$ and diverges for when $x \neq 5$.

□

Example 8.6 (Power Series Converges | Interval of Convergence)

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n} x^n$ convergent ?

Solution.
Let $a_n = \frac{(-1)^n}{3^n n} x^n$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{3^{n+1}(n+1)}}{\frac{(-1)^n x^n}{3^n n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{3^n n x^{n+1}}{3^{n+1}(n+1)x^n}\right| \\
&= \lim_{n\to\infty}\frac{1}{3}\frac{n}{n+1}|x| \\
&= \frac{1}{3}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x| \\
&= \frac{1}{3}|x|
\end{aligned}$$

Hence, by the Ratio Test, we know that the given series converges absolutely if $\frac{|x|}{3} < 1$ and diverges if $\frac{|x|}{3} > 1$. Therefore, we know that the series converges when $-3 < x < 3$ and diverges when $x < -3$ or $x > 3$. Since the Ratio Test give no conclusion when $|x| = 3$, we must test the endpoints ($x = \pm 3$) separately.

For $x = -3$, the series becomes

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n}(-3)^n &= \sum_{n=1}^{\infty}\frac{(-1)^n}{\cancel{3^n} n}(-1)^n \cancel{3^n} \\
&= \sum_{n=1}^{\infty}\frac{1}{n} \qquad (-1)^n(-1)^n = (-1)^{2n} = 1
\end{aligned}$$

This is the harmonic series which we know is divergent.

For $x = 3$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\cancel{3^n} n}\cancel{3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

which is -1 times the alternating harmonic series, which we know converges (see example 7.66).

Therefore, the given power series converges for $-3 < x \leqslant 3$.

□

In the previous examples we saw how a power series might converge. In example 8.4 the power series was convergent everywhere. In example 8.5, the power series was convergent at only one point. Finally, in example 8.6, the power series was convergent on a finite interval. The following theorem says that these are the only three possibilities.

Theorem 8.7 (Convergence Theorem for Power Series)

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$*, there are only three possibilities:*

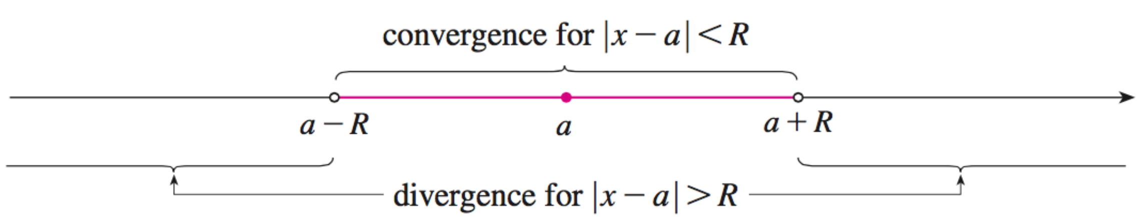
(1) *The series converges only when* $x = a$.

(2) *The series converges for all* x.

(3) *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

The constant R in case (3) is called the radius of convergence. If the series converges for all values of x, we say its radius of convergence is infinite ($R = \infty$) and if it converges only at $x = a$, we say its radius of convergence is zero ($R = 0$). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. So in case (1), the interval consist of a single point and in case (2) the interval of convergence is $(-\infty, \infty)$. In case (3), the power series converges absolutely for $|x-a| < R$ but anything could happen at the end points ($x = a \pm R$) that is, depending on the particular the series, it may or may not converge at either of the endpoints. Hence, in case (3), the interval of convergence may be open, closed, or half-open:

$$(a-R, a+R), \quad [a-R, a+R], \quad (a-R, a+R], \quad [a-R, a+R)$$



In general, the Ratio Test is used to determine the radius of convergence of a power series. However, if the radius of convergence is finite we must test the endpoints for convergence or divergence separately using some other test since the Ratio Test will always fail at the endpoints.

Example 8.8 (Power Series | Radius of Convergence)

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}(x-2)^n$.

Solution.

We use the Ratio Test to find the radius of convergence. Let $a_n = \frac{(-1)^n}{\sqrt{n+1}}(x-2)^n$. Then,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x-2)^{n+1}}{\sqrt{n+2}}}{\frac{(-1)^n(x-2)^n}{\sqrt{n+1}}}\right| \\
&= \lim_{n\to\infty}\left|(-1)\frac{\sqrt{n+1}}{\sqrt{n+2}}(x-2)\right| \\
&= \lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n+2}}|x-2| \\
&= \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{2}{n}}}|x-2| \\
&= |x-2|
\end{aligned}$$

By the Ratio Test, the given series converges if $|x-2|<1$ and diverges if $|x-2|>1$. Therefore, the radius of convergence is 1.

We know that the series converges for $1<x<3$. We must now test the convergence of the endpoints.

If $x=1$, the series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}(1-2)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}(-1)^n=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$$

which diverges since it is a p-series with $p=\frac{1}{2}<1$.

If $x=3$, the series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}(3-2)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$$

which we can show is convergent by the Alternating Series Test.

Hence, the given power series converges for $1<x\leqslant 3$. Therefore, the interval of convergence is $(1,3]$.

□

§8.1 Representations of Function as Power Series

Now that we know to find where a power series is convergent. We will focus our attention on how to represent functions as power series. We start with a familiar equation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad |x| < 1 \tag{30}$$

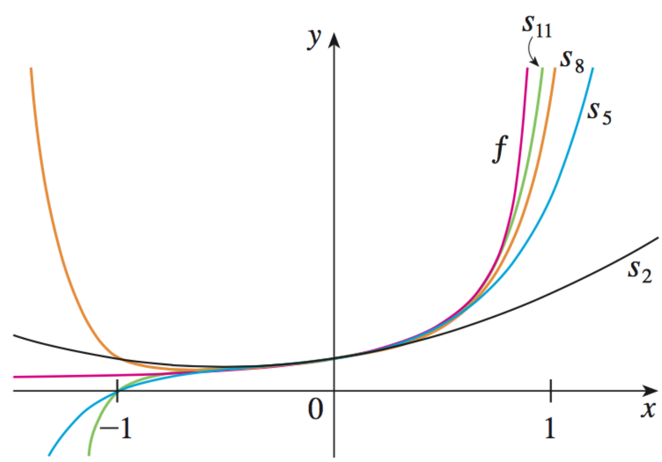
So far we have used equation (30) as a formula for the sum of the series. But now we are going to change our point of view: we now think of (30) as expressing the function $\frac{1}{1-x}$ as a sum of a power series. Notice that the nth partial sum of the series in (30) is a polynomial of the form

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

and since the sum of a series is the limit of the sequence of partial sums, we have that

$$\frac{1}{1-x} = \lim_{n\to\infty} P_n(x)$$

So as n increases $P_n(x)$ becomes a better approximation for $-1 < x < 1$. For values of x near zero, we need to take only few terms to get a good approximation. As we get closer to the endpoints of the interval of convergence of the series (i.e. $x = \pm 1$), we need more and more terms. The figure shows the graph of $f(x) = \frac{1}{1-x}$ and some approximating polynomials $s_n = P_n(x)$. Notice that the function $f(x) = \frac{1}{1-x}$ has a vertical asymptote $x = 1$ and therefore, these approximations do not apply when $x \geqslant 1$.



Here we have realised that there is a series that is equivalent to a known function on an interval! You may be wondering why this is useful. In most situations it is definitely easier to work with the function $\frac{1}{1-x}$ directly than with its power series representation. However, this approach can be very useful if we are faced with more complicated functions. For instance, we will later see that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Why would this be useful at all? Of course, if you want to find the value of $e^{1.42331}$, you would definitely use a calculator. But, have you ever wondered how calculators compute it? They use a power series representation to find an approximation to e^x! To find such an approximation, they compute the value of a partial sum S_n which is very easy to do since the partial sums of the series are just polynomials.

Starting with the geometric series we can obtain power series representations for many other functions.

Example 8.9

Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution.
Replacing x by $-x^2$ in equation (30) we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Because this is a geometric series, it converges when $|-x^2| < 1$ that is $x^2 < 1$. Therefore, the interval of convergence is $(-1, 1)$.

We could have also used the Ratio Test but all that work is not necessary here.

□

Differentiation and Integration of Power Series

As we have already discussed, the sum of a power series is a function $f(x) = \sum_{n-0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence. A natural question to ask is: can we differentiate and integrate these functions? The answer is yes and it turns out that it is very easy to do so. We just have to differentiate or integrate each term of the series just as we would do for polynomials. This is called term-by-term differentiation and integration.

Theorem 8.10 (Term-by-term Differentiation)
If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on $(a-R, a+R)$ and we obtain the derivative by differentiating the original series term by term:

$$f'(x) = \sum_{n=0}^{\infty} c_n n(x-a)^{n-1}$$

and the radius of convergence is also R.

This can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}[c_n(x-a)^n]$$

We know that for finite sums the derivative of the sum is the sum of the derivatives. This theorem tells us that this is also true for power series.

Remark 8.11

Although the theorem says that the radius of convergence remains the same when a power series is differentiated, this doesn't mean that the interval of convergence remains the same.

Example 8.12

Express $\frac{1}{(1-x)^2}$ as a power series. What is the radius of convergence?

Solution.
Notice that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, differentiating term by term we get

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$$

□

It is also true that power series can be integrated term by term.

Theorem 8.13 (Term-by-term Integration)
If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

can be integrated by integrating the original series term by term:

$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

and the radius of convergence is also R.

Example 8.14

Find a power series representation for $f(x) = \tan^{-1}(x)$.

Solution.
Notice that $f'(x) = \frac{1}{1+x^2}$ so we can find the required series by integrating the power series for $\frac{1}{1+x^2}$.

Recall that in example 8.9 we saw that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

So, integrating term by term we get

$$\begin{aligned}\int f'(x)\mathrm{d}x &= \int \frac{1}{1+x^2}\mathrm{d}x \\ &= \int \left[\sum_{n=0}^{\infty}(-1)^n x^{2n}\right]\mathrm{d}x \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C\end{aligned}$$

Hence, $\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$.

Notice that for $x = 0$ all the terms of the series are 0. So, to find the value of C we just need to put $x = 0$ into the equation to get $\tan^{-1}(0) = 0 + C$. So $C = 0$. Therefore,

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence of $\frac{1}{1+x^2}$ is 1, the radius of convergence of this series for $\tan^{-1}(x)$ is also 1. □

Example 8.15

Identify the function

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \quad -1 < x < 1 \tag{31}$$

Solution.
We differentiate the original series term by term to get

$$\begin{aligned}f(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}\cancel{n}\frac{x^{n-1}}{\cancel{n}} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1} \\ &= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \\ &= \sum_{n=0}^{\infty}(-x)^n\end{aligned}$$

This is a geometric series with $a = 1$ and $r = -x$. Therefore, for $-1 < x < 1$,

$$f'(x) = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

We can now integrate $f'(x) = \dfrac{1}{1+x}$ to get

$$\int f'(x)\mathrm{d}x = \int \frac{1}{1+x}\mathrm{d}x = \ln(1+x) + C$$

Notice that the series for $f(x)$ is zero when $x = 0$ so we have

$$0 = f(0) = \ln(1+0) + C$$

so $C = 0$. Therefore,

$$f'(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \ln(1+x) \quad -1 < x < 1$$

It can also be proved that the series in (31) converges at $x = 1$ to $\ln(2)$, but that was not guaranteed by theorem 8.13.

□

§8.2 Taylor and Maclaurin Series

So far we have been able to find power series representations for some functions. But the tools we have so far are very limited and cannot be used to find a power series representation for most function. We would like to consider a more general problem: given a function, is it possible to express it as a power series on an interval? If so, what will be the coefficients?

We can answer the last question if we assume that the function f can be represented as a power series.

$$\begin{aligned} f(x) &= \sum_{n=0} c_n(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \end{aligned}$$

and converges for $|x-a| < R$.

Notice that if we put $x = a$ into the equation, we have that all the terms in the series are 0 except for the first one. So

$$f(a) = c_0$$

To find the other coefficients c_n, we will use term by term differentiation

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

so plugging in $x = a$, we obtain

$$f'(a) = c_1$$

Differentiating again, we have

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3\cdot c_4(x-a)^2 + \cdots$$

so

$$f''(a) = 2c_2$$

Differentiating one more time, we have

$$f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + \cdots$$

so

$$f'''(a) = 3 \cdot 2c_3$$

Can you see the pattern?

Notice that if we continue differentiating our function and plugging in $x = a$, we obtain

$$f^{(n)}(a) = n!c_n$$

so solving for c_n we have

$$c_n = \frac{f^{(n)}(a)}{n!}$$

If we adopt the conventions that $0! = 1$ and $f^{(0)} = f$, then

$$c_n = \frac{f^{(n)}(a)}{n!} \text{ for all } n \geqslant 0$$

We have proved the following theorem.

Theorem 8.16
If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad |x-a| < R$$

then there is only one such series and its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

so if f has a series representation, then the series must be

$$f(x) = \sum_{n=0} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{32}$$

But if we start with an arbitrary function f that is infinitely differentiable on an interval centred at a and used to generated a power series like the one in equation (32), will the series converge to $f(x)$? The answer is, sometimes it does, sometimes it doesn't.

The power series in equation (32) is a very important one and it has a special name.

Definition 8.17 (Taylor and Maclaurin Series)
Let f be a function with derivative of all orders on an interval $(a-R, a+R)$ for some $R > 0$. Then, the **Taylor Series of f at a (or centred at a)** is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

For the special case $a = 0$, the Taylor series centred at 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

and it is often called **Maclaurin Series**.

Remark 8.18

We have shown that if f can be represented as a power series centred at a, then f is the sum of its Taylor Series. However, there are functions with derivatives of all orders that are not equal to their Taylor series.

Example 8.19

Find the Maclaurin series of $f(x) = e^x$.

Solution.
We first need to find a formula for the nth derivative. Note that $f'(x) = e^x$, $f''(x) = e^x$, $\cdots$.
So, $f^{(n)}(x) = e^x$ for all n. Hence, $f^{(n)}(0) = e^0 = 1$ for all n.

Therefore, the Maclaurin series for $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

And, in example 8.4 we showed that this series converges for all $x \in \mathbb{R}$. □

In example 8.19 we found the Taylor series of $f(x) = e^x$. But the only conclusion we can draw from this example is that if e^x has a power series representation at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We would like to know if the Taylor Series actually converges to the function. To study this issue we need to consider the partial sums of the Taylor series.

Notice that the nth partial sum of a Taylor series is simply a polynomial

$$\begin{aligned}P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n\end{aligned}$$

This polynomial is called **Taylor polynomial of order n** of f at $x = a$. It is called Taylor polynomial of order n instead of degree n because $f^{(n)}(a)$ may be zero. For example, we will soon see that the first two Taylor polynomials of $f(x) = \cos x$ at $x = 0$, are $P_0(x) = 1$ and $P_1(x) = 1$. The first-order Taylor polynomial of $\cos x$ has degree zero, not one!

Example 8.20 (Constructing and Graphing Taylor Polynomials of e^x)

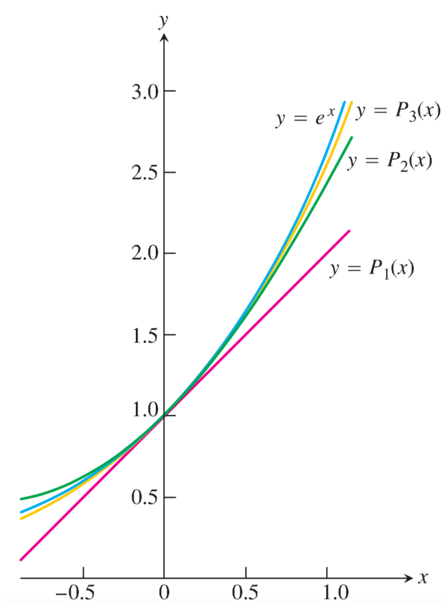
The Taylor polynomials of $f(x) = e^x$ at $x = 0$ are

$$P_1(x) = 1 + x, \quad P_2(x) = 1 + x + \frac{x^2}{2!}, \quad P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad \cdots$$

so, in general, the Taylor polynomial of order n of e^x at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

The figure shows the graph of $f(x) = e^x$ together with some Taylor polynomials. Notice that as n gets larger, the graph of $P_n(x)$ seems to be approaching the graph of $f(x) = e^x$. The graphical evidence suggests that the partial sums are approaching e^x. That is, it suggests that e^x is equal to the sum of its Taylor series. With some work, it can be proved that this is actually the case (if you are curious, you can find the proof in your textbook).



Picture taken from Thomas' Calculus 12th Edition.

Example 8.21 (Taylor Series of $\cos x$ at $x = 0$)

(1) Find the Taylor series of $f(x) = \cos x$ at $x = 0$.

(2) Find its radius of convergence.

Solution.

(1) First, we compute some derivatives and their value at $x = 0$.

$f(x) = \cos x$	$f(0) = 1$
$f'(x) = -\sin x$	$f'(0) = 0$
$f''(x) = -\cos x$	$f''(0) = -1$
$f'''(x) = \sin x$	$f'''(0) = 0$
$f^{(4)}(x) = \cos x$	$f^{(4)}(0) = 1$

Notice that the derivatives repeat in a cycle of four and $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$. Therefore, the Taylor series of $f(x) = \cos x$ centred at $x = 0$ is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= 1 + 0 \cdot x - \frac{1}{2}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 - \frac{1}{6!}x^6 + \cdots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

This is also the Maclaurin series for $\cos x$. Notice that only even powers of x occur in the Taylor series of $\cos x$. This agrees with the fact that cosine is an even function.

(2) We use the Ratio Test to find the radius of convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{x^{2(n+1)}}{(2(n+1))!}}{(-1)^n\frac{x^{2n}}{(2n)!}}\right| \\ &= \lim_{n\to\infty}\frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}|x^2| \\ &= \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} \\ &= 0 \qquad \text{for all } x\end{aligned}$$

Therefore, the series converges for all x by the Ratio Test.

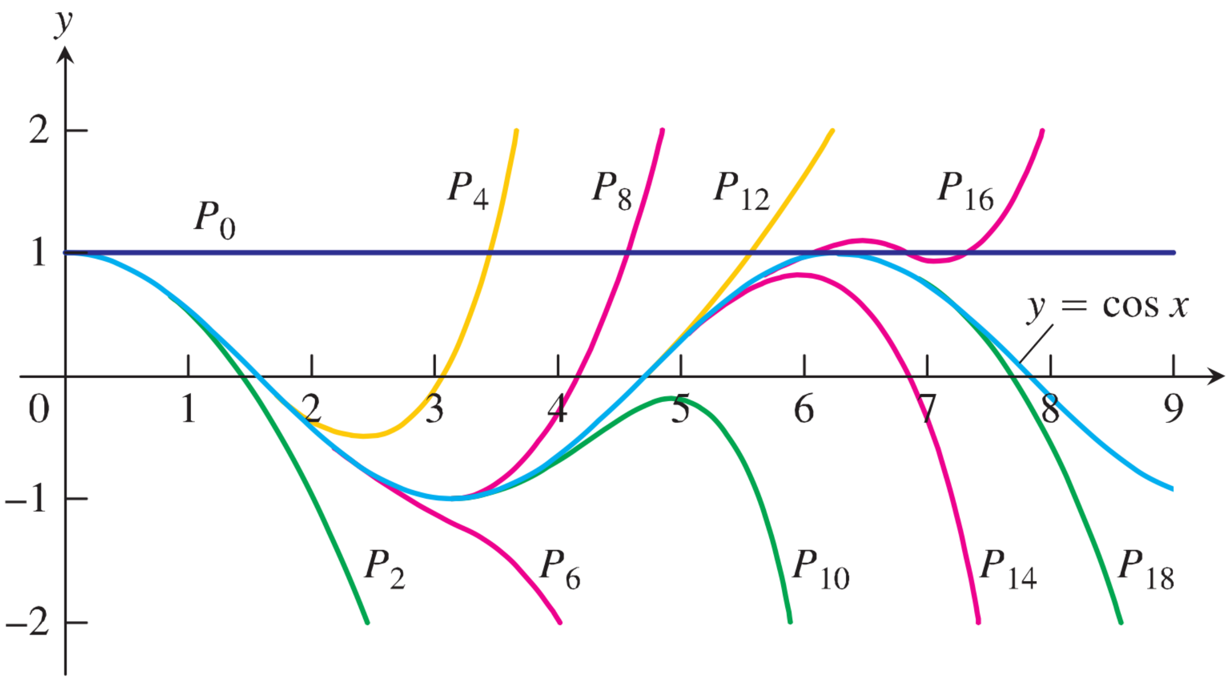
□

Example 8.22 (Taylor polynomials of $f(x) = \cos x$ at 0)

In example 8.21 we found that $f^{(2n+1)}(0) = 0$. Therefore, the Taylor polynomials of order $2n$ and $2n+1$ coincide:

$$P_{2n+1} = P_{2n} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$

In the figure we show the graph of $f(x) = \cos x$ and some Taylor polynomials. We only show the graph for $x \geqslant 0$ because the graphs are symmetric about the y-axis. Notice that these polynomials approximate $f(x) = \cos x$ very well near $x = 0$. Also, notice that as n increases, the interval over which P_{2n} provides a close approximation to $f(x) = \cos x$ also increases.



Picture taken from Thomas' Calculus 12th Edition.

The graphical evidence suggests that the sequence of Taylor polynomials is approaching $\cos x$. In fact, it can be proved that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

What is really fascinating about this is that we can deduce the behaviour of $\cos x$, even for large values of x, from simply knowing the values of cosine and its derivatives at $x = 0$.

Example 8.23 (Maclaurin series of $f(x) = \sin x$)

Find the Maclaurin series (Taylor series centred at 0) of $f(x) = \sin x$.

Solution.
We could proceed directly as in example 8.21 but i is much easier if we use term by term differentiation to differentiate the Maclaurin series for $\cos x$ obtained in example 8.21.

$$\begin{aligned}
\sin x &= -\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\left[1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right] \\
&= -\left[-2\frac{x}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots\right] \\
&= -\left[-x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\right] \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Since the Maclaurin series for $\cos x$ converges for all x, we have that the differentiated series also converges for all x. Hence, the Maclaurin of $\sin x$ converges for all x.

□

Example 8.24 (Taylor series for e^{2x} centred at $x = -1$)

Find the Taylor series for $f(x) = e^{2x}$ centred at $x = -1$.

Solution.
We first need to find a formula for the nth derivative, $f^{(n)}(x)$.

$$\begin{aligned}
f(x) &= e^{2x} \\
f'(x) &= 2e^{2x} \\
f''(x) &= 4e^{2x} = 2^2 e^{2x} \\
f'''(x) &= 8e^{2x} = 2^3 e^{2x} \\
&\vdots \\
f^{(n)}(x) &= 2^n e^{2x}
\end{aligned}$$

Hence, $f^{(n)}(-1) = 2^n e^{-2}$. Therefore, the Taylor series for $f(x) = e^{2x}$ centred at $x = -1$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x - (-1))^n = \sum_{n=0}^{\infty} \frac{2^n e^{-2}}{n!}(x+1)^n$$

□

The following table summarises some important Maclaurin series that we have seen in this section and the preceding one, as well as their their radii of convergence.

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$R = \infty$
$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$	$R = 1$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$R = 1$

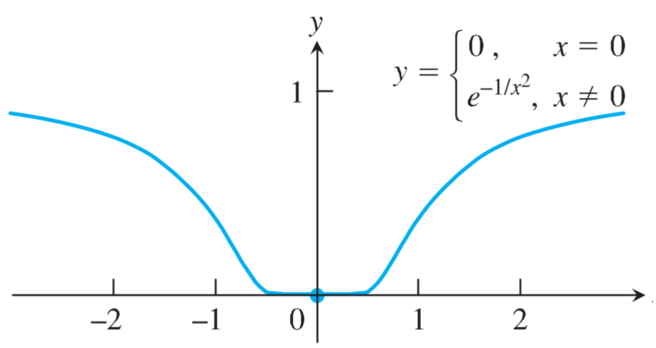
In all the examples we have considered so far, the function is equal to its Taylor series. However, there are some functions that have derivatives of all order that are not equal to their Taylor series.

Example 8.25 (A function that is not equal to its Taylor Series)

With a bit of work, it can be shown that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

has derivative of all orders at $x = 0$ and $f^{(n)}(0) = 0$ for all n.



Picture taken from Thomas' Calculus 12th Edition.

Therefore, its Taylor series at $x = 0$ is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots \\ &= 0 + 0 \cdot x + 0 \cdot x^2 + \cdots \\ &= 0 \end{aligned}$$

So this series is convergent for all x (its sum is 0). However, it doesn't converge to $f(x)$ since $f(x) \neq 0$ for $x \neq 0$. Hence, $f(x)$ is not equal to its Taylor series $x = 0$.

§8.3 Applications of Taylor Series

In the previous section we learned how to find the Taylor series expansion of several functions. Here, we will learn how Taylor series can be used to solve several problems. In this section we will use Taylor series to approximate values of transcendental functions, find the exact value of infinite sums, evaluate limits and integrals.

Example 8.26 (Using Taylor Polynomials to approximate a cosine value)

Use the Taylor polynomial of order 4 to approximate $\cos(0.1)$.

Solution.
We know that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Therefore,

$$\begin{aligned}\cos(0.1) &= \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{2n}}{(2n)!} \\ &= 1 - \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!} - \frac{(0.1)^6}{6!} + \cdots \\ &= 1 - \frac{1}{10^2 \cdot 2!} + \frac{1}{10^3 \cdot 3!} + \frac{1}{10^4 \cdot 4!} - \frac{1}{10^6 \cdot 6!} + \cdots\end{aligned}$$

So we have that

$$\cos(0.1) \approx 1 - \frac{1}{10^2 \cdot 2!} + \frac{1}{10^3 \cdot 3!} + \frac{1}{10^4 \cdot 4!} = \frac{238801}{240000} \approx 0.9950041666$$

If you compute $\cos(0.1)$ using a calculator, you will obtain: $0.99500416527\ldots$.

As you can see, this Taylor polynomial of order 4 gives us a very good approximation of $\cos(0.1)$. □

Example 8.27 (Using Taylor series to find the sum of a series)

Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$.

This series might look a little bit scary at first, but a closer look reveals that it looks very similar to the Taylor series of $\cos x$. Indeed, notice that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

We can also use Taylor series to evaluate some complicated limits that otherwise would require the use of L'Hôpital's rule.

Example 8.28 (Using Taylor series to evaluate limits)

Evaluate $\lim_{x \to 0} \dfrac{\sin(x^3) - x^3}{x^9}$.

Solution.
We first need to find the Taylor series of $\sin(x^3)$. Plugging x^3 in the Taylor series for $\sin x$, we obtain

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots$$

So we have

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9} &= \lim_{x \to 0} \frac{\left(\cancel{x^3} - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots\right) - \cancel{x^3}}{x^9} \\
&= \lim_{x \to 0} \frac{\left(-\frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots\right)}{x^9} \\
&= \lim_{x \to 0} \left(-\frac{1}{3!} + \frac{x^6}{5!} - \frac{x^{12}}{7!} + \cdots\right) \\
&= -\frac{1}{3!} \\
&= -\frac{1}{6}
\end{aligned}$$

Notice that you could also have evaluated this limit using L'Hôpital's Rule three times.

□

We can also use Taylor series to integrate functions. We have seen that there are some functions that don't have an anti-derivative that can be expressed in terms of elementary functions. Now we are able to use Taylor series to express non-elementary integrals in terms of power series.

Example 8.29 (Using Taylor Series to Evaluate an Integral)

Evaluate $\displaystyle\int e^{-x^2} \mathrm{d}x$ as an infinite series.

Solution.
As we have mentioned before, we are not able to find an anti-derivative of e^{-x^2} in terms of functions that we already know. However, we can use Taylor series and term by term integration of power series to evaluate this integral. We first need to find a power series representation of e^{-x^2}. We could do this directly, but it is much easier to replace x with $-x^2$ in the Taylor series for e^x. So we have

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

and integrating term by term we obtain

$$\begin{aligned}\int e^{-x^2}\,\mathrm{d}x &= \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\right] \mathrm{d}x \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\cdot n!} \\ &= C + x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\end{aligned}$$

The series converges for all x because the series for e^x converges for all x.

□

§A

Sigma Notation

Sigma notation is used to write sums with many terms in a compact form

$$\sum_{i=m}^{n} a_i := a_m + a_{m+1} + \ldots + a_{n-1} + a_n \ .$$

It reads "*the sum from $i = m$ to n of a_i*".

Example A.1

We can write

$$1^2 + 2^2 + 3^2 + 4^2 = \sum_{i=1}^{4} i^2 \ .$$

$$\frac{2}{3} + \frac{3}{4} + \ldots + \frac{99}{100} = \sum_{k=2}^{99} \frac{k}{k+1} \ .$$

$$f(1) + f(2) + \ldots + f(n) = \sum_{i=1}^{n} f(i) \ .$$

Example A.2

Find $\sum_{i=1}^{n} 1$.

Solution.

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \ldots + 1}_{n \text{ times}} = n.$$

□

Theorem A.3 (Algebra of Finite Sums)

If c is any constant (that does not depend on i), then

(1) $\displaystyle\sum_{i=m}^{n} c a_i = c \sum_{i=m}^{n} a_i$.

(2) $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$.

Some useful formulas:

(1) $\displaystyle\sum_{i=1}^{n} 1 = n$

(2) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(3) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(4) $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

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