MAT 2322 D   Winter 2017   February 16th, 14:30

TEST #1

Max = 25

Name: Solutions

Student Number: ______________________

• Time: 80 min.

• Verify that your copy of the midterm has exactly 6 pages (including this one).

• There are 5 questions worth 5 marks each for a total of 25 points.

• Only basic scientific calculators are permitted: non-programmable, non-graphing.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature: ______________________

1. Find and classify the critical points of the following function.

$$f(x,y) = 4xy - x^4 - y^4$$

$f_x = 4y - 4x^3 = 4(y-x^3) = 0$ if $y = x^3$

$f_y = 4x - 4y^3 = 4(x-y^3) = 0$ if $x = y^3$

ie $x = x^9$ or $x^9 - x = 0$ or $x(x^8-1) = 0$

or $x(x^4-1)(x^4+1) = 0$ or

or $x(x^2-1)(x^2+1)(x^4+1) = 0 \Rightarrow x = 0, \pm 1$

so have 3 critical pts $\boxed{(0,0),\ (-1,-1) \text{ and } (1,1)}$

$f_{xx} = -12x^2$, $f_{xy} = 4$, $f_{yy} = -12y^2$

so $D = f_{xx}f_{yy} - f_{xy} = 144x^2y^2 - 16$

$D(0,0) < 0 \Rightarrow$ $\boxed{(0,0) \text{ is a saddle pt}}$

$$\left.\begin{array}{l} D(-1,-1) = D(1,1) > 0 \\ f_{xx}(-1,-1) = f_{xx}(1,1) < 0 \end{array}\right\} \quad \boxed{\begin{array}{l} (-1,-1) \text{ and} \\ (1,1) \text{ are local max's} \end{array}}$$

2. Use the method of Lagrange multipliers to determine the absolute minimum and maximum values of the function $f(x,y) = x^2 + 8y + y^2$ on the circle $x^2 + y^2 = 9$.

$\nabla f = \lambda \nabla g \Rightarrow \quad f_x = \lambda g_x \Rightarrow \quad 2x = 2\lambda x$

$x = \lambda x$ ← so $\underline{x=0}$

$f_y = \lambda g_y \Rightarrow \quad 8 + 2y = 2\lambda y$

$4 + y = \lambda y \Rightarrow \underline{\lambda \neq 1}$

if $x=0$, $y^2 = 9 \Rightarrow y = \pm 3$

and have $\underline{2 \text{ points}}$ $\boxed{(0,3) \text{ and } (0,-3)}$

$f(0,3) = 8(3) + (3)^2 = 33$

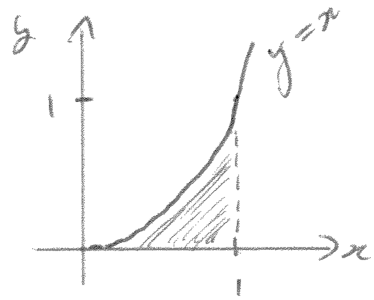
$f(0,-3) = 8(-3) + (3)^2 = -15$

so $\boxed{\begin{array}{l} \text{max is } 33 \text{ at } (0,3) \\ \text{and min is } -15 \text{ at } (0,-3) \end{array}}$

3. Calculate the following integral.

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(\pi x^3)\, dx\, dy$$

can't do x integration $\Rightarrow$ change order



x goes from $\sqrt{y}$ to 1
then y from 0 to 1

so y goes from 0 to x^2
then x from 0 to 1

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(\pi x^3)\, dx\, dy = \int_0^1 \int_0^{x^2} \sin(\pi x^3)\, dy\, dx$$

$$= \int_0^1 \sin(\pi x^3)\left(y\Big|_0^{x^2}\right) dx$$

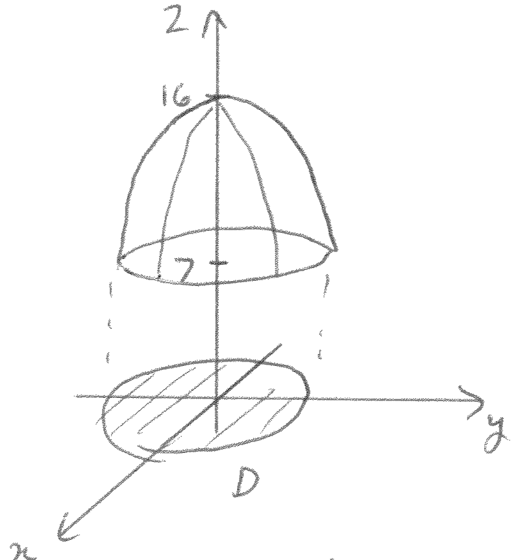
$$= \int_0^1 x^2 \sin(\pi x^3)\, dx$$

$$= \frac{-1}{3\pi} \cos(\pi x^3)\Big|_0^1$$

$$= \frac{-1}{3\pi}\left(-1 - 1\right)$$

$$= \boxed{2/3\pi}$$

4. Compute the volume of the solid bounded by the paraboloid $z = 16 - x^2 - y^2$ and the plane $z = 7$.



$16 - x^2 - y^2 = 7 \Rightarrow x^2 + y^2 = 9$

so D is disk of radius 3

(use polar coords!)

The volume is

$$V = \iint_D \left((16 - x^2 - y^2) - 7 \right) dA$$

$$= \int_0^{2\pi} \int_0^3 (9 - r^2) r \, dr \, d\theta$$

$$= 2\pi \int_0^3 (9r - r^3) dr$$

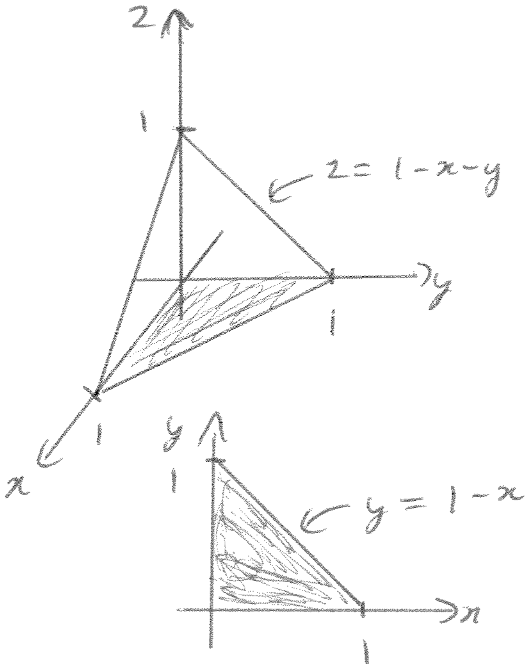
$$= 2\pi \left(\frac{9}{2} r^2 - \frac{1}{4} r^4 \Big|_0^3 \right)$$

$$= 2\pi \left(\frac{81}{2} - \frac{81}{4} \right)$$

$$= 2\pi \left(\frac{81}{4} \right)$$

$$= \boxed{81\pi/2}$$

5. Calculate the total mass of the tetrahedron which is in the first octant (*ie* all coordinates are ≥ 0) and under the plane $x + y + z = 1$ where the density function is $\rho(x, y, z) = 2z$.



$$\text{mass} = \iiint \rho(x,y,z)\, dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 2z \; dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(z^2 \Big|_0^{(1-x-y)} \right) dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy\, dx$$

$$= \int_0^1 \left(-\frac{1}{3} (1-x-y)^3 \Big|_0^{1-x} \right) dx$$

$$= -\frac{1}{3} \int_0^1 \left((1-x-(1-x))^3 - (1-x)^3 \right) dx$$

$$= \frac{1}{3} \int_0^1 (1-x)^3 \, dx$$

$$= \frac{-1}{12} (1-x)^4 \Big|_0^1$$

$$= \boxed{1/12}$$