MATH1005H — Solution-Test 1 —- 14:35–15:25, Sep. 30 2016

Total: 20 marks
Closed book, no GRAPHING calculator!

Question 1. [2] The equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ can be expressed as an equation of the type

(a) Separable (b) Homogeneous (**) (c) Linear (d) None of these

Question 2. [2] The solution of the initial-value problem $y' = \dfrac{8x^3}{3y^2}$, $y(0) = 1$, satisfy $y(1) =$

(a) $\sqrt[3]{3}$ (**) (b) $-\sqrt{2}$ (c) $\sqrt{3}$ (d) $\sqrt[3]{2}$

Question 3. Consider the equation $2x + y - xy' = 0$.

(a) Solve it as homogeneous equation. [3]

(b) Solve it as a linear equation. [3]

(c) Find the orthogonal trajectories of the one-parameter family of cures defined by general solution.[3]

Solution:
(a) $y' = 2 + \frac{y}{x};\ u = \frac{y}{x} \to y' = u + u'x;$ then $u + u'x = 2 + u \to u'x = 2 \to \int du = \int \frac{2}{x} dx$
$u = 2\ln|x| + c \to y = 2x(\ln|x| + c)$
(b) $y' - \frac{1}{x}y = 2;\ I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$
$y = x\int 2(\frac{1}{x})dx = 2x(\ln|x| + c)$

(c) $y' = \frac{2x+y}{x} \to y'_{OR} = \frac{-x}{2x+y}$, which needs to solve by using Homogeneous method
$y' = \frac{-1}{2+u};\ u + xu' = \frac{-1}{2+u} \to xu' = \frac{-1}{2+u} - u = \frac{-1-2u-u^2}{2+u}$
$\int \frac{2+u}{u^2+2u+1}du = \int \frac{-1}{x}dx;$ by using partial fraction method
$\frac{2+u}{(u+1)(u+1)} = \frac{A}{u+1} + \frac{B}{(u+1)^2} = \frac{Au+A+B}{(u+1)^2},\ A = 1, B = 1$
$\int \frac{2+u}{u^2+2u+1}du = \int \frac{1}{u+1}du + \int \frac{1}{(u+1)^2}du = \ln|u+1| - \frac{1}{(u+1)} = -\ln|x| + c$
Then, $\ln|\frac{y}{x} + 1| - \frac{1}{(\frac{y}{x}+1)} = -\ln|x| + c$

Question 4. Let $f(x, y) = \sin(xy) + \sqrt{xy}$, $x(t) = e^t + 1$ and $y(t) = \ln(t)$. Determine $\dfrac{d}{dt}f(x(t), y(t))$. [3]

Solution:

$$\frac{d}{dt}f(x(t),y(t)) = (y\cos(xy) + \frac{\sqrt{y}}{2\sqrt{x}})(e^t) + (x\cos(xy) + \frac{\sqrt{x}}{2\sqrt{y}})(\frac{1}{t})$$

Question 5. Solve the initial-value problem [4]

$$3y' - 2y = \frac{1}{y^2}, \qquad y(0) = 2$$

Solution:
'Bernoulli Equation'
$\alpha = -2, u = y^3 \rightarrow y = u^{\frac{1}{3}} \rightarrow y' = \frac{1}{3}u^{-\frac{2}{3}}u'$
$u^{-\frac{2}{3}}u' - 2u^{1/3} = u^{-2/3} \rightarrow u' - 2u = 1$ which is linear where
$I(x) = e^{\int -2dx} = e^{-2x} \rightarrow u = e^{2x}\int e^{-2x}dx = e^{2x}[-\frac{1}{2}e^{-2x} + c] = -\frac{1}{2} + ce^{2x}$
Then $y = (-\frac{1}{2} + ce^{2x})^{\frac{1}{3}} \rightarrow 2 = (-\frac{1}{2} + c)^{\frac{1}{3}} \rightarrow 8 = -\frac{1}{2} + c \rightarrow c = \frac{15}{2}$