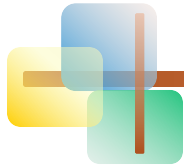


Statistics for Business and Economics

8th Edition



Chapter 9

Hypothesis Testing: Single Population



Chapter Goals

After completing this chapter, you should be able to:

- Formulate null and alternative hypotheses for applications involving
 - a single population mean from a normal distribution
 - a single population proportion (large samples)
 - the variance of a normal distribution
- Formulate a decision rule for testing a hypothesis
- Know how to use the critical value and p-value approaches to test the null hypothesis (for both mean and proportion problems)
- Define Type I and Type II errors and assess the power of a test
- Use the chi-square distribution for tests of the variance of a normal distribution

Concepts of Hypothesis Testing

- A hypothesis is a claim (assumption) about a population parameter:



- population mean

Example: The mean monthly cell phone bill of this city is $\mu = \$52$

- population proportion

Example: The proportion of adults in this city with cell phones is $P = .88$

The Null Hypothesis, H_0

- States the assumption (numerical) to be tested

Example: The average number of TV sets in U.S. Homes is equal to three ($H_0 : \mu = 3$)

- Is always about a population parameter, not about a sample statistic

$$H_0 : \mu = 3$$

$$H_0 : \bar{x} = 3$$



The Null Hypothesis, H_0

(continued)

- Begin with the assumption that the null hypothesis is true
 - Similar to the notion of innocent until proven guilty
- Refers to the status quo
- Always contains “=”, “≤” or “≥” sign
- May or may not be rejected



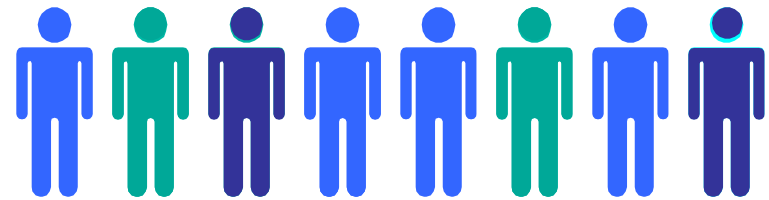


The Alternative Hypothesis, H_1

- Is the opposite of the null hypothesis
 - e.g., The average number of TV sets in U.S. homes is not equal to 3 ($H_1: \mu \neq 3$)
- Challenges the status quo
- Never contains the “=”, “≤” or “≥” sign
- May or may not be supported
- Is generally the hypothesis that the researcher is trying to support

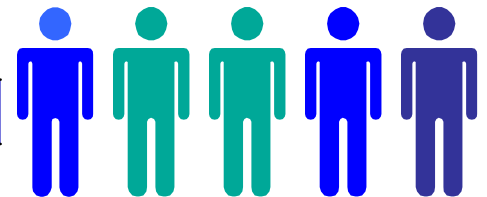
Hypothesis Testing Process

Claim: the population mean age is 50.
(Null Hypothesis:
 $H_0: \mu = 50$)



Population

Now select a random sample



Sample

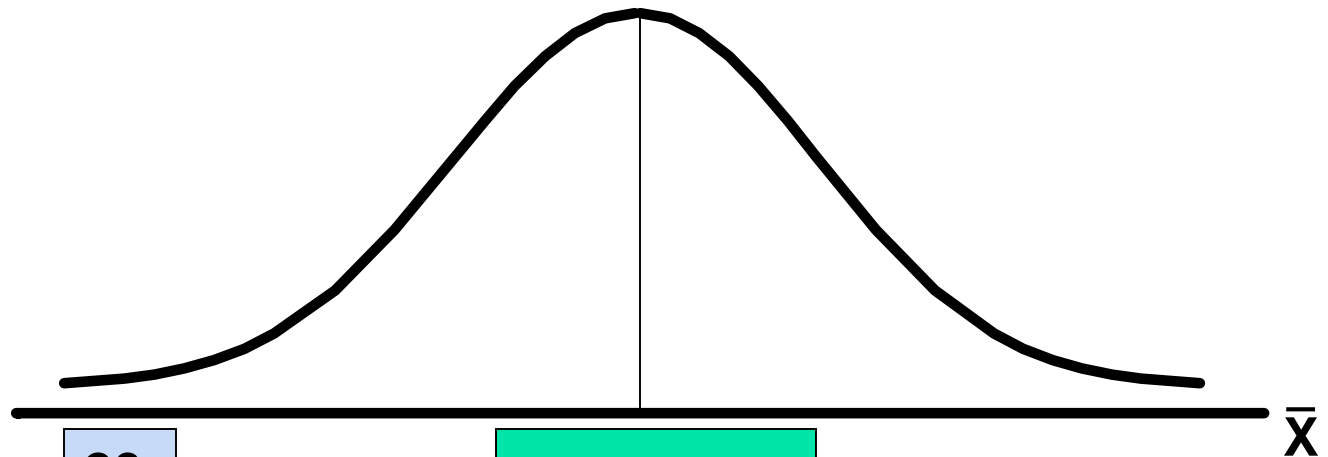
Is $\bar{x}=20$ likely if $\mu = 50$?

If not likely,
REJECT
Null Hypothesis

Suppose the sample mean age is 20: $\bar{x} = 20$

Reason for Rejecting H_0

Sampling Distribution of \bar{X}



20

$\mu = 50$
If H_0 is true

\bar{X}

If it is unlikely that we would get a sample mean of this value ...

... if in fact this were the population mean...

... then we reject the null hypothesis that $\mu = 50$.



Level of Significance, α

- **Defines the unlikely values of the sample statistic if the null hypothesis is true**
 - Defines **rejection region** of the sampling distribution
- Is designated by **α** , (level of significance)
 - Typical values are 0.01, 0.05, or 0.10
- Is selected by the researcher at the beginning
- Provides the **critical value(s)** of the test

Level of Significance and the Rejection Region

Level of significance = α

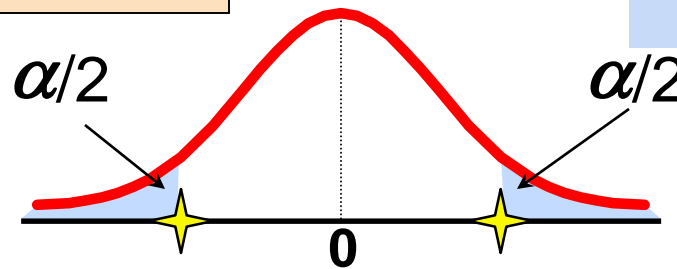
✦ Represents critical value

Rejection region is shaded

$$H_0: \mu = 3$$

$$H_1: \mu \neq 3$$

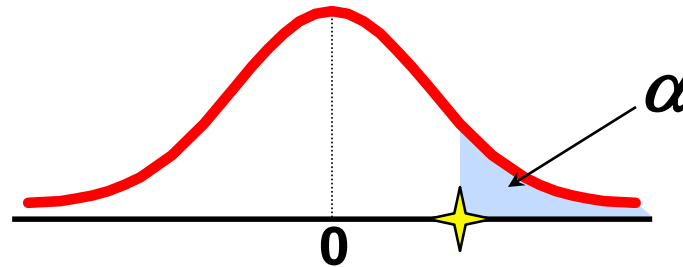
Two-tail test



$$H_0: \mu \leq 3$$

$$H_1: \mu > 3$$

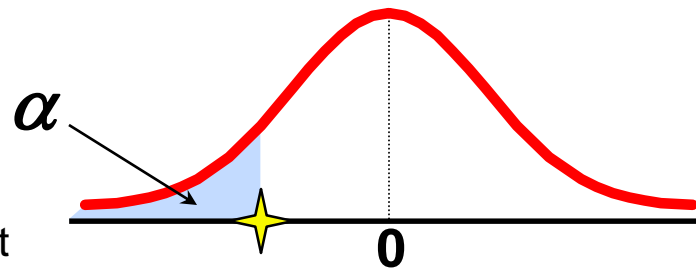
Upper-tail test



$$H_0: \mu \geq 3$$

$$H_1: \mu < 3$$

Lower-tail test





Errors in Making Decisions

- **Type I Error**

- Reject a true null hypothesis
- Considered a serious type of error

The probability of Type I Error is α

- Called **level of significance** of the test
- Set by researcher in advance



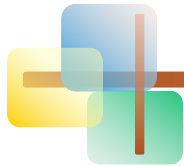
Errors in Making Decisions

(continued)

- **Type II Error**
 - Fail to reject a false null hypothesis

The probability of Type II Error is β

Outcomes and Probabilities



Possible Hypothesis Test Outcomes

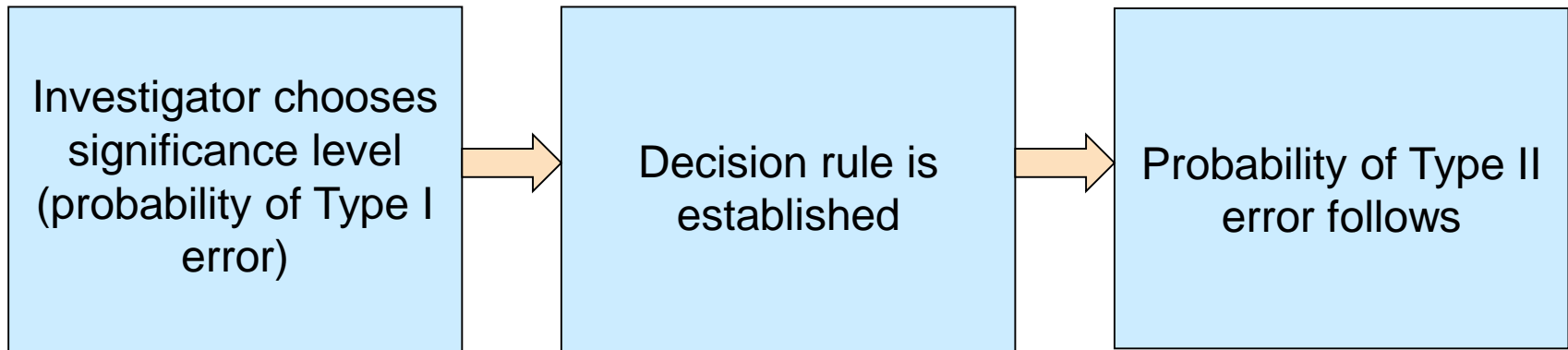
| | Actual Situation | |
|----------------------|--------------------------------------|-------------------------------------|
| Decision | H_0 True | H_0 False |
| Fail to Reject H_0 | Correct Decision ($1 - \alpha$) | Type II Error (β) |
| Reject H_0 | Type I Error (α) | Correct Decision ($1 - \beta$) |

Key:
Outcome
(Probability)

($1 - \beta$) is called the power of the test

Consequences of Fixing the Significance Level of a Test



- Once the significance level α is chosen (generally less than 0.10), the probability of Type II error, β , can be found.













Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
 - Type I error can only occur if H_0 is **true**
 - Type II error can only occur if H_0 is **false**

If Type I error probability (α) , then
Type II error probability (β) 

Factors Affecting Type II Error

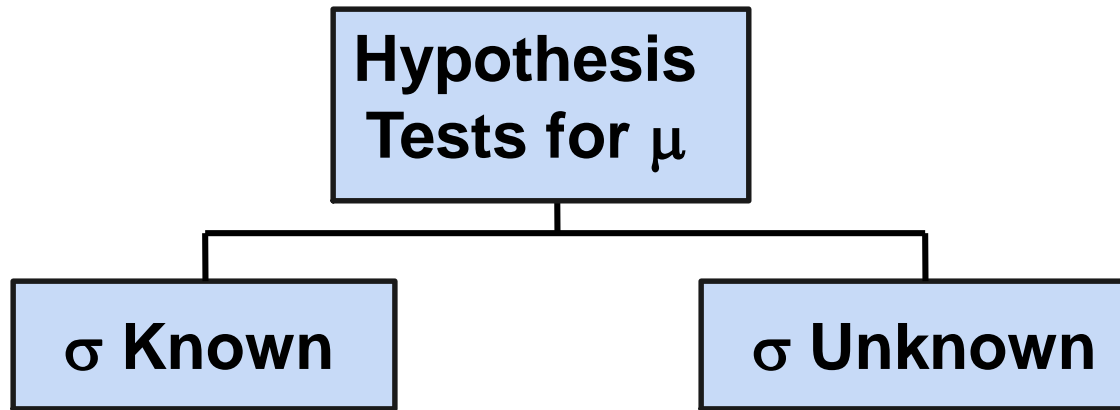
- All else equal,
 - β  when the difference between hypothesized parameter and its true value 
 - β  when α 
 - β  when σ 
 - β  when n 



Power of the Test

- The **power** of a test is the probability of rejecting a null hypothesis that is false
- i.e., $\text{Power} = P(\text{Reject } H_0 \mid H_1 \text{ is true})$
 - Power of the test increases as the sample size increases

Hypothesis Tests for the Mean



Tests of the Mean of a Normal Distribution (σ Known)

9.2

- Convert sample result (\bar{x}) to a **z value**

Hypothesis Tests for μ

σ Known

σ Unknown

Consider the test

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The **decision rule** is:

$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$$

Decision Rule

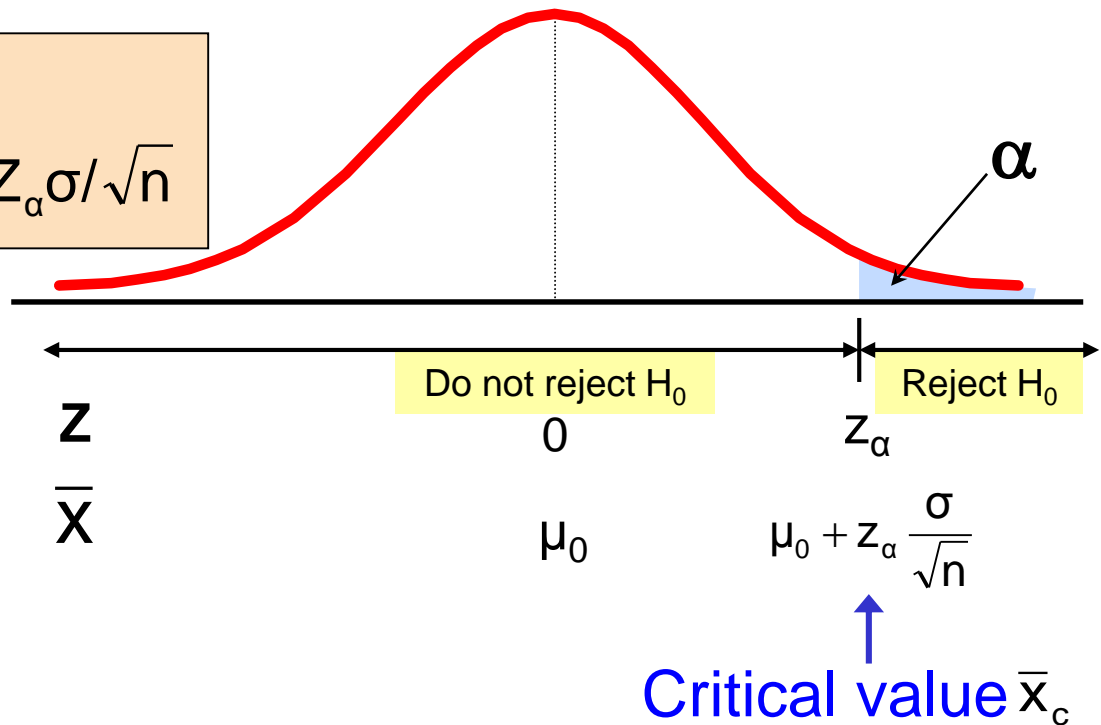
Reject H_0 if $z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$

$$H_0: \mu = \mu_0$$

$$H_1: \mu > \mu_0$$

Alternate rule:

Reject H_0 if $\bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$





p-Value

- **p-value**: Probability of obtaining a test statistic more extreme (\leq or \geq) than the observed sample value **given H_0 is true**
 - Also called **observed level of significance**
 - Smallest value of α for which H_0 can be rejected

p-Value Approach to Testing

- Convert sample result (e.g., \bar{X}) to test statistic (e.g., z statistic)
- Obtain the **p-value**

- For an upper tail test:

$$\begin{aligned} \text{p-value} &= P\left(z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}, \text{ given that } H_0 \text{ is true}\right) \\ &= P\left(z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_0\right) \end{aligned}$$

- **Decision rule:** compare the **p-value** to α

- If $\text{p-value} < \alpha$, reject H_0
- If $\text{p-value} \geq \alpha$, do not reject H_0

Example: Upper-Tail Z Test for Mean (σ Known)

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over \$52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)



Form hypothesis test:

$H_0: \mu \leq 52$ the average is **not** over \$52 per month

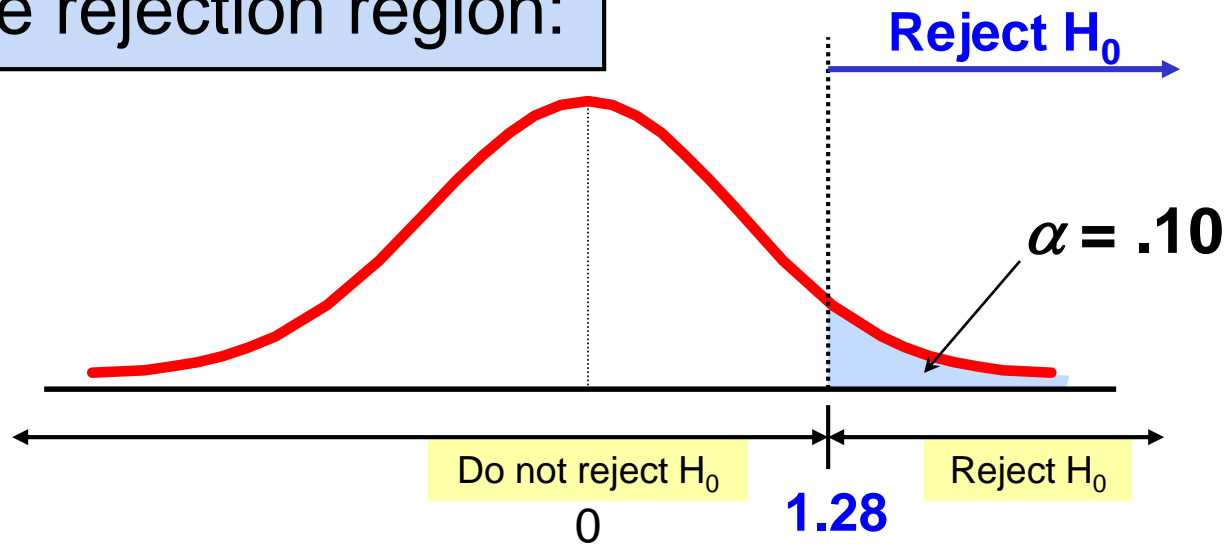
$H_1: \mu > 52$ the average **is** greater than \$52 per month
(i.e., sufficient evidence exists to support the manager's claim)

Example: Find Rejection Region

(continued)

- Suppose that $\alpha = .10$ is chosen for this test

Find the rejection region:



$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > 1.28$$



Example: Sample Results

(continued)

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: $n = 64$, $\bar{x} = 53.1$ ($\sigma = 10$ was assumed known)

- Using the sample results,

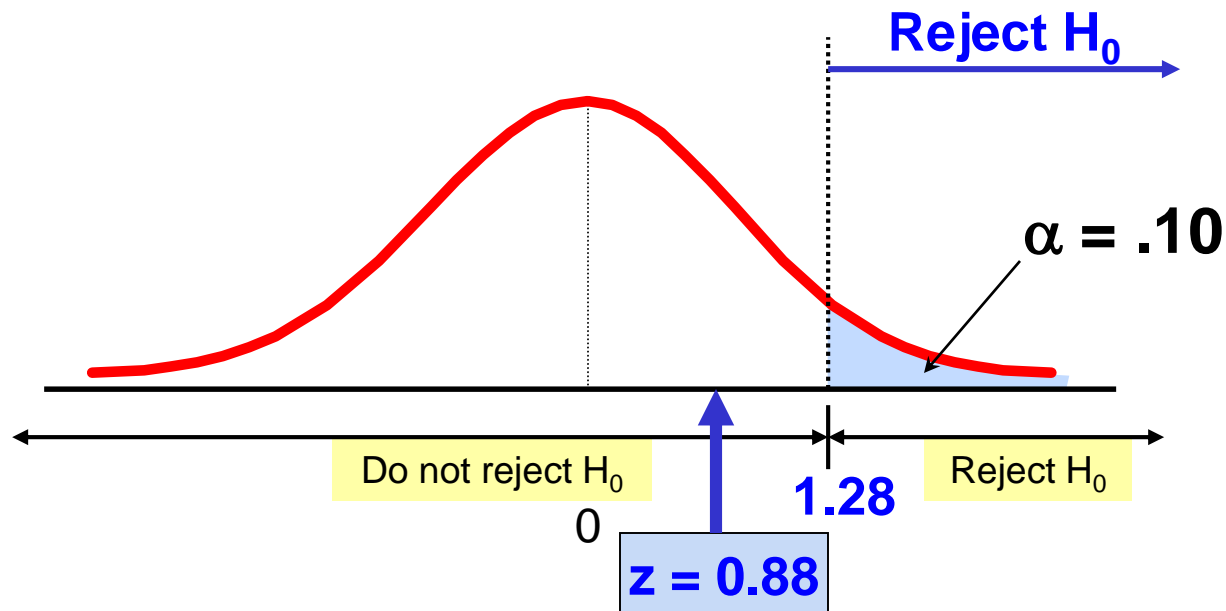
$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$



Example: Decision

(continued)

Reach a decision and interpret the result:



Do not reject H_0 since $z = 0.88 < 1.28$

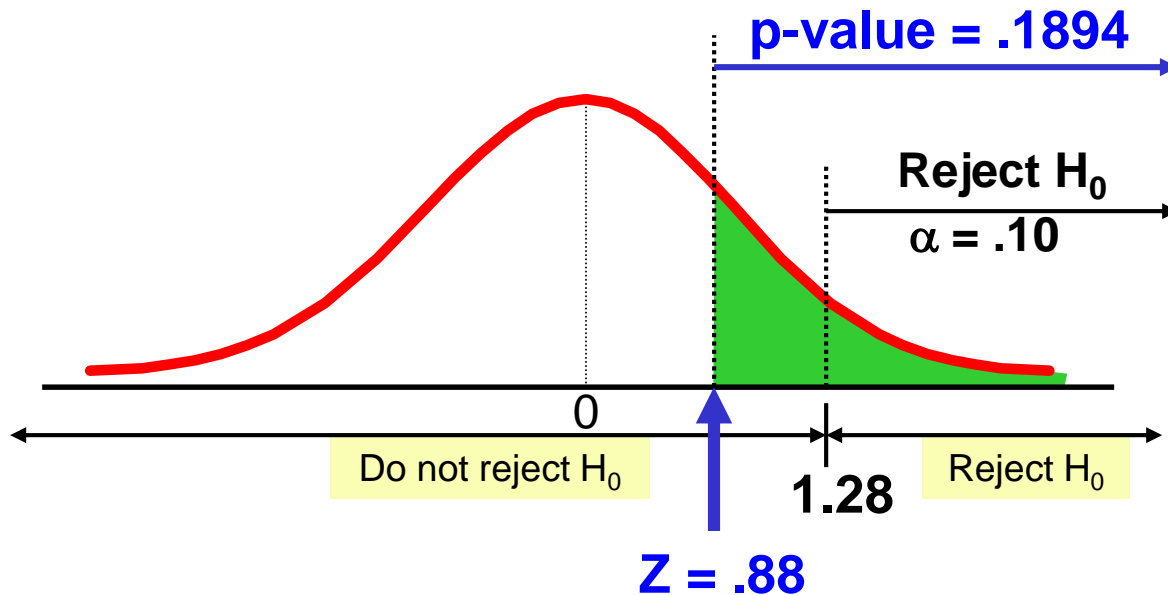
i.e.: there is not sufficient evidence that the mean bill is over \$52



Example: p-Value Solution

(continued)

Calculate the p-value and compare to α
(assuming that $\mu = 52.0$)



$$P(\bar{x} \geq 53.1 | \mu = 52.0)$$

$$= P\left(z \geq \frac{53.1 - 52.0}{10/\sqrt{64}}\right)$$

$$= P(z \geq 0.88) = 1 - .8106$$

$$= .1894$$

Do not reject H_0 since p-value = .1894 > $\alpha = .10$

One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

$$H_0: \mu \leq 3$$

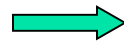
$$H_1: \mu > 3$$



This is an **upper**-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

$$H_0: \mu \geq 3$$

$$H_1: \mu < 3$$



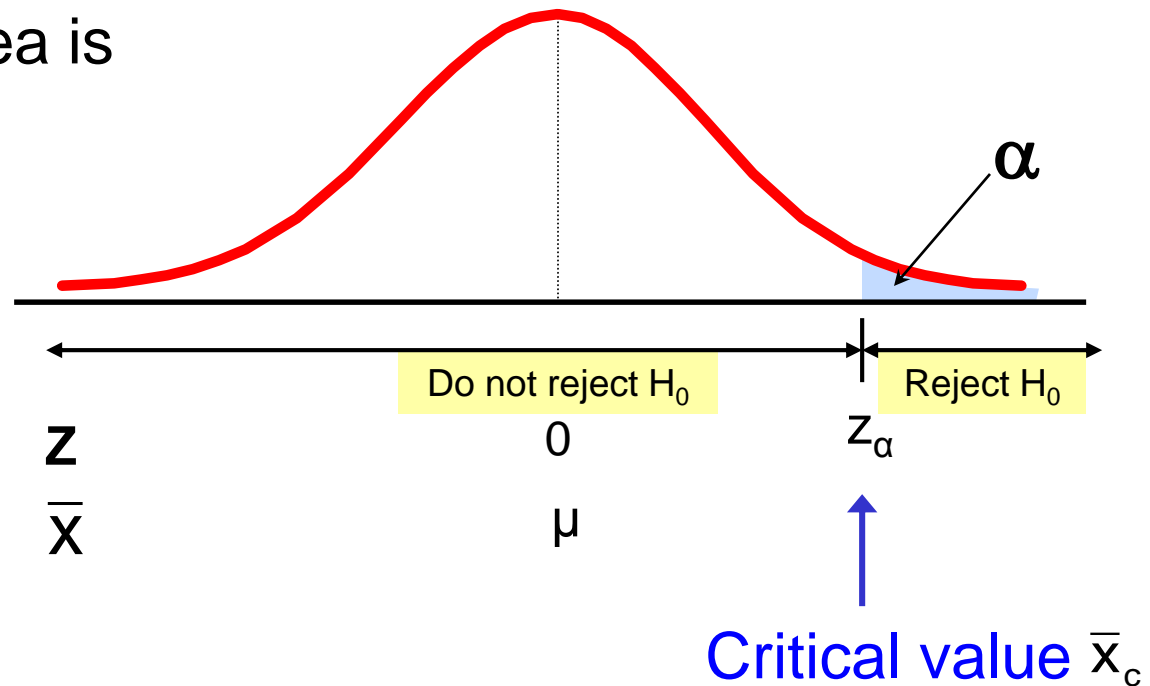
This is a **lower**-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

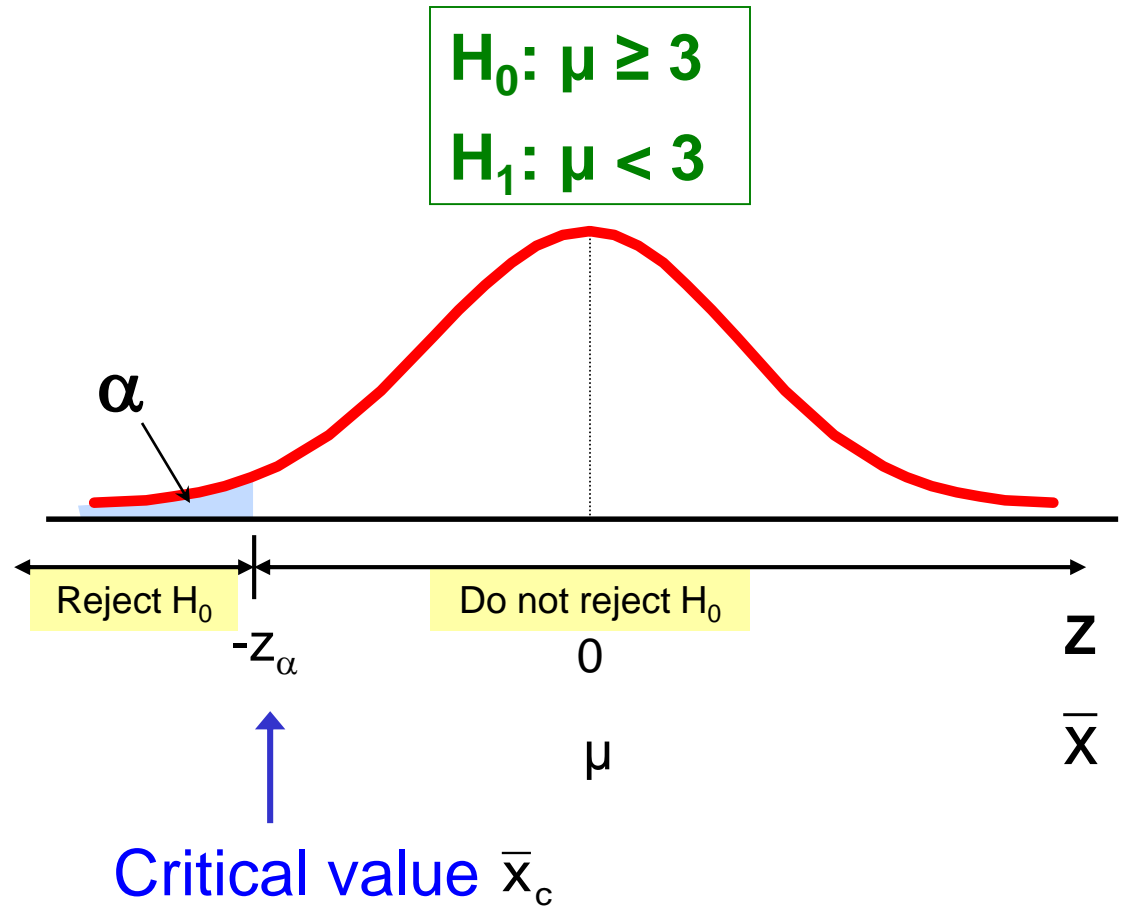
$$H_0: \mu \leq 3$$

$$H_1: \mu > 3$$



Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

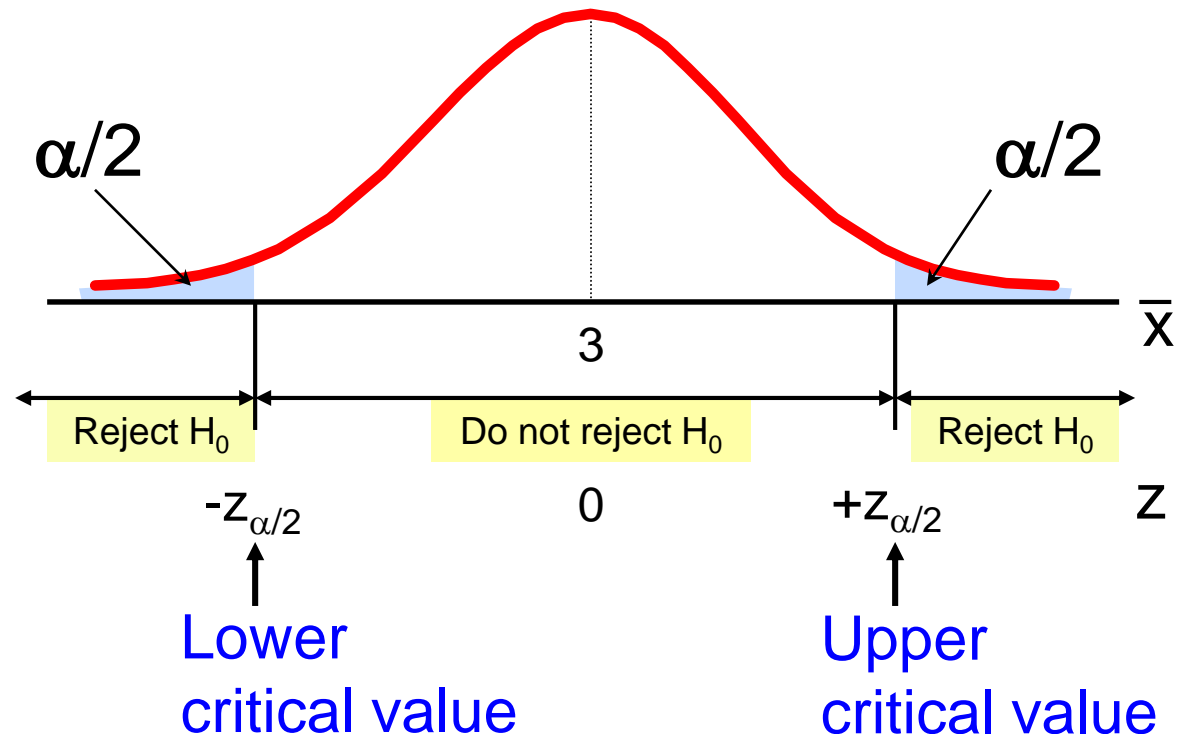


Two-Tail Tests

- In some settings, the alternative hypothesis does not specify a unique direction

$$H_0: \mu = 3$$
$$H_1: \mu \neq 3$$

- There are two critical values, defining the two regions of rejection



Hypothesis Testing Example

**Test the claim that the true mean # of TV sets in US homes is equal to 3.
(Assume $\sigma = 0.8$)**

- State the appropriate null and alternative hypotheses
 - $H_0: \mu = 3$, $H_1: \mu \neq 3$ (This is a two tailed test)
- Specify the desired level of significance
 - Suppose that $\alpha = .05$ is chosen for this test
- Choose a sample size
 - Suppose a sample of size $n = 100$ is selected



Hypothesis Testing Example

(continued)

- Determine the appropriate technique
 - σ is known so this is a z test
- Set up the critical values
 - For $\alpha = .05$ the critical z values are ± 1.96
- Collect the data and compute the test statistic
 - Suppose the sample results are
 $n = 100$, $\bar{X} = 2.84$ ($\sigma = 0.8$ is assumed known)

So the test statistic is:

$$z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

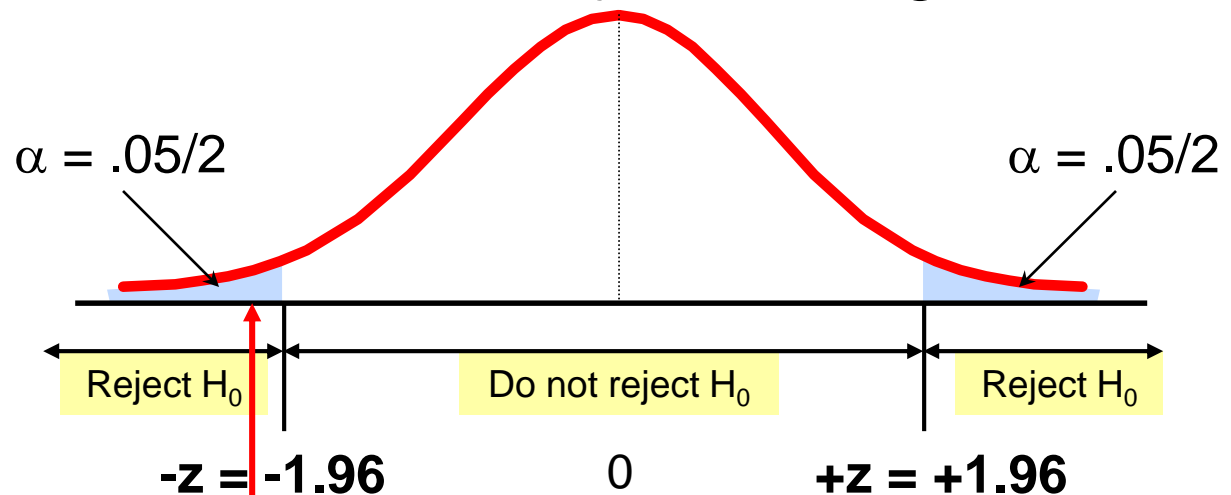


Hypothesis Testing Example

(continued)

- Is the test statistic in the rejection region?

Reject H_0 if
 $z < -1.96$ or
 $z > 1.96$;
otherwise
do not
reject H_0



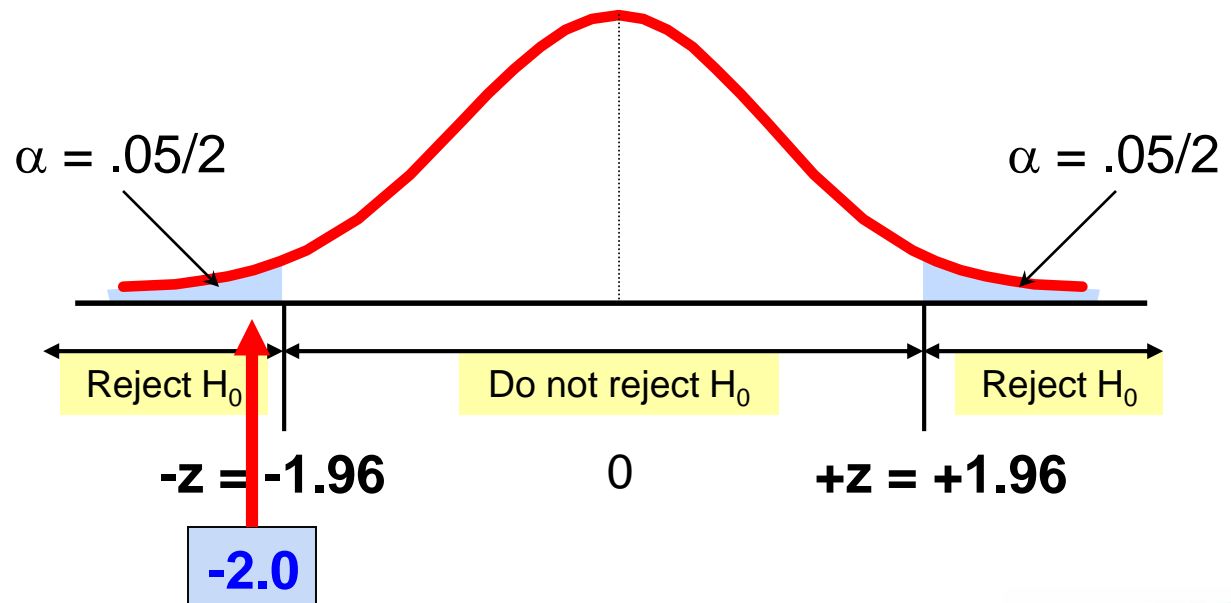
Here, $z = -2.0 < -1.96$, so the test statistic is in the rejection region



Hypothesis Testing Example

(continued)

- Reach a decision and interpret the result



Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TVs in US homes is not equal to 3



Example: p-Value

- **Example:** How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is $\mu = 3.0$?

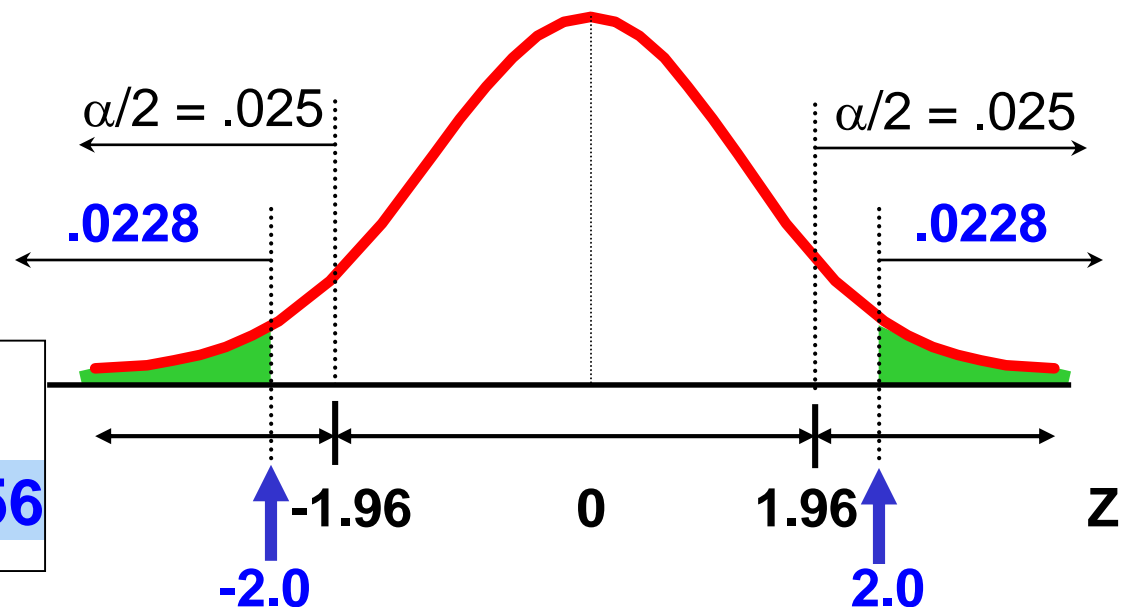
$\bar{x} = 2.84$ is translated to a z score of $z = -2.0$

$$P(z < -2.0) = .0228$$

$$P(z > 2.0) = .0228$$

p-value

$$= .0228 + .0228 = .0456$$



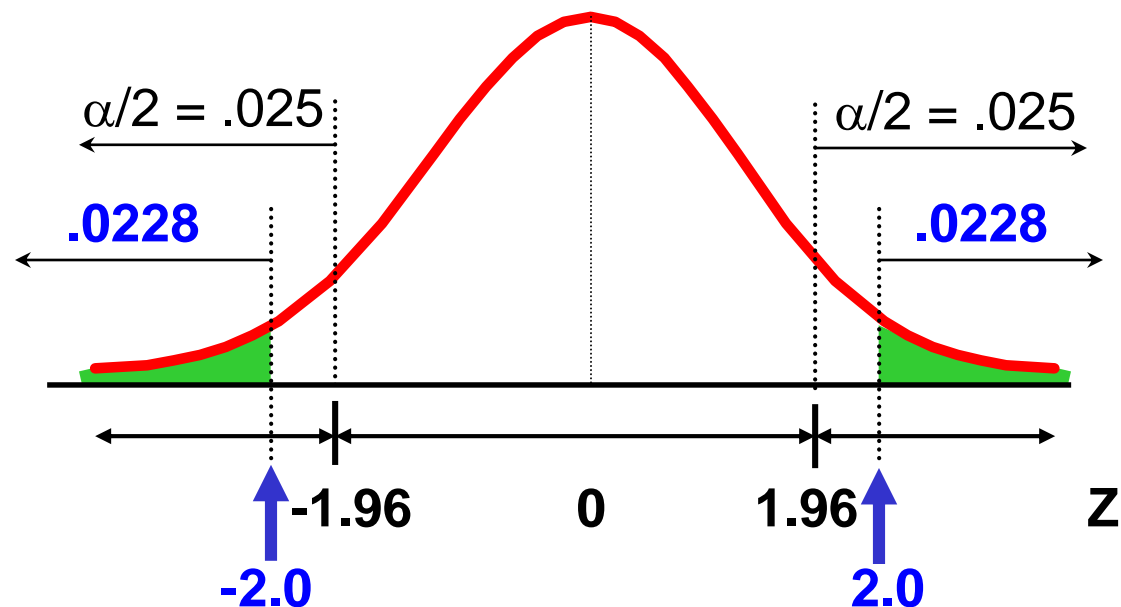
Example: p-Value

(continued)

- Compare the p-value with α
 - If p-value $< \alpha$, reject H_0
 - If p-value $\geq \alpha$, do not reject H_0

Here: p-value = .0456
 $\alpha = .05$

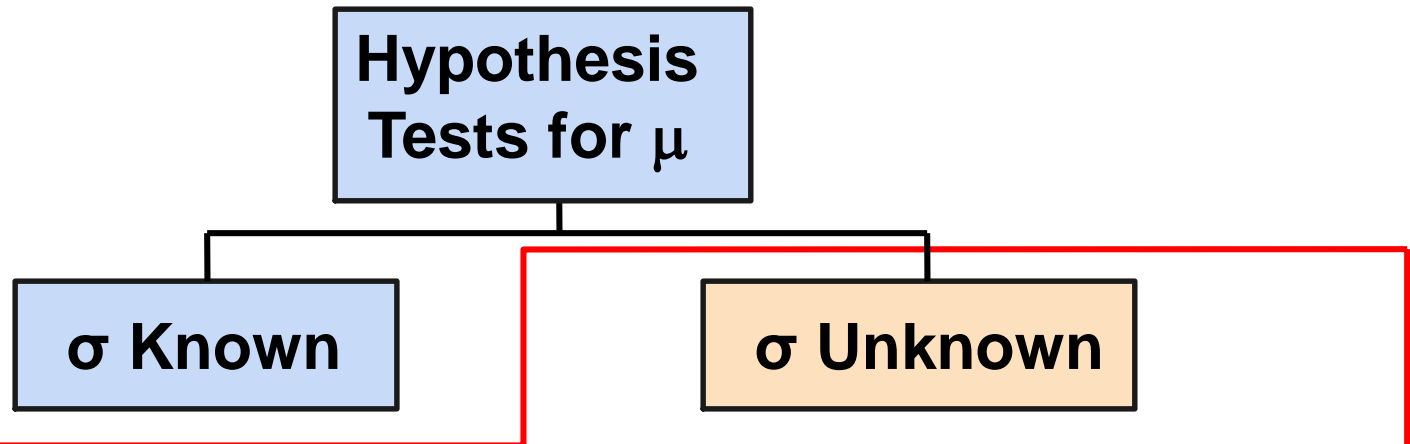
Since $.0456 < .05$, we
reject the null
hypothesis



Tests of the Mean of a Normal Population (σ Unknown)

9.3

- Convert sample result (\bar{x}) to a **t test statistic**



Consider the test

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The **decision rule** is:

$$\text{Reject } H_0 \text{ if } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1, \alpha}$$

Tests of the Mean of a Normal Population (σ Unknown)

(continued)

- For a two-tailed test:

Consider the test

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu \neq \mu_0$$

(Assume the population is normal, and the population variance is unknown)

The **decision rule** is:

Reject H_0 if

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} < -t_{n-1, \alpha/2}$$

or if

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1, \alpha/2}$$

Example: Two-Tail Test (σ Unknown)

The average cost of a hotel room in Chicago is said to be \$168 per night. A random sample of 25 hotels resulted in $\bar{x} = \$172.50$ and $s = \$15.40$. Test at the $\alpha = 0.05$ level.

(Assume the population distribution is normal)



$$H_0: \mu = 168$$

$$H_1: \mu \neq 168$$

Example Solution: Two-Tail Test

$$H_0: \mu = 168$$

$$H_1: \mu \neq 168$$

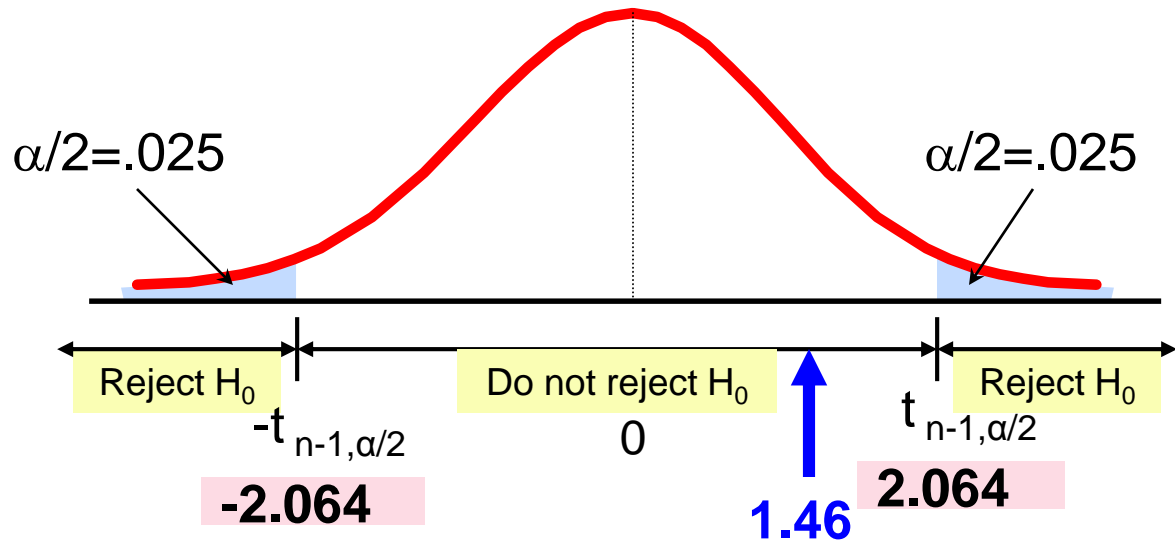
- $\alpha = 0.05$

- $n = 25$

- σ is unknown, so use a **t statistic**

- Critical Value:**

$$t_{24, .025} = \pm 2.064$$



$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

Do not reject H_0 : not sufficient evidence that true mean cost is different than \$168

Tests of the Population Proportion

- Involves **categorical variables**
- Two possible outcomes
 - “Success” (a certain characteristic is present)
 - “Failure” (the characteristic is not present)
- Fraction or proportion of the population in the “success” category is denoted by P
- Assume sample size is large



Tests of the Population Proportion

(continued)

- The sample proportion in the success category is denoted by \hat{p}

- $$\hat{p} = \frac{x}{n} = \frac{\text{number of successes in sample}}{\text{sample size}}$$

- When $nP(1 - P) > 5$, \hat{p} can be approximated by a normal distribution with mean and standard deviation

- $$\mu_{\hat{p}} = P$$

$$\sigma_{\hat{p}} = \sqrt{\frac{P(1-P)}{n}}$$

Hypothesis Tests for Proportions

- The sampling distribution of \hat{p} is approximately normal, so the test statistic is a z value:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1 - P_0)}{n}}}$$

Hypothesis Tests for P

$$nP(1 - P) > 5$$

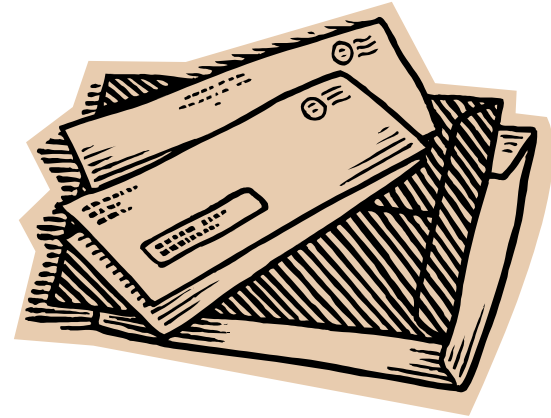
$$H_0 : P = P_0$$
$$H_1 : P > P_0$$

$$nP(1 - P) < 5$$

Not discussed
in this chapter

Example: Z Test for Proportion

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the $\alpha = .05$ significance level.



Check:

Our approximation for P is

$$\hat{p} = 25/500 = .05$$

$$\begin{aligned} nP(1 - P) &= (500)(.05)(.95) \\ &= 23.75 > 5 \end{aligned}$$



Z Test for Proportion: Solution

$$H_0: P = .08$$

$$H_1: P \neq .08$$

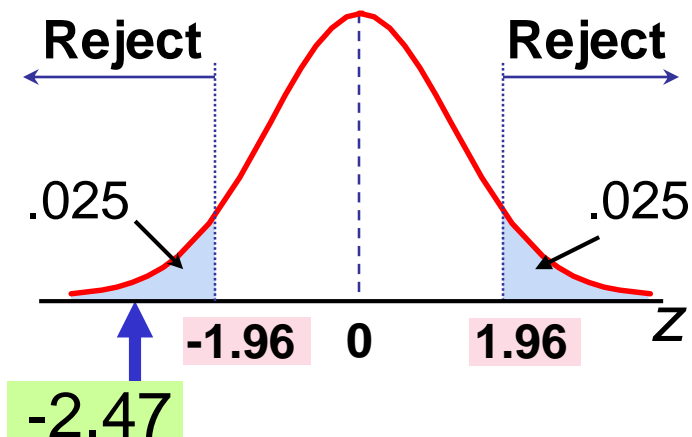
$$\alpha = .05$$

$$n = 500, \hat{p} = .05$$

Test Statistic:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Critical Values: ± 1.96



Decision:

Reject H_0 at $\alpha = .05$

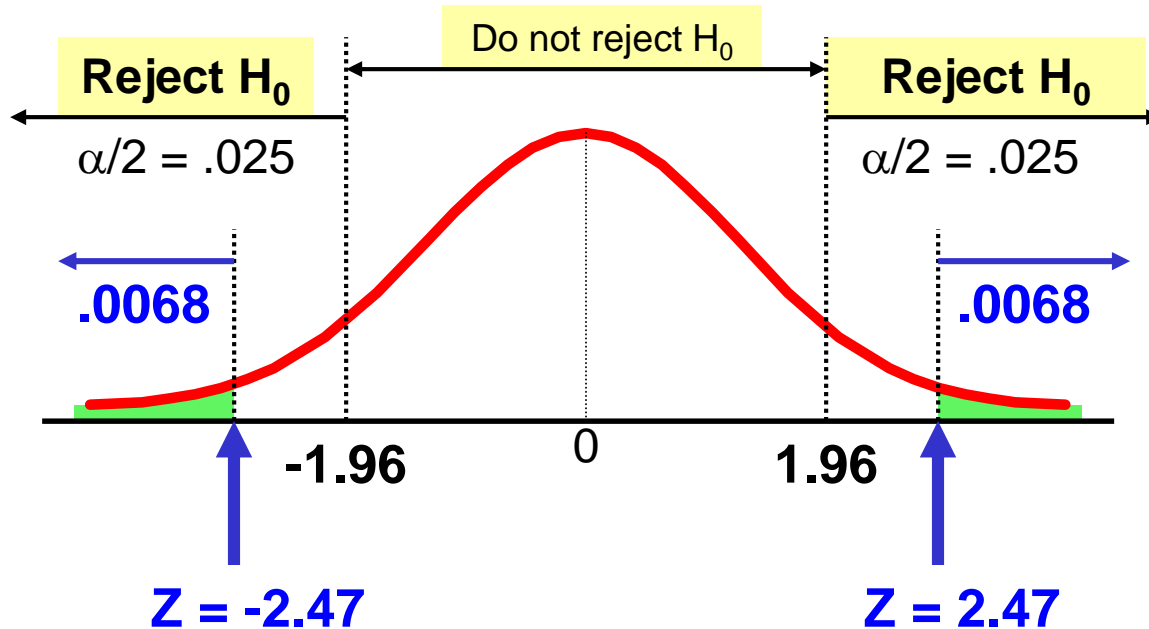
Conclusion:

There is sufficient evidence to reject the company's claim of 8% response rate.

p-Value Solution

(continued)

Calculate the p-value and compare to α
(For a two sided test the p-value is always two sided)



p-value = .0136:

$$P(Z \leq -2.47) + P(Z \geq 2.47) \\ = 2(.0068) = 0.0136$$

Reject H_0 since p-value = .0136 < α = .05

Assessing the Power of a Test

- Recall the possible hypothesis test outcomes:

Key:
Outcome
(Probability)

| | Actual Situation | |
|---------------------|--------------------------------------|-------------------------------------|
| Decision | H_0 True | H_0 False |
| Do Not Reject H_0 | Correct Decision ($1 - \alpha$) | Type II Error (β) |
| Reject H_0 | Type I Error (α) | Correct Decision ($1 - \beta$) |

- β denotes the probability of Type II Error
- $1 - \beta$ is defined as the **power of the test**

Power = $1 - \beta$ = the probability that a false null hypothesis is rejected



Type II Error

Assume the population is normal and the population variance is known. Consider the test

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu > \mu_0$$

The decision rule is:

$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha \quad \text{or} \quad \text{Reject } H_0 \text{ if } \bar{x} > \bar{x}_c = \mu_0 + z_\alpha \sigma / \sqrt{n}$$

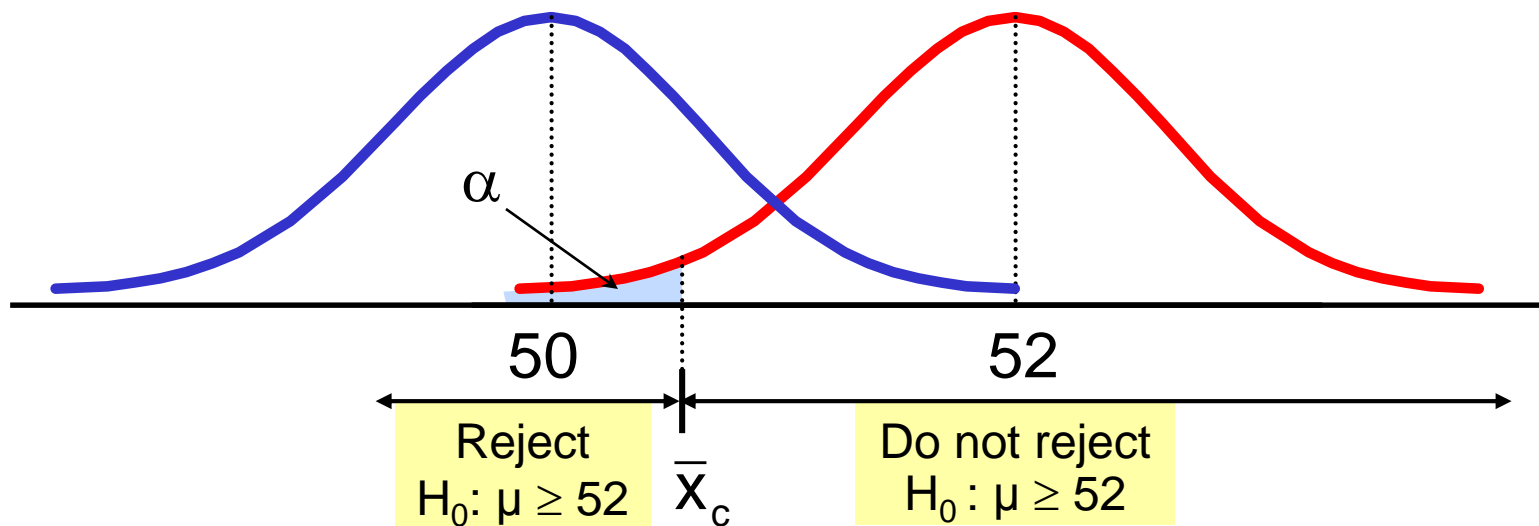
If the null hypothesis is false and the true mean is μ^* , then the probability of type II error is

$$\beta = P(\bar{x} < \bar{x}_c \mid \mu = \mu^*) = P\left(z < \frac{\bar{x}_c - \mu^*}{\sigma / \sqrt{n}}\right)$$

Type II Error Example

- Type II error is the probability of failing to reject a false H_0

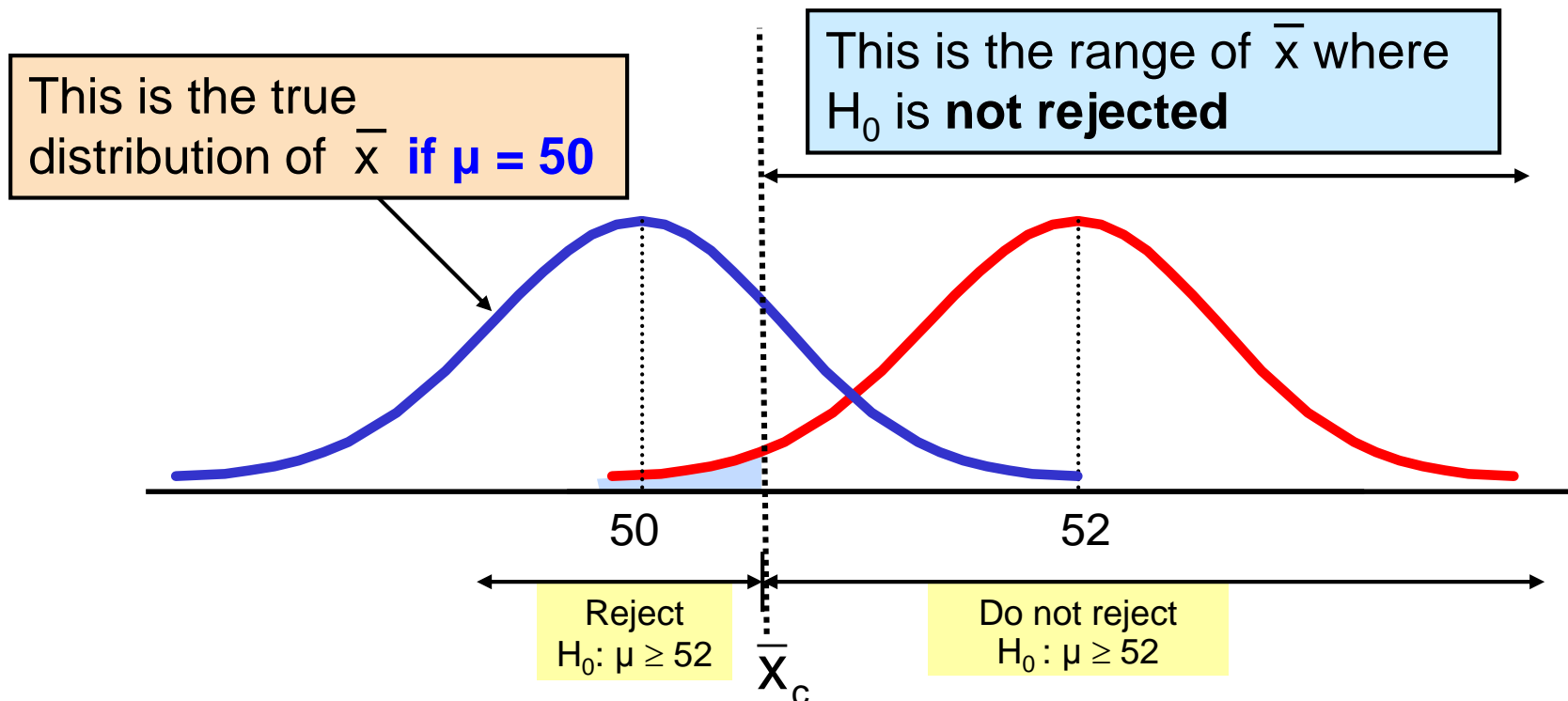
Suppose we fail to reject $H_0: \mu \geq 52$
when in fact the true mean is $\mu^* = 50$



Type II Error Example

(continued)

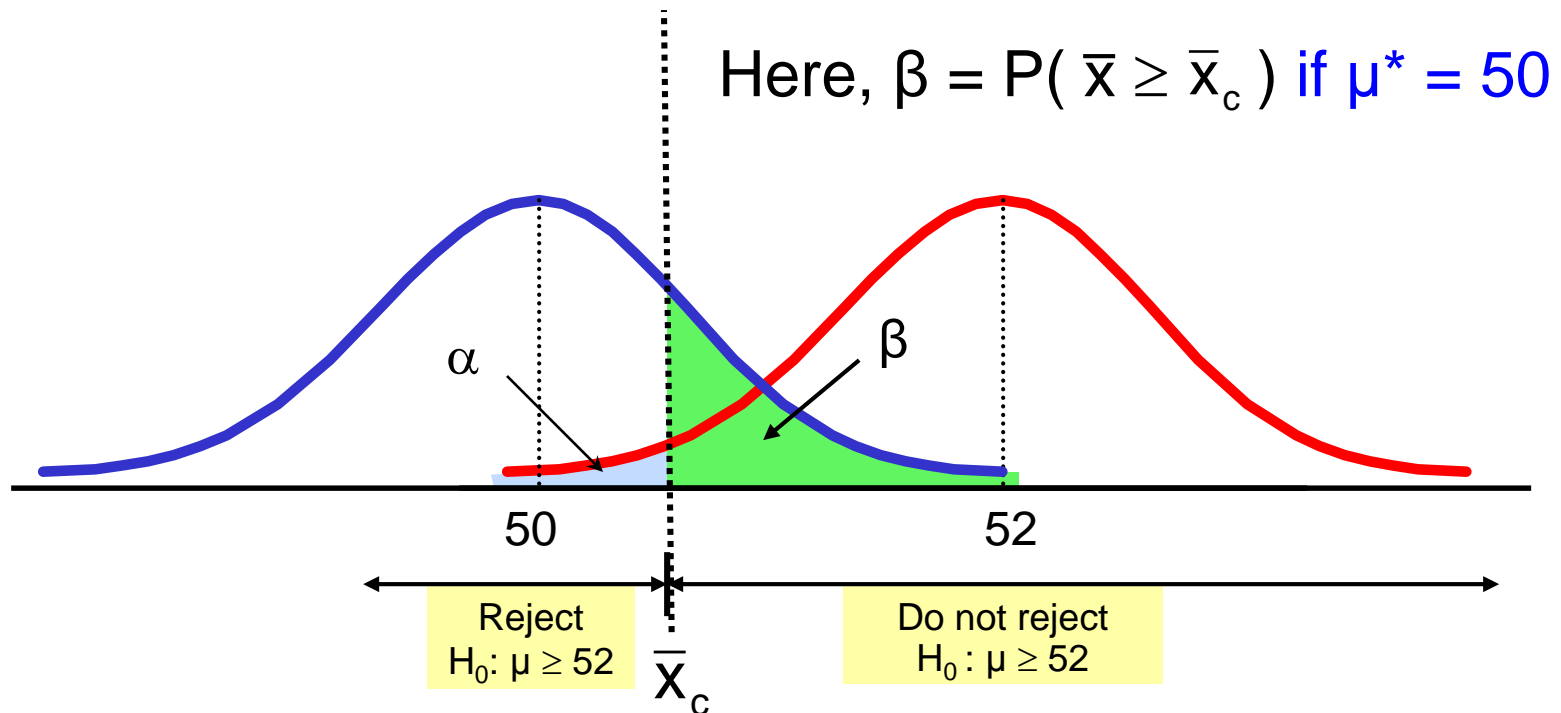
- Suppose we do not reject $H_0: \mu \geq 52$ when in fact the true mean is $\mu^* = 50$



Type II Error Example

(continued)

- Suppose we do not reject $H_0: \mu \geq 52$ when in fact the true mean is $\mu^* = 50$



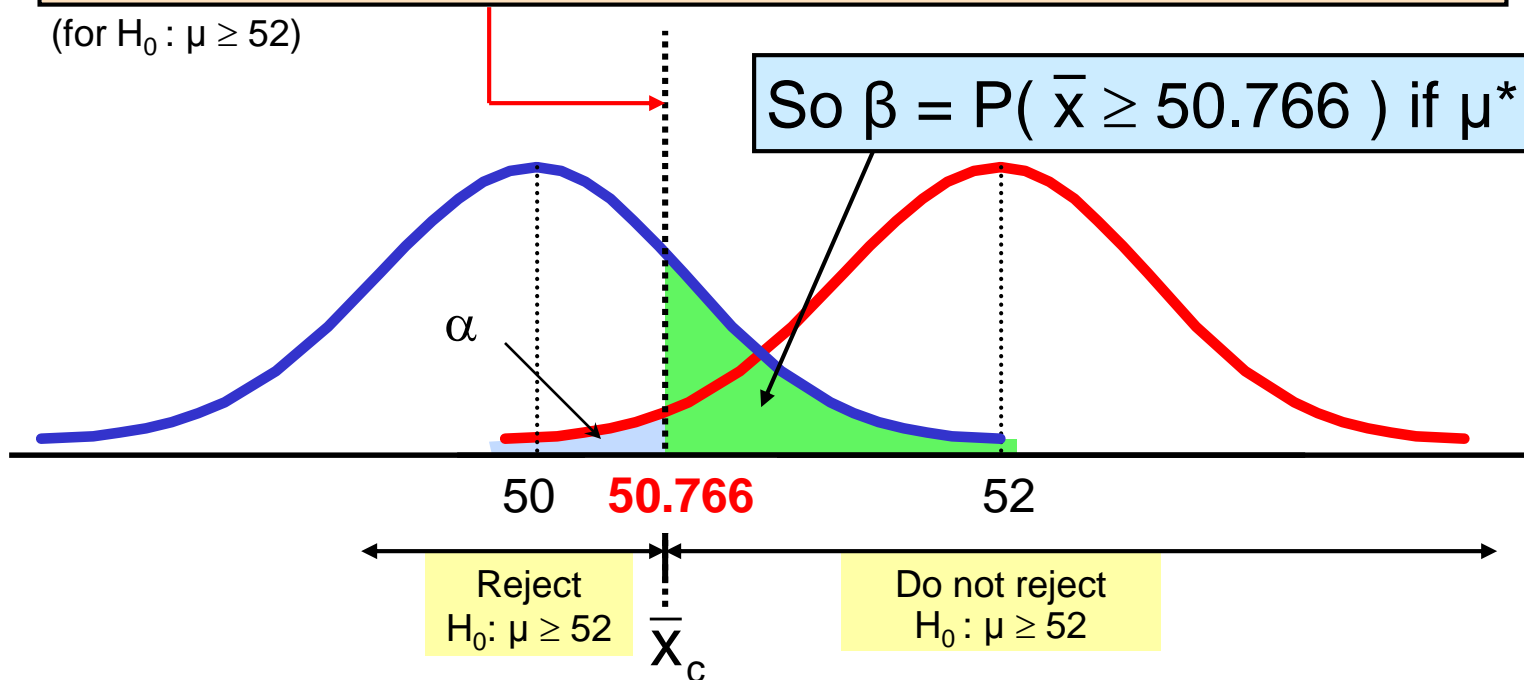
Calculating β

- Suppose $n = 64$, $\sigma = 6$, and $\alpha = .05$

$$\bar{X}_c = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 52 - 1.645 \frac{6}{\sqrt{64}} = 50.766$$

(for $H_0: \mu \geq 52$)

So $\beta = P(\bar{x} \geq 50.766)$ if $\mu^* = 50$

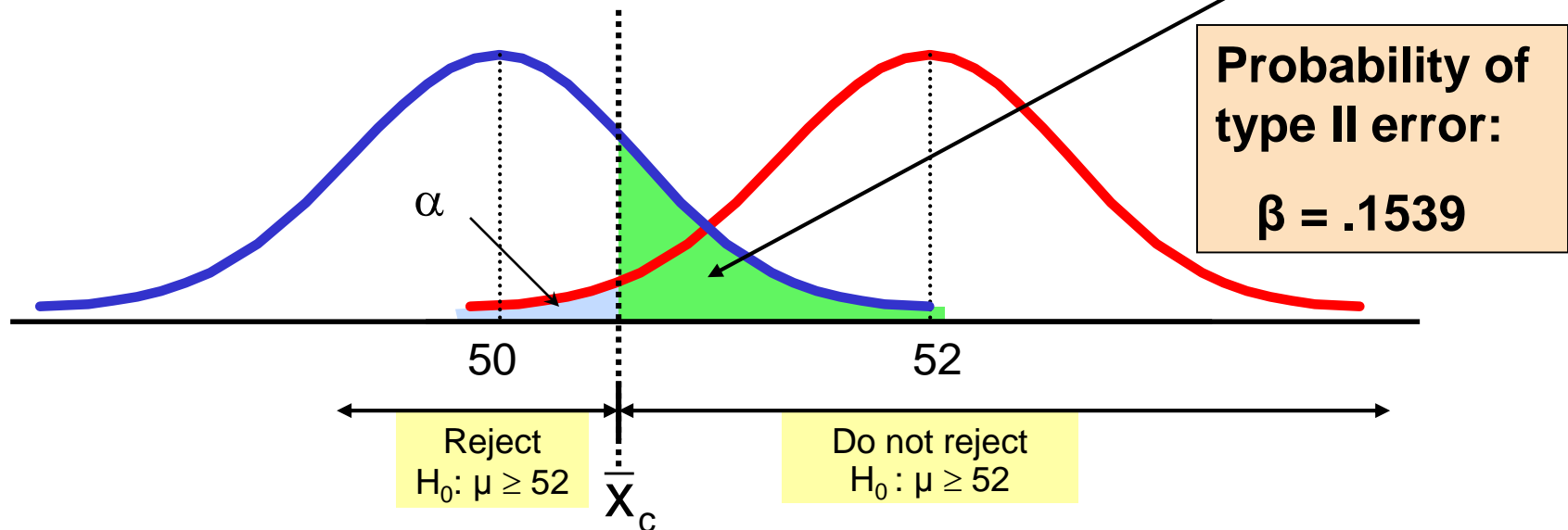


Calculating β

(continued)

- Suppose $n = 64$, $\sigma = 6$, and $\alpha = .05$

$$P(\bar{x} \geq 50.766 | \mu^* = 50) = P\left(z \geq \frac{50.766 - 50}{\frac{6}{\sqrt{64}}}\right) = P(z \geq 1.02) = .5 - .3461 = .1539$$



Power of the Test Example

If the true mean is $\mu^* = 50$,

- The probability of Type II Error = $\beta = 0.1539$
- The power of the test = $1 - \beta = 1 - 0.1539 = 0.8461$

Key:
Outcome
(Probability)

| | Actual Situation | |
|---------------------|--|---|
| Decision | H_0 True | H_0 False |
| Do Not Reject H_0 | Correct Decision $1 - \alpha = 0.95$ | Type II Error $\beta = 0.1539$ |
| Reject H_0 | Type I Error $\alpha = 0.05$ | Correct Decision $1 - \beta = 0.8461$ |

(The value of β and the power will be different for each μ^*)

Tests of the Variance of a Normal Distribution

9.6

- **Goal:** Test hypotheses about the population variance, σ^2 (e.g., $H_0: \sigma^2 = \sigma_0^2$)
- If the population is normally distributed,

$$\chi_{n-1}^2 = \frac{(n-1)s^2}{\sigma^2}$$

has a chi-square distribution with $(n - 1)$ degrees of freedom

Tests of the Variance of a Normal Distribution

(continued)

The test statistic for hypothesis tests about one population variance is

$$\chi_{n-1}^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

Decision Rules: Variance

Population variance

Lower-tail test:

$$H_0: \sigma^2 \geq \sigma_0^2$$

$$H_1: \sigma^2 < \sigma_0^2$$

Upper-tail test:

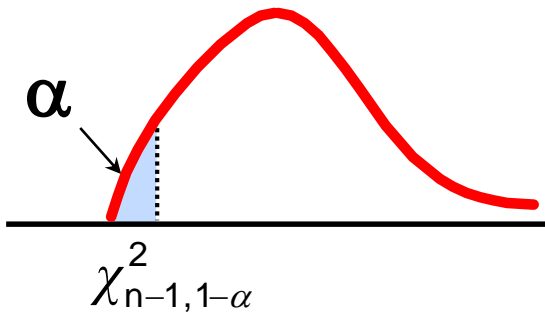
$$H_0: \sigma^2 \leq \sigma_0^2$$

$$H_1: \sigma^2 > \sigma_0^2$$

Two-tail test:

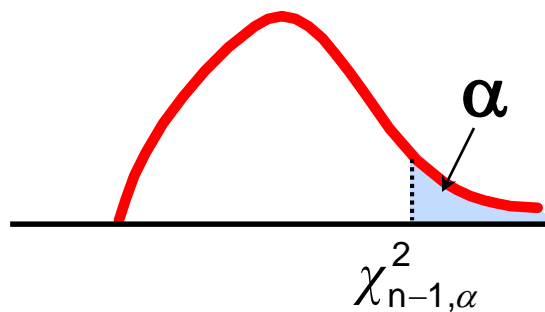
$$H_0: \sigma^2 = \sigma_0^2$$

$$H_1: \sigma^2 \neq \sigma_0^2$$



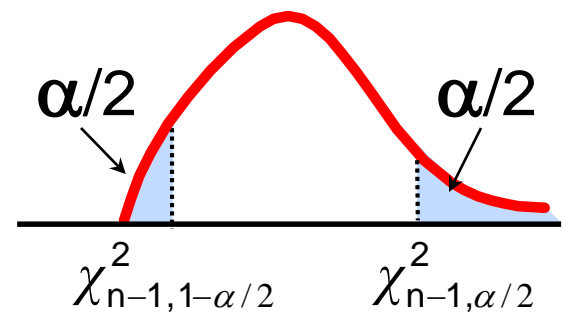
Reject H_0 if

$$\chi_{n-1}^2 < \chi_{n-1, 1-\alpha}^2$$



Reject H_0 if

$$\chi_{n-1}^2 > \chi_{n-1, \alpha}^2$$



Reject H_0 if

or $\chi_{n-1}^2 > \chi_{n-1, \alpha/2}^2$
 $\chi_{n-1}^2 < \chi_{n-1, 1-\alpha/2}^2$



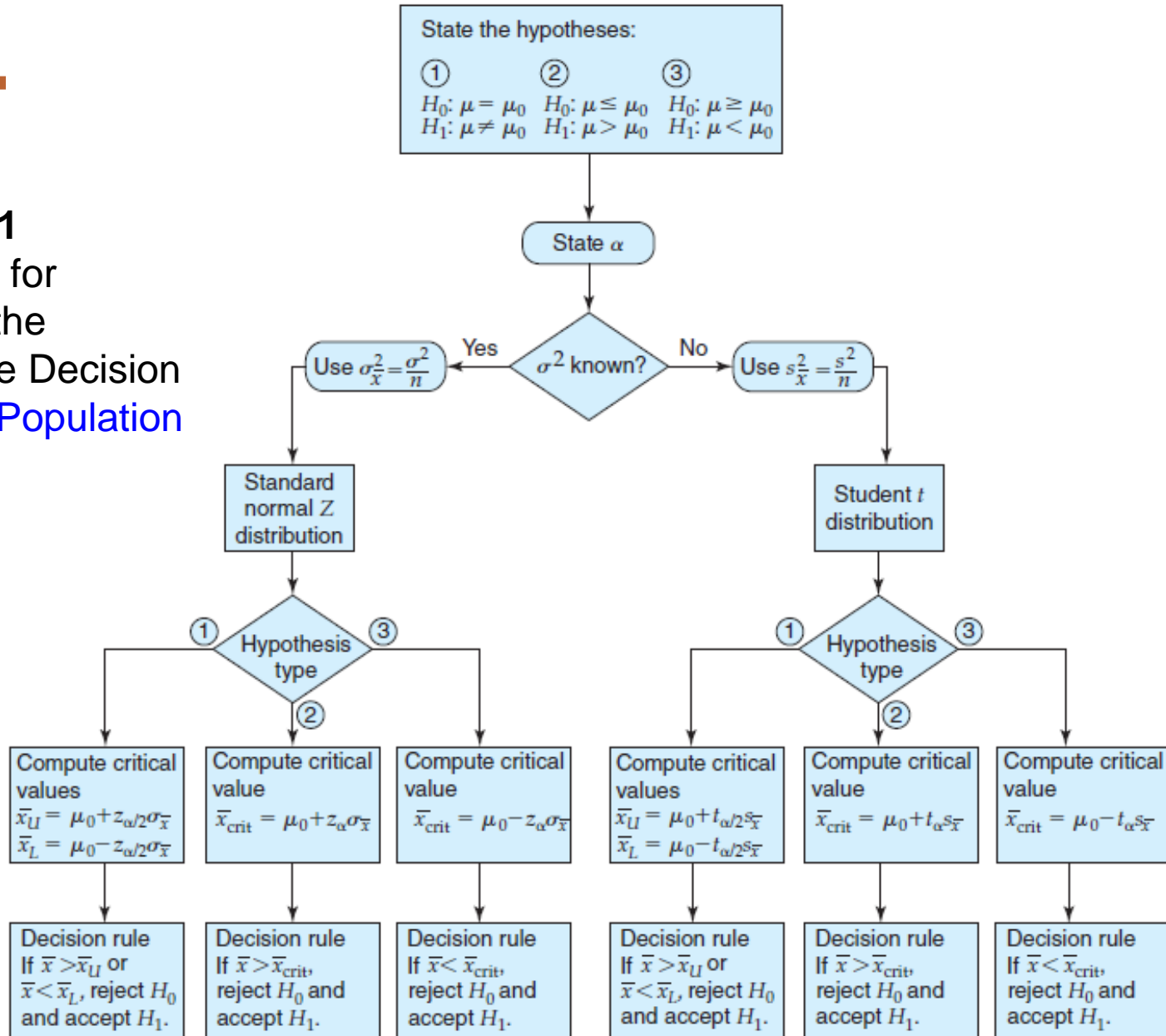
Chapter Summary

- Addressed hypothesis testing methodology
- Performed z Test for the mean (σ known)
- Discussed critical value and p-value approaches to hypothesis testing
- Performed one-tail and two-tail tests
- Performed t test for the mean (σ unknown)
- Performed z test for the proportion
- Discussed type II error and power of the test
- Performed a hypothesis test for the variance (χ^2)

Appendix: Guidelines for Decision Rule

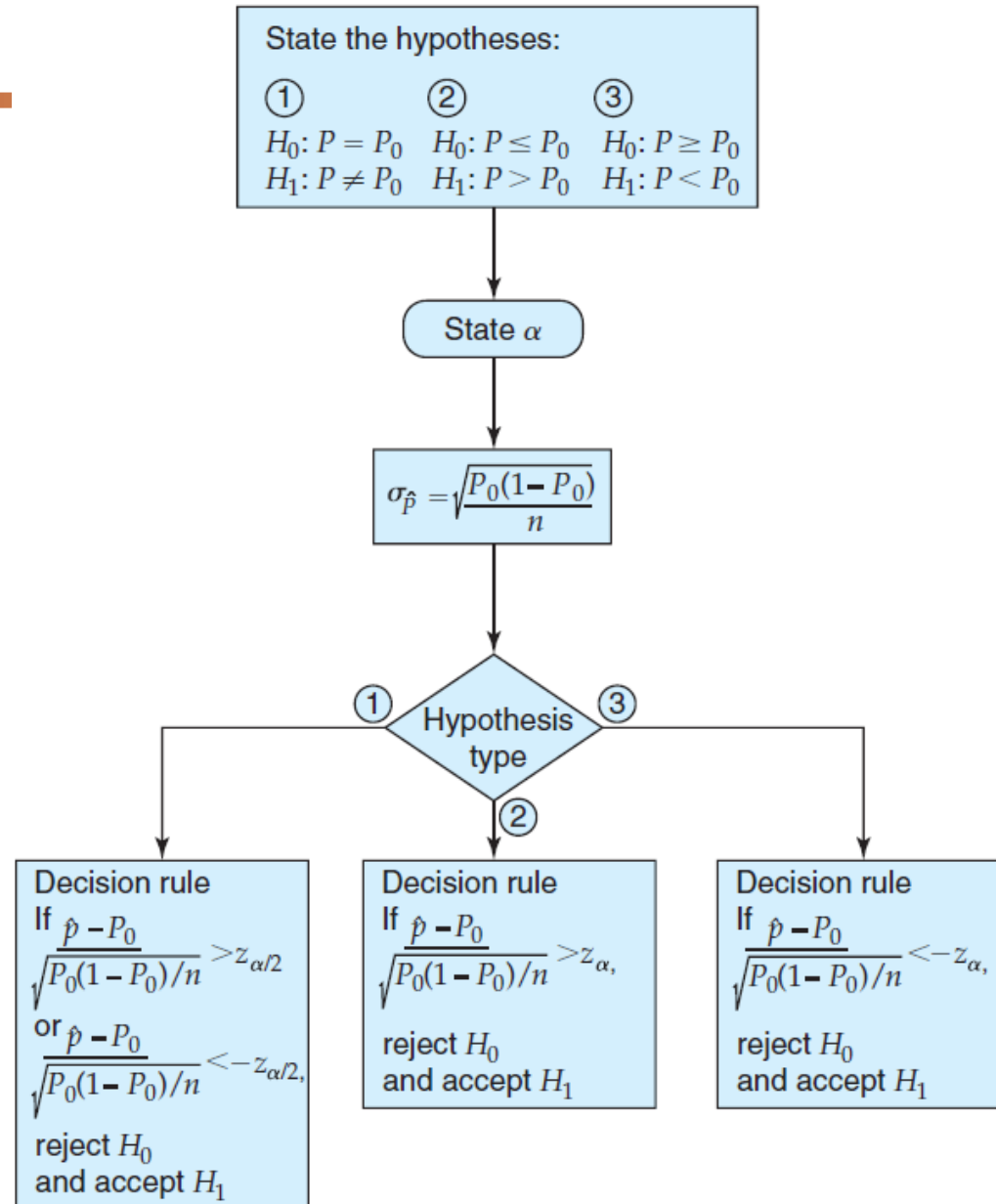



Figure 9.11
Guidelines for
Choosing the
Appropriate Decision
Rule for a **Population
Mean**




Appendix: Guidelines for Decision Rule

Figure 9.12
Guidelines for
Choosing the
Appropriate Decision
Rule for a **Population
Proportion**







This work is protected by United States copyright laws and is provided solely for the use of instructors in teaching their courses and assessing student learning. Dissemination or sale of any part of this work (including on the World Wide Web) will destroy the integrity of the work and is not permitted. The work and materials from it should never be made available to students except by instructors using the accompanying text in their classes. All recipients of this work are expected to abide by these restrictions and to honor the intended pedagogical purposes and the needs of other instructors who rely on these materials.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher.

Printed in the United States of America.