Statistics for Business and Economics
8th Edition

Chapter 5

Continuous Random Variables and Probability Distributions

Chapter Goals

After completing this chapter, you should be able to:

- Explain the difference between a discrete and a continuous random variable
- Describe the characteristics of the uniform and normal distributions
- Translate normal distribution problems into standardized normal distribution problems
- Find probabilities using a normal distribution table

Chapter Goals

(continued)

After completing this chapter, you should be able to:

- Evaluate the normality assumption
- Use the normal approximation to the binomial distribution
- Recognize when to apply the exponential distribution
- Explain jointly distributed variables and linear combinations of random variables
- Explain examples to Financial Investment Portfolios

Probability Distributions

Probability Distributions

- Ch. 4 — **Discrete** Probability Distributions
  - Binomial
  - Hypergeometric
  - Poisson
- Ch. 5 — **Continuous** Probability Distributions
  - Uniform
  - Normal
  - Exponential

5.1 Continuous Random Variables

- A continuous random variable is a variable that can assume any value in an interval
  - thickness of an item
  - time required to complete a task
  - temperature of a solution
  - height, in inches
- These can potentially take on any value, depending only on the ability to measure accurately.

Cumulative Distribution Function

- The cumulative distribution function, F(x), for a continuous random variable X expresses the probability that X does not exceed the value of x

$$F(x) = P(X \le x)$$

- Let a and b be two possible values of X, with a < b. The probability that X lies between a and b is

$$P(a < X < b) = F(b) - F(a)$$

Probability Density Function

The probability density function, f(x), of random variable X has the following properties:

1. $f(x) > 0$ for all values of x
2. The area under the probability density function f(x) over all values of the random variable X within its range, is equal to 1.0
3. The probability that X lies between two values is the area under the density function graph between the two values

$$P(a < X < b) = \int_a^b f(x)dx$$

Probability Density Function

(continued)

The probability density function, f(x), of random variable X has the following properties:

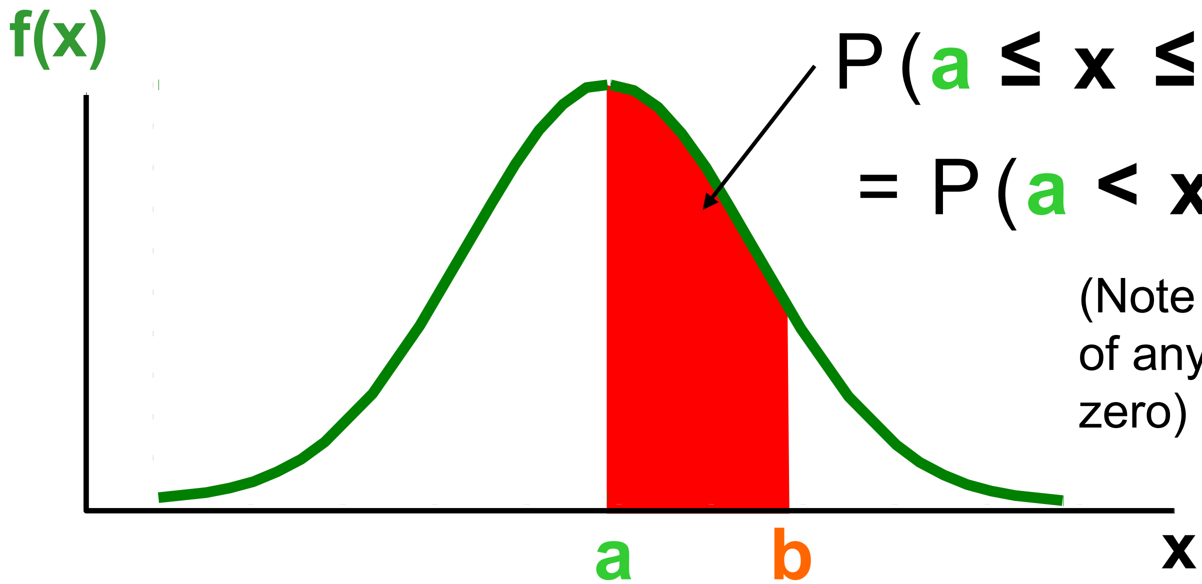
4. The cumulative density function $F(x_0)$ is the area under the probability density function $f(x)$ from the minimum x value up to x_0

$$F(x_0) = \int_{x_m}^{x_0} f(x)dx$$

where x_m is the minimum value of the random variable x

Probability as an Area

Shaded area under the curve is the probability that X is between a and b



$P(a \leq x \leq b)$

$= P(a < x < b)$

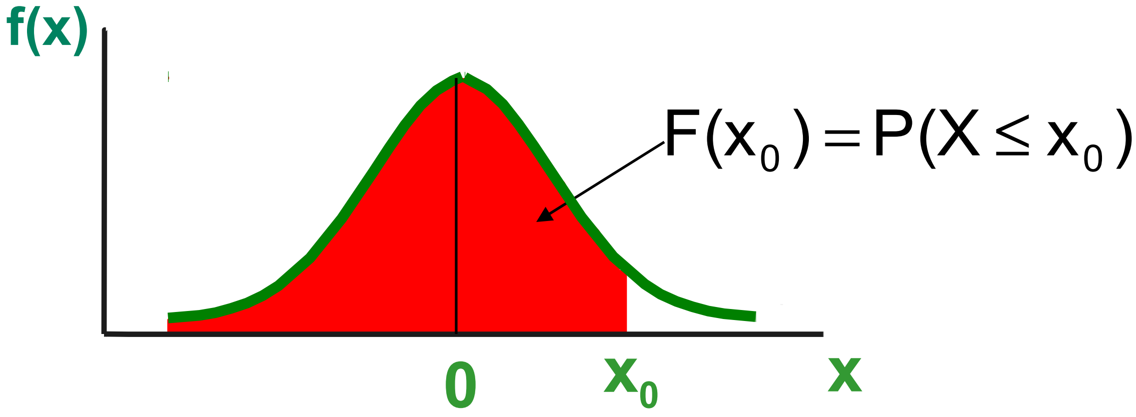
(Note that the probability of any individual value is zero)

Probability as an Area

(continued)

1. The total area under the curve $f(x)$ is 1

2. The area under the curve $f(x)$ to the left of x_0 is $F(x_0)$, where x_0 is any value that the random variable can take.

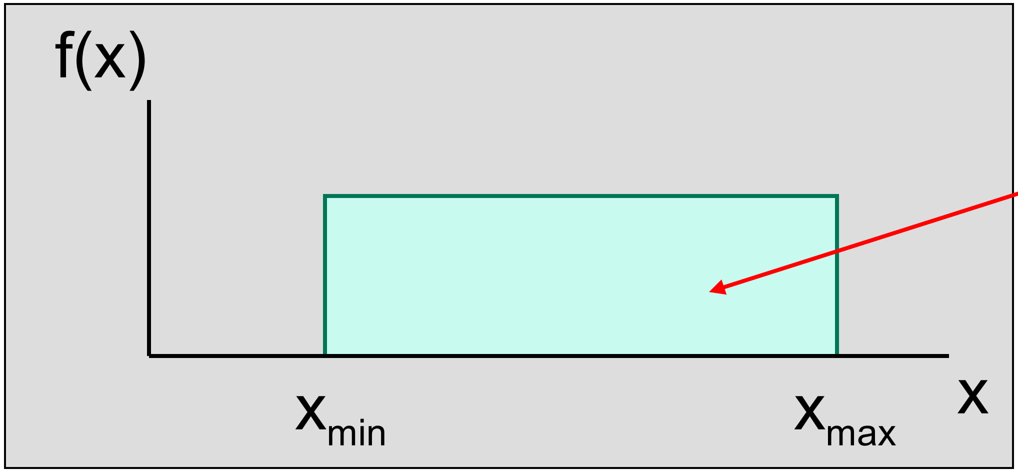


The Uniform Distribution

- **Probability Distributions**
  - **Continuous** Probability Distributions
    - Uniform
    - Normal
    - Exponential

The Uniform Distribution

- The uniform distribution is a probability distribution that has equal probabilities for all equal-width intervals within the range of the random variable



Total area under the uniform probability density function is 1.0

The Uniform Distribution

(continued)

The Continuous Uniform Distribution:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where

f(x) = value of the density function at any x value

a = minimum value of x

b = maximum value of x

5.2 Expectations for Continuous Random Variables

- The mean of X, denoted μ_X, is defined as the expected value of X

$$\mu_X = E[X]$$

- The variance of X, denoted σ_X^2, is defined as the expectation of the squared deviation, $(X - \mu_X)^2$, of a random variable from its mean

$$\sigma_X^2 = E[(X - \mu_X)^2]$$

Mean and Variance of the Uniform Distribution

- The **mean** of a uniform distribution is

$$\mu = \frac{a+b}{2}$$

- The **variance** is

$$\sigma^2 = \frac{(b-a)^2}{12}$$

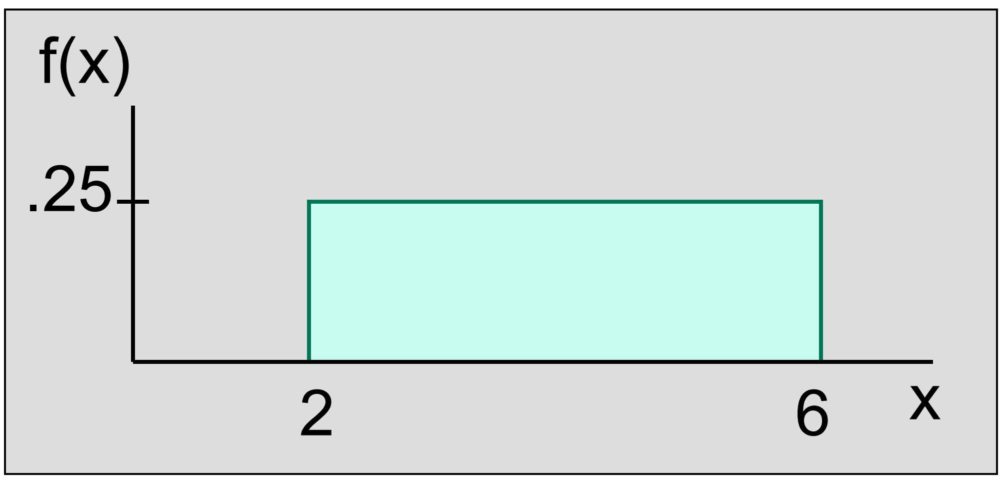
Where a = minimum value of x

b = maximum value of x

Uniform Distribution Example

Example: Uniform probability distribution over the range $2 \le x \le 6$:

$$f(x) = \frac{1}{6-2} = .25 \quad \text{for } 2 \le x \le 6$$



$$\mu = \frac{a+b}{2} = \frac{2+6}{2} = 4$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(6-2)^2}{12} = 1.333$$

Linear Functions of Random Variables

- Let $W = a + bX$, where X has mean μ_X and variance σ_X^2, and a and b are constants
- Then the mean of W is

$$\mu_W = E[a + bX] = a + b\mu_X$$

- the variance is

$$\sigma_W^2 = \mathrm{Var}[a + bX] = b^2\sigma_X^2$$

- the standard deviation of W is

$$\sigma_W = |b|\sigma_X$$

Linear Functions of Random Variables

(continued)

- An important special case of the result for the linear function of random variable is the standardized random variable

$$Z = \frac{X - \mu_X}{\sigma_X}$$

- which has a mean 0 and variance 1

5.3

The Normal Distribution

- **Probability Distributions**
  - **Continuous** Probability Distributions
    - Uniform
    - Normal
    - Exponential

The Normal Distribution

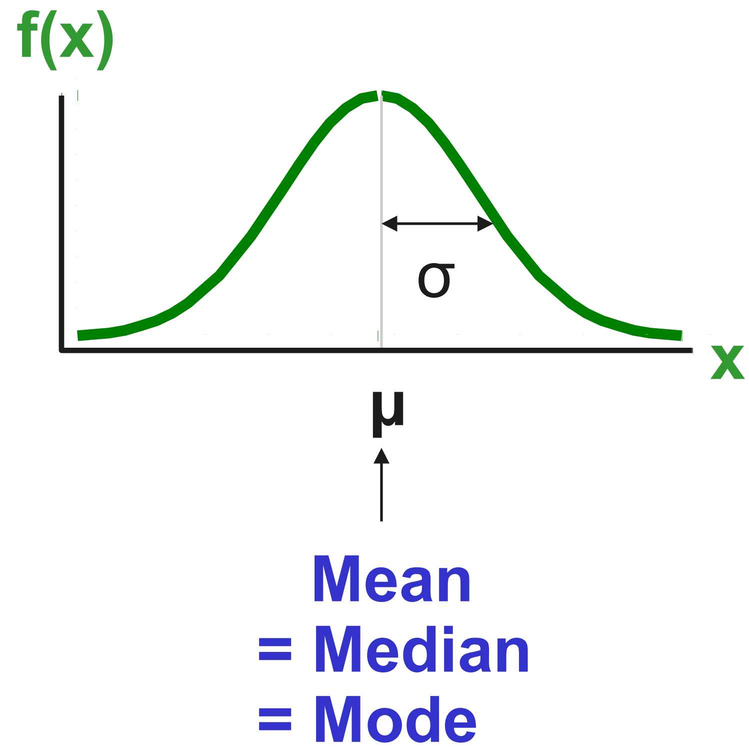
(continued)

- **Bell Shaped**
- **Symmetrical**
- **Mean, Median and Mode are Equal**

Location is determined by the mean, μ

Spread is determined by the standard deviation, σ

The random variable has an infinite theoretical range: + ∞ to − ∞

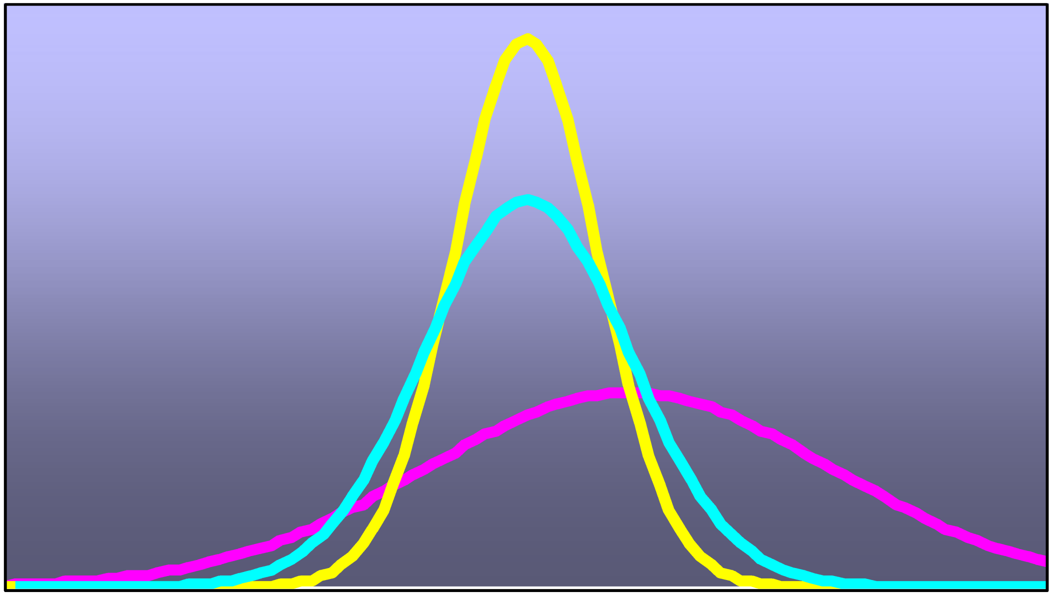


The Normal Distribution

(continued)

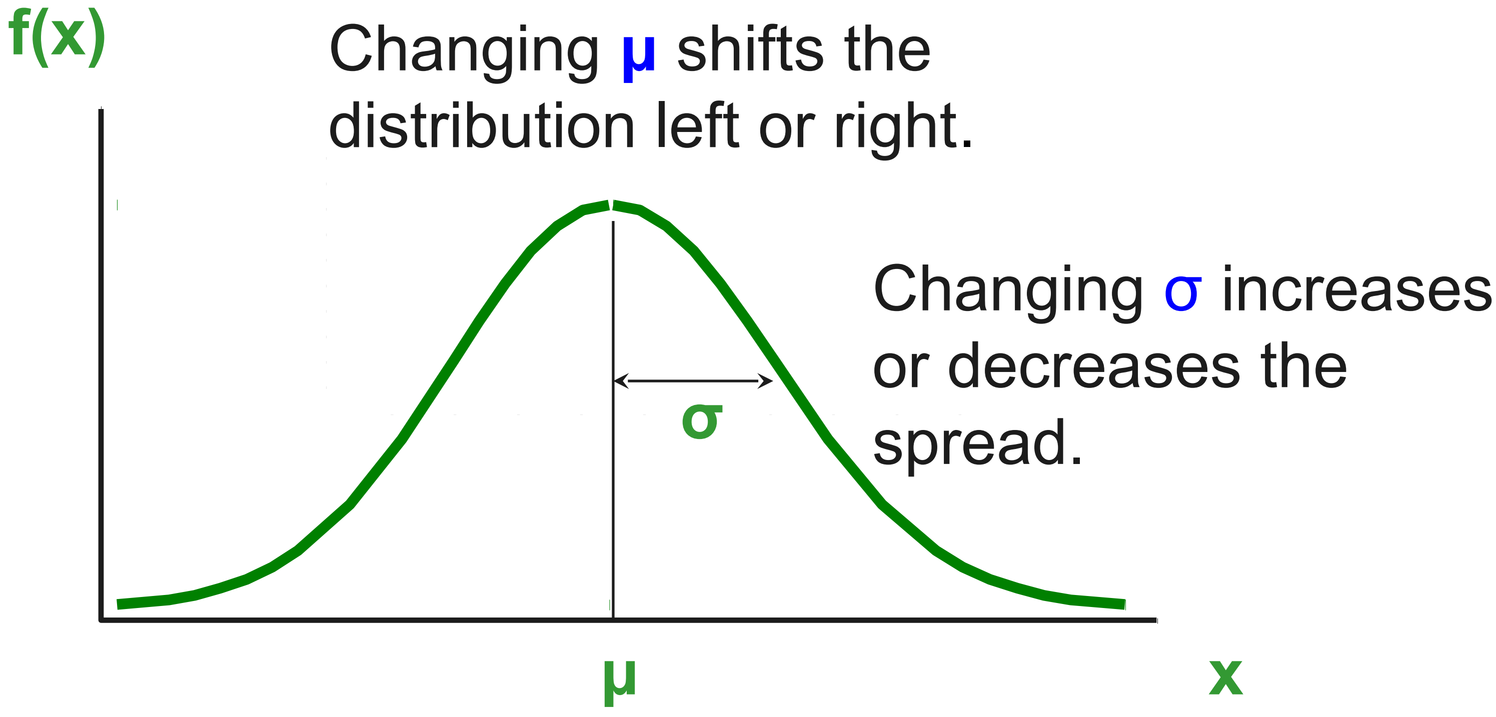
- The normal distribution closely approximates the probability distributions of a wide range of random variables
- Distributions of sample means approach a normal distribution given a "large" sample size
- Computations of probabilities are direct and elegant
- The normal probability distribution has led to good business decisions for a number of applications

Many Normal Distributions



By varying the parameters μ and σ, we obtain different normal distributions

The Normal Distribution Shape



Given the mean μ and variance σ^2 we define the normal distribution using the notation

$$X \sim N(\mu, \sigma^2)$$

The Normal Probability Density Function

- The formula for the normal probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

μ = the population mean

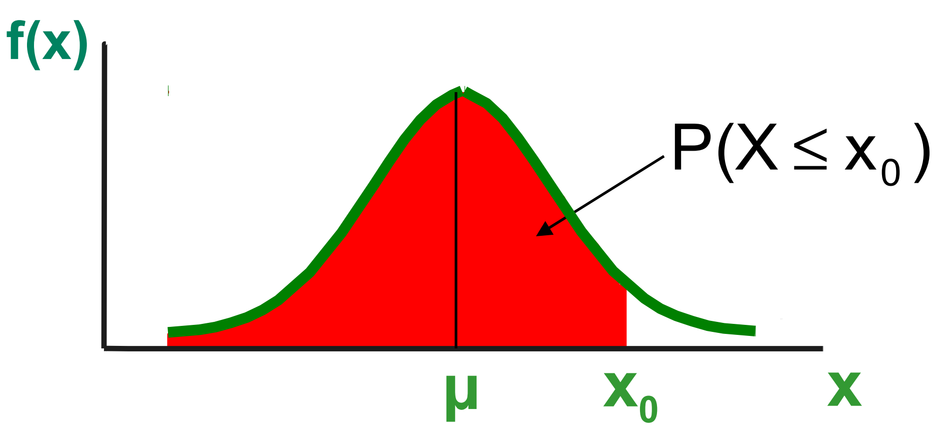
σ^2 = the population variance

x = any value of the continuous variable, $-\infty < x < \infty$

Cumulative Normal Distribution

- For a normal random variable X with mean μ and variance σ^2 , i.e., $X \sim N(\mu, \sigma^2)$, the cumulative distribution function is

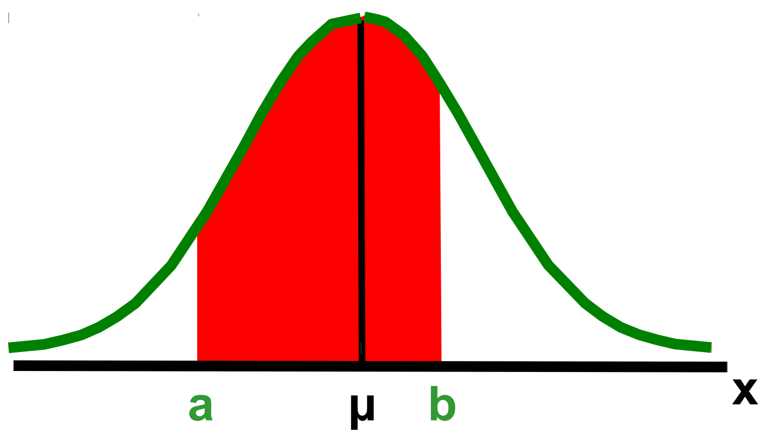
$$F(x_0) = P(X \le x_0)$$



Finding Normal Probabilities

The probability for a range of values is measured by the area under the curve

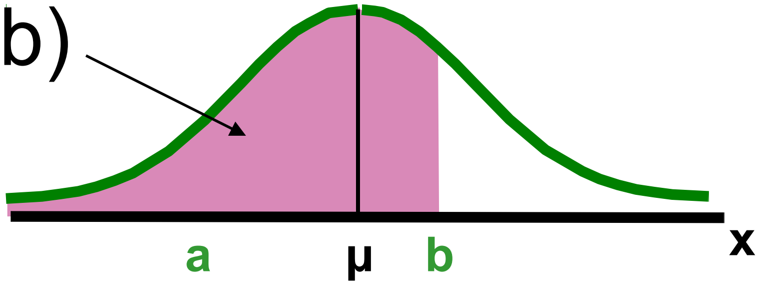
$$P(a < X < b) = F(b) - F(a)$$



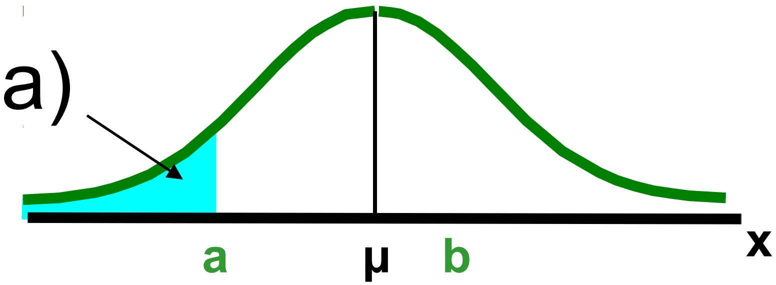
Finding Normal Probabilities

(continued)

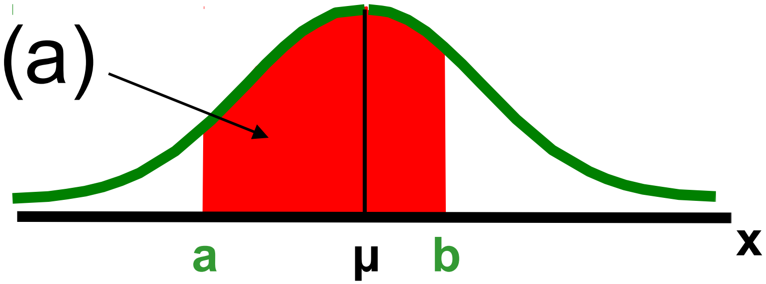
$$F(b) = P(X < b)$$



$$F(a) = P(X < a)$$



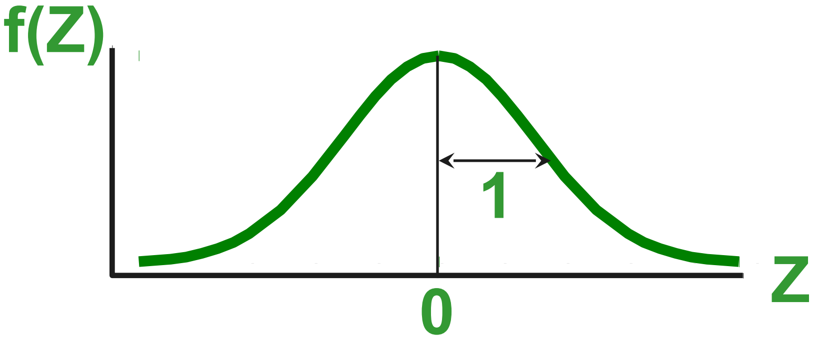
$$P(a < X < b) = F(b) - F(a)$$



The Standard Normal Distribution

- Any normal distribution (with any mean and variance combination) can be transformed into the standardized normal distribution (Z), with mean 0 and variance 1

$$Z \sim N(0,1)$$



- Need to transform X units into Z units by subtracting the mean of X and dividing by its standard deviation

$$Z = \frac{X - \mu}{\sigma}$$

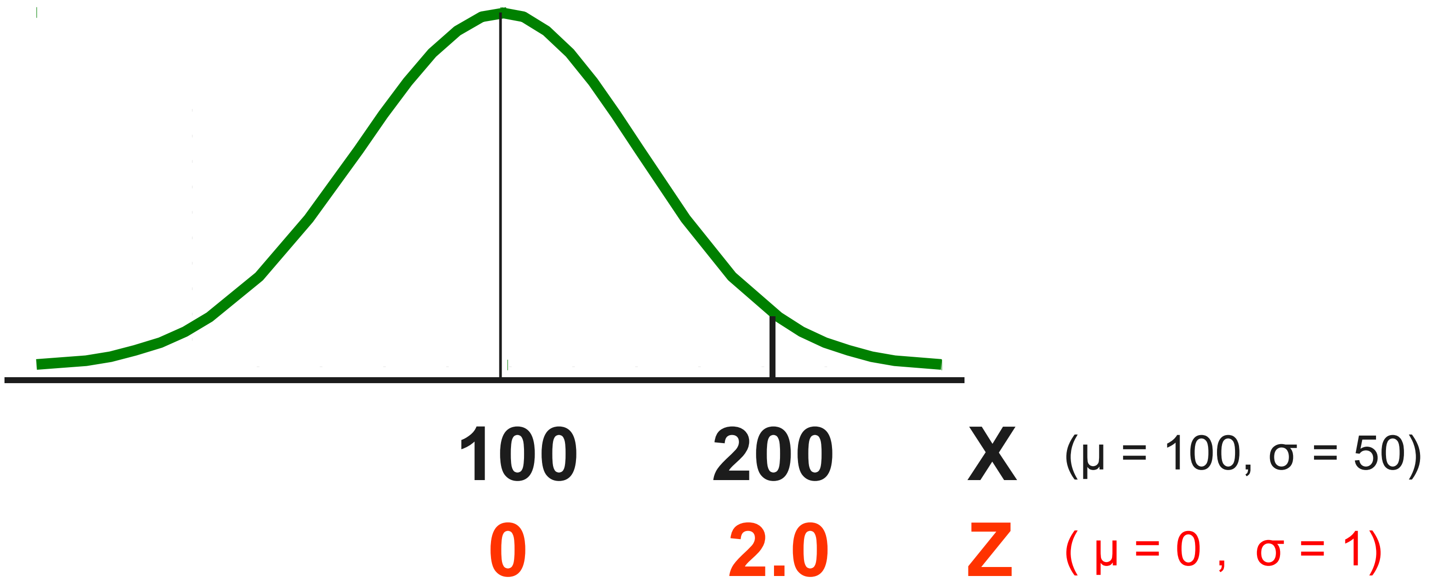
Example

- If X is distributed normally with mean of 100 and standard deviation of 50, the Z value for X = 200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{200 - 100}{50} = 2.0$$

- This says that X = 200 is two standard deviations (2 increments of 50 units) above the mean of 100.

Comparing X and Z units

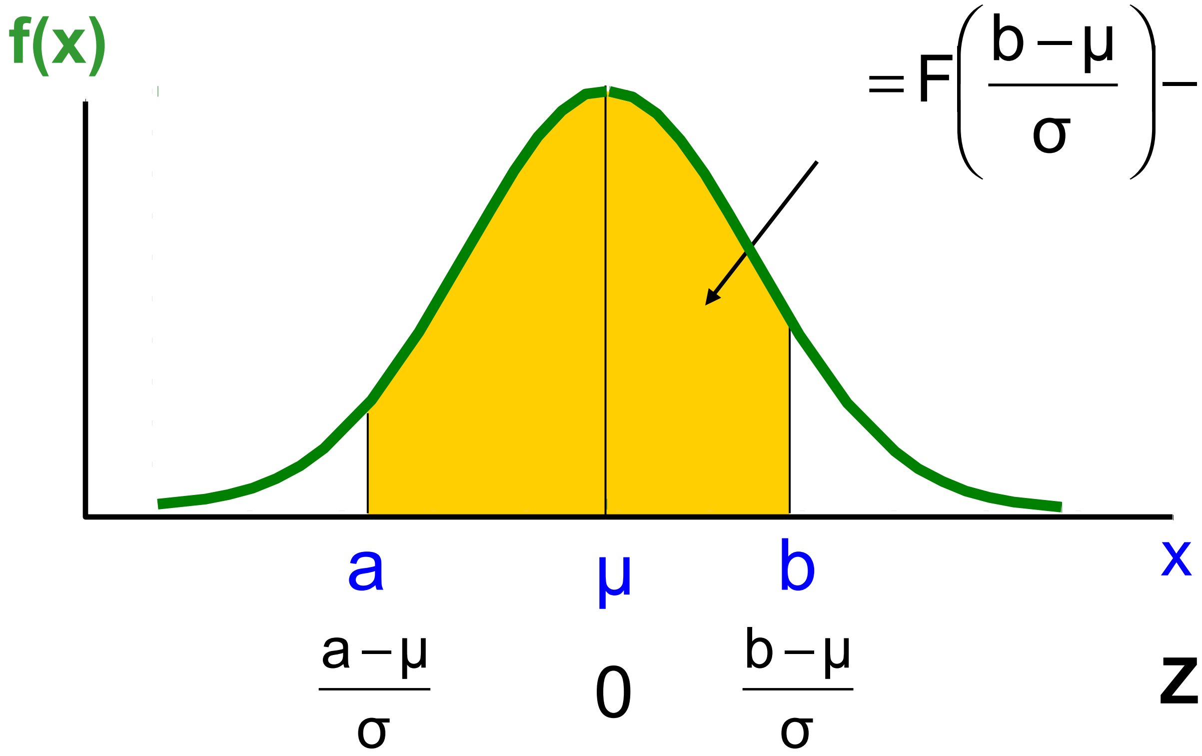


Note that the distribution is the same, only the scale has changed. We can express the problem in original units (X) or in standardized units (Z)

Finding Normal Probabilities

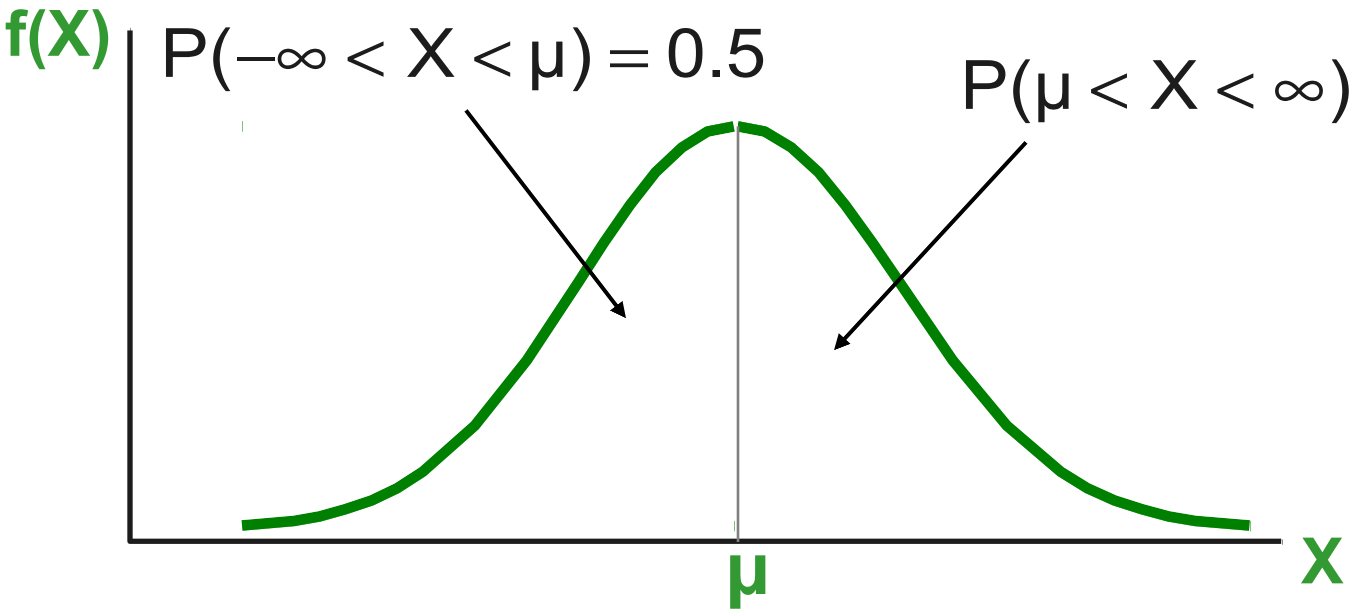
$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$



Probability as Area Under the Curve

The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below

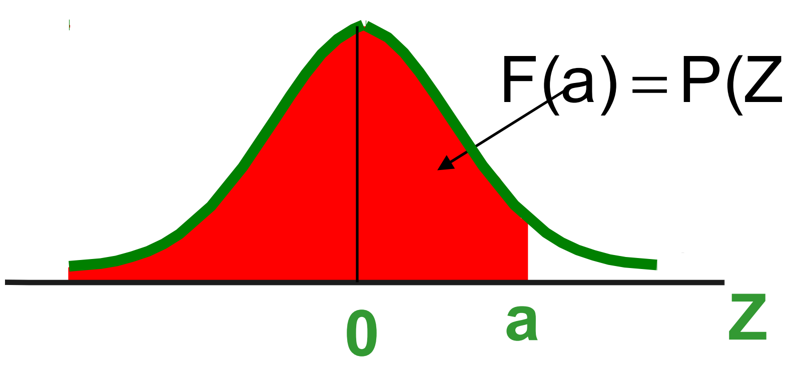


$$P(-\infty < X < \infty) = 1.0$$

Appendix Table 1

- The Standard Normal Distribution table in the textbook (Appendix Table 1) shows values of the cumulative normal distribution function

- For a given Z-value a , the table shows F(a)
  (the area under the curve from negative infinity to a)

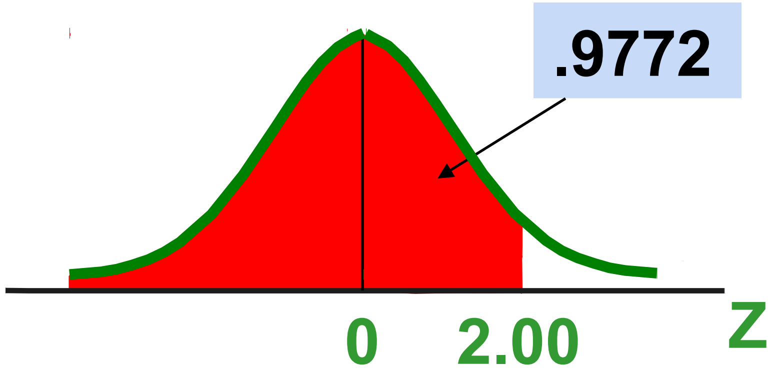


The Standard Normal Table

- Appendix Table 1 gives the probability F(a) for any value a

Example:

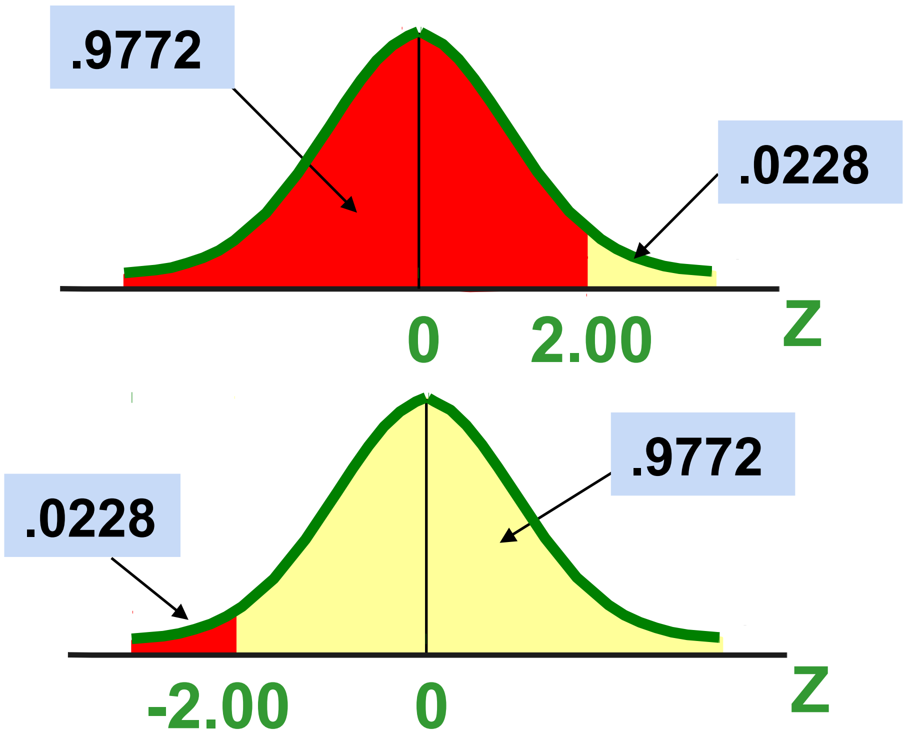
P(Z < 2.00) = .9772



The Standard Normal Table

(continued)

- For negative Z-values, use the fact that the distribution is symmetric to find the needed probability:



Example:

P(Z < -2.00) = 1 – 0.9772

= 0.0228

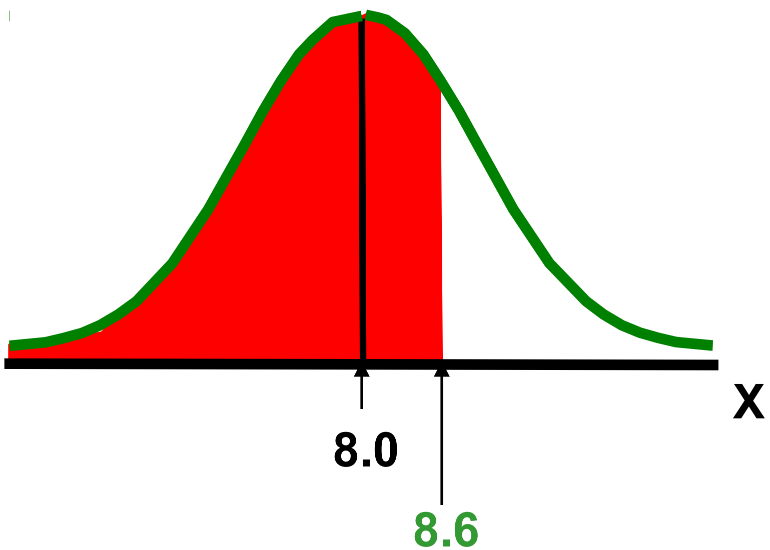
General Procedure for Finding Probabilities

To find $P(a < X < b)$ when X is distributed normally:

- Draw the normal curve for the problem in terms of X
- Translate X-values to Z-values
- Use the Cumulative Normal Table

Finding Normal Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0
- Find $P(X < 8.6)$

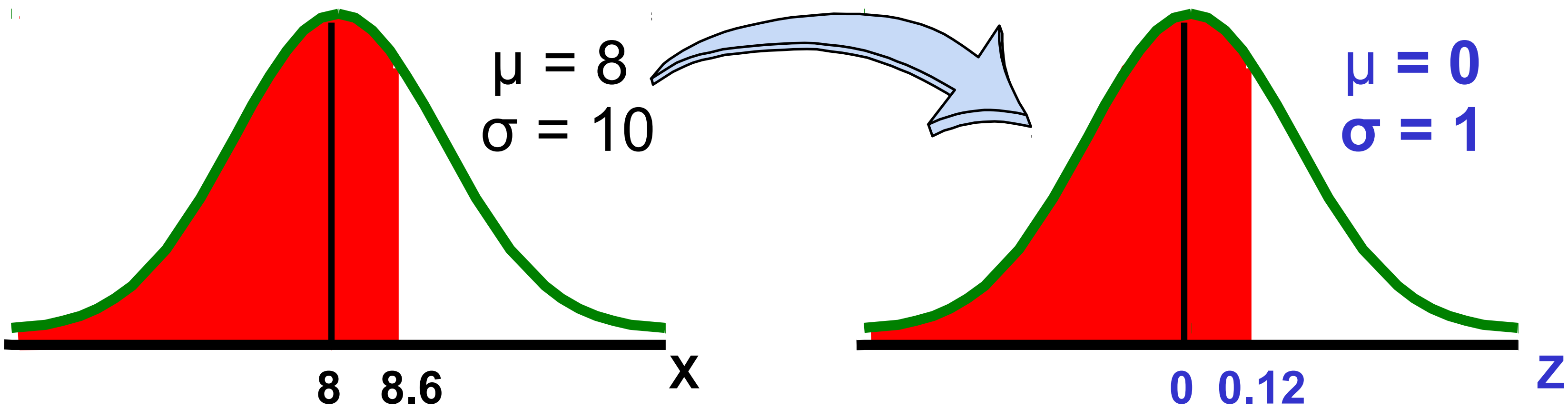


Finding Normal Probabilities

(continued)

- Suppose X is normal with mean 8.0 and standard deviation 5.0. Find P(X < 8.6)

$$Z = \frac{X - \mu}{\sigma} = \frac{8.6 - 8.0}{5.0} = 0.12$$



P(X < 8.6) P(Z < 0.12)

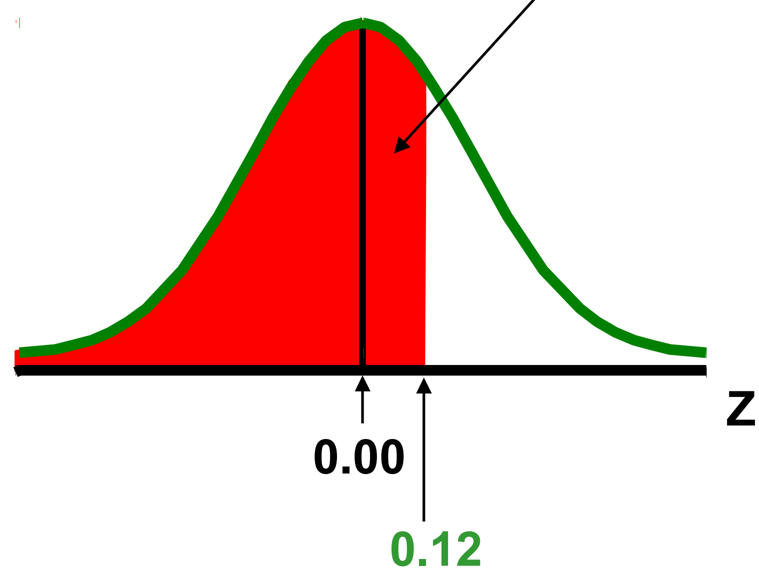
Solution: Finding P(Z < 0.12)

Standardized Normal Probability Table (Portion)

z	F(z)
.10	.5398
.11	.5438
.12	.5478
.13	.5517

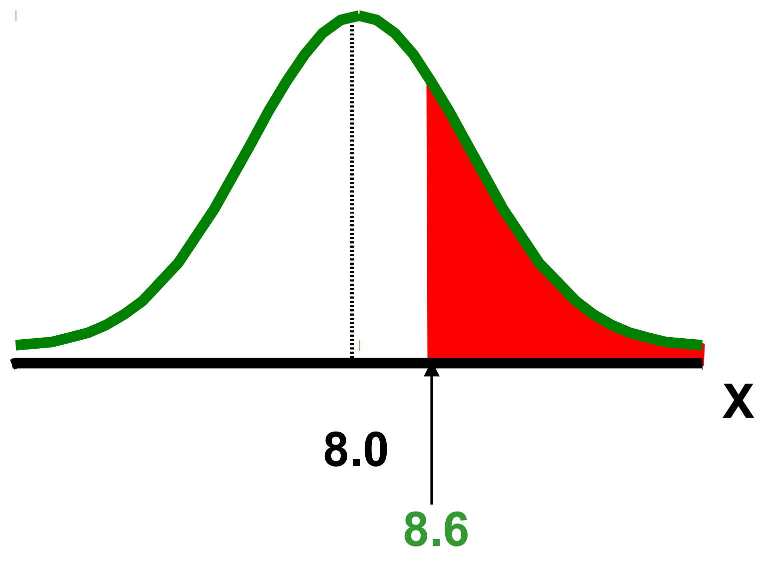
P(X < 8.6)
= P(Z < 0.12)

F(0.12) = 0.5478



Upper Tail Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now Find P(X > 8.6)



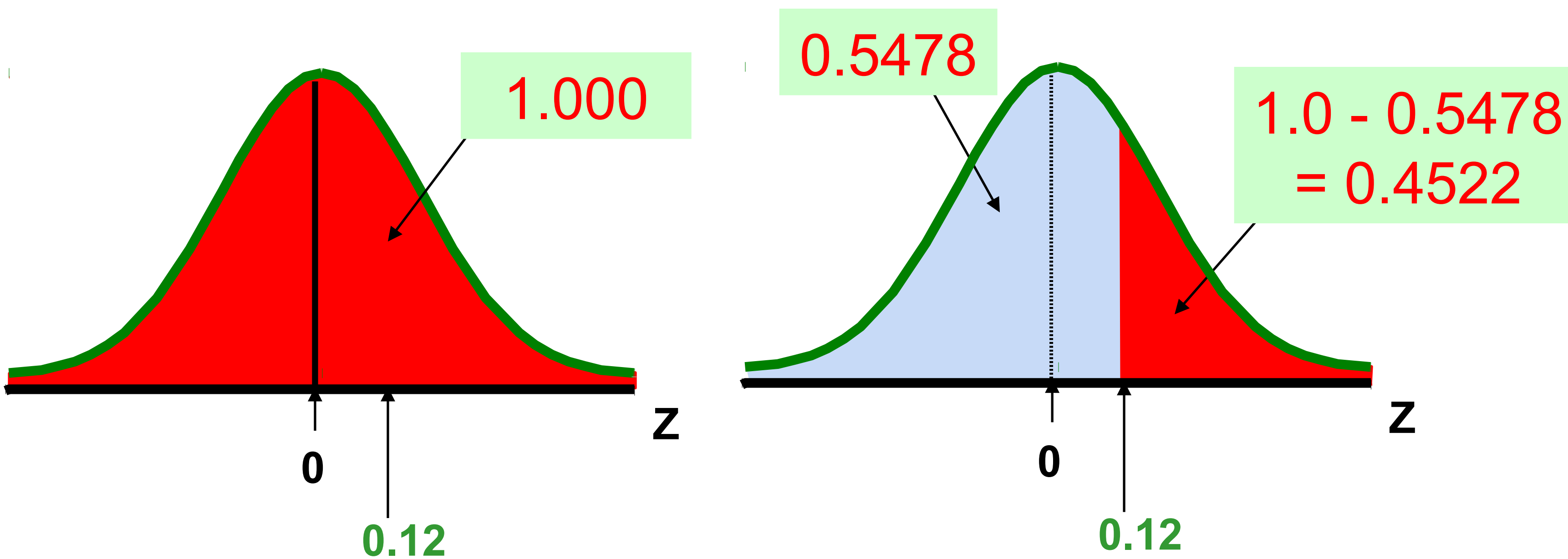
Upper Tail Probabilities

(continued)

- Now Find P(X > 8.6)…

$$P(X > 8.6) = P(Z > 0.12) = 1.0 - P(Z \le 0.12)$$

$$= 1.0 - 0.5478 = 0.4522$$



Finding the X value for a Known Probability

- Steps to find the X value for a known probability:
  1. Find the Z value for the known probability
  2. Convert to X units using the formula:

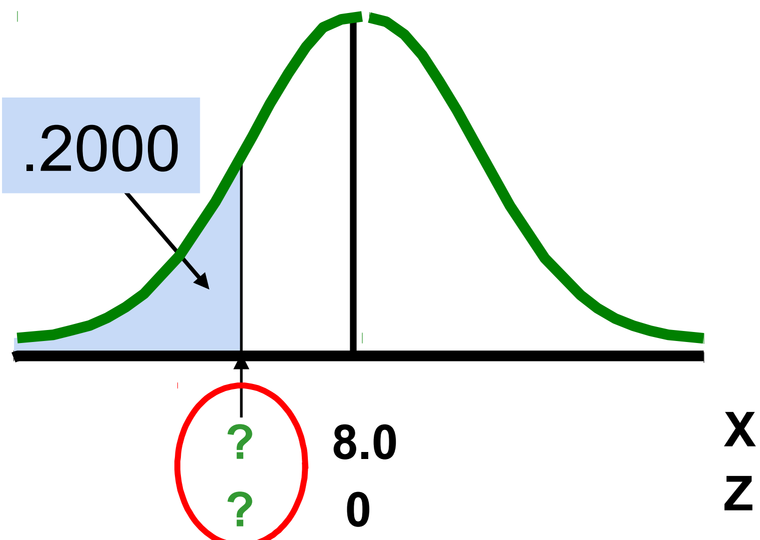
$$X = \mu + Z\sigma$$

Finding the X value for a Known Probability

(continued)

Example:

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now find the X value so that only 20% of all values are below this X



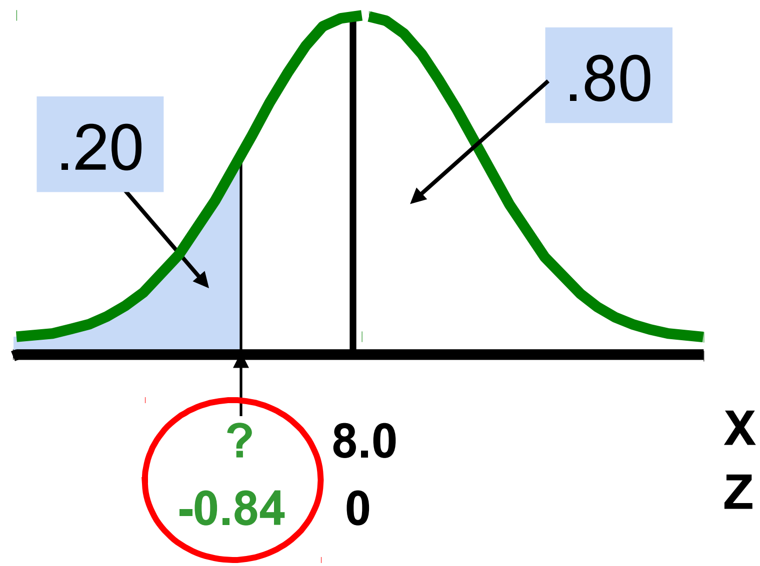
Find the Z value for 20% in the Lower Tail

1. Find the Z value for the known probability

Standardized Normal Probability Table (Portion)

z	F(z)
.82	.7939
.83	.7967
.84	.7995
.85	.8023

- 20% area in the lower tail is consistent with a Z value of -0.84



Finding the X value

2. Convert to X units using the formula:

$$X = \mu + Z\sigma$$

$$= 8.0 + (-0.84)5.0$$

$$= 3.80$$

So 20% of the values from a distribution with mean 8.0 and standard deviation 5.0 are less than 3.80

Assessing Normality

- Not all continuous random variables are normally distributed
- It is important to evaluate how well the data is approximated by a normal distribution

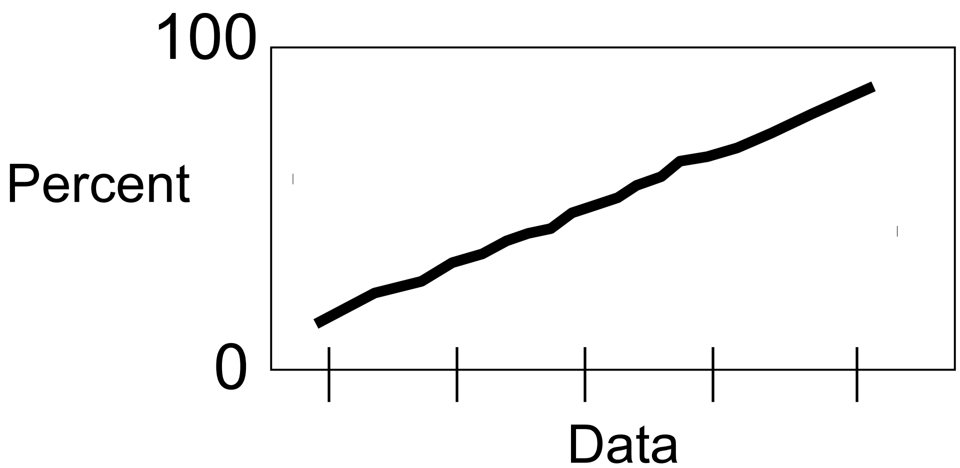
The Normal Probability Plot

- Normal probability plot
  - Arrange data from low to high values
  - Find cumulative normal probabilities for all values
  - Examine a plot of the observed values vs. cumulative probabilities (with the cumulative normal probability on the vertical axis and the observed data values on the horizontal axis)
  - Evaluate the plot for evidence of linearity

The Normal Probability Plot

(continued)

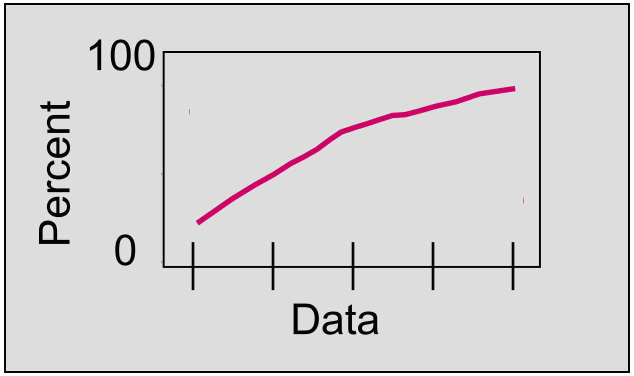
A normal probability plot for data from a normal distribution will be approximately linear:



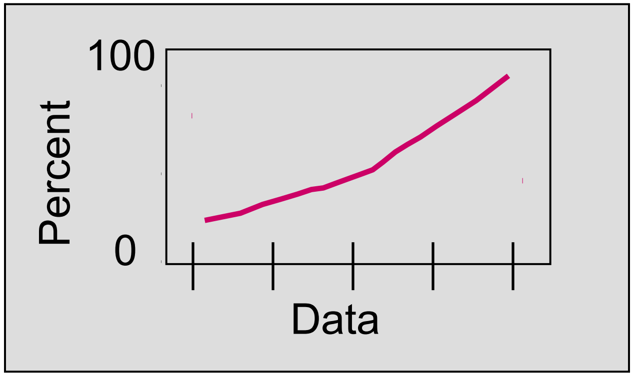
The Normal Probability Plot

(continued)

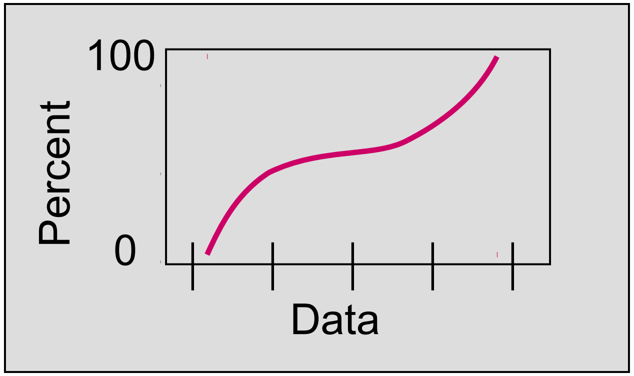
Left-Skewed



Right-Skewed



Uniform



Nonlinear plots indicate a deviation from normality

5.4

Normal Distribution Approximation for Binomial Distribution

- Recall the binomial distribution:
  - n independent trials
  - probability of success on any given trial = P
- Random variable X:
  - X_i =1 if the i^{th} trial is "success"
  - X_i =0 if the i^{th} trial is "failure"

$$E[X] = \mu = nP$$

$$Var(X) = \sigma^2 = nP(1-P)$$

Normal Distribution Approximation for Binomial Distribution

(continued)

- The shape of the binomial distribution is approximately normal if n is large
- The normal is a good approximation to the binomial when $nP(1 - P) > 5$
- Standardize to Z from a binomial distribution:

$$Z = \frac{X - E[X]}{\sqrt{Var(X)}} = \frac{X - np}{\sqrt{nP(1-P)}}$$

Normal Distribution Approximation for Binomial Distribution

(continued)

- Let X be the number of successes from n independent trials, each with probability of success P.

- If nP(1 - P) > 5,

$$P(a < X < b) = P\left(\frac{a - nP}{\sqrt{nP(1-P)}} \leq Z \leq \frac{b - nP}{\sqrt{nP(1-P)}} \right)$$

Binomial Approximation Example

- 40% of all voters support ballot proposition A. What is the probability that between 76 and 80 voters indicate support in a sample of n = 200 ?
  - $E[X] = \mu = nP = 200(0.40) = 80$
  - $Var(X) = \sigma^2 = nP(1 – P) = 200(0.40)(1 – 0.40) = 48$
    (note: $nP(1 – P) = 48 > 5$)

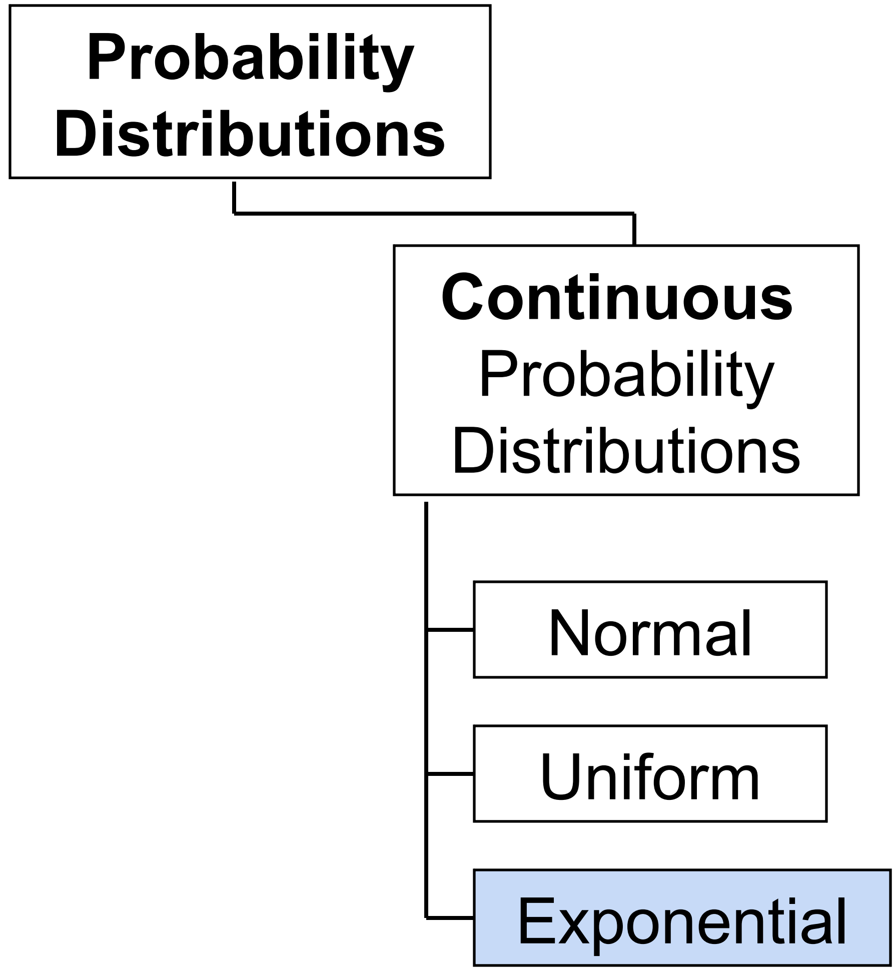
$$P(76 < X < 80) = P\left(\frac{76-80}{\sqrt{200(0.4)(1-0.4)}} \le Z \le \frac{80-80}{\sqrt{200(0.4)(1-0.4)}}\right)$$

$$= P(-0.58 < Z < 0)$$

$$= F(0) - F(-0.58)$$

$$= 0.5000 - 0.2810 = 0.2190$$

The Exponential Distribution



The Exponential Distribution

- Used to model the length of time between two occurrences of an event (the time between arrivals)

  - Examples:
    - Time between trucks arriving at an unloading dock
    - Time between transactions at an ATM Machine
    - Time between phone calls to the main operator

The Exponential Distribution

(continued)

- The exponential random variable T (t>0) has a probability density function

$$f(t) = \lambda e^{-\lambda t} \quad \text{for } t > 0$$

- Where
  - λ is the mean number of occurrences per unit time
  - t is the number of time units until the next occurrence
  - e = 2.71828
- T is said to follow an exponential probability distribution

The Exponential Distribution

(continued)

- Defined by a single parameter, its mean λ (lambda)

- The cumulative distribution function (the probability that an arrival time is less than some specified time t) is

$$F(t) = 1 - e^{-\lambda t}$$

where e = mathematical constant approximated by 2.71828

λ = the population mean number of arrivals per unit

t = any value of the continuous variable where t > 0

Exponential Distribution Example

Example: Customers arrive at the service counter at the rate of 15 per hour. What is the probability that the arrival time between consecutive customers is less than three minutes?

- The mean number of arrivals per hour is 15, so $\lambda = 15$
- Three minutes is .05 hours
- P(arrival time < .05) $= 1 - e^{-\lambda X} = 1 - e^{-(15)(.05)} =$ 0.5276
- So there is a 52.76% probability that the arrival time between successive customers is less than three minutes

Jointly Distributed Continuous Random Variables

5.6

- Let $X_1, X_2, \ldots, X_k$ be continuous random variables
- Their joint cumulative distribution function,

  $F(x_1, x_2, \ldots, x_k)$

  defines the probability that simultaneously X_1 is less than x_1, X_2 is less than x_2, and so on; that is

$$F(x_1, x_2, \ldots, x_k) = P(X_1 < x_1 \cap X_2 < x_2 \cap \cdots X_k < x_k)$$

Jointly Distributed Continuous Random Variables

(continued)

- The cumulative distribution functions

  $F(x_1), F(x_2), \ldots, F(x_k)$

  of the individual random variables are called their marginal distribution functions

- The random variables are independent if and only if

$$F(x_1, x_2, \ldots, x_k) = F(x_1)F(x_2)\cdots F(x_k)$$

Covariance

- Let X and Y be continuous random variables, with means μ_x and μ_y
- The expected value of $(X - \mu_x)(Y - \mu_y)$ is called the covariance between X and Y

$$Cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$$

- An alternative but equivalent expression is

$$Cov(X, Y) = E[XY] - \mu_x \mu_y$$

- If the random variables X and Y are independent, then the covariance between them is 0. However, the converse is not true.

Correlation

- Let X and Y be jointly distributed random variables.
- The correlation between X and Y is

$$\rho = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Sums of Random Variables

Let $X_1, X_2, \ldots, X_k$ be k random variables with means $\mu_1, \mu_2, \ldots, \mu_k$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. Then:

- The mean of their sum is the sum of their means

$$E[(X_1 + X_2 + \cdots + X_k)] = \mu_1 + \mu_2 + \cdots + \mu_k$$

Sums of Random Variables

(continued)

Let $X_1, X_2, \ldots, X_k$ be k random variables with means $\mu_1, \mu_2, \ldots, \mu_k$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. Then:

- If the covariance between every pair of these random variables is 0, then the variance of their sum is the sum of their variances

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_k) = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2$$

- However, if the covariances between pairs of random variables are not 0, the variance of their sum is

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_k) = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2 + 2\sum_{i=1}^{K-1}\sum_{j=i+1}^{K}\mathrm{Cov}(X_i, X_j)$$

Differences Between a Pair of Random Variables

For two random variables, X and Y

- The mean of their difference is the difference of their means; that is

$$E[X-Y] = \mu_X - \mu_Y$$

- If the covariance between X and Y is 0, then the variance of their difference is

$$Var(X-Y) = \sigma_X^2 + \sigma_Y^2$$

- If the covariance between X and Y is not 0, then the variance of their difference is

$$Var(X-Y) = \sigma_X^2 + \sigma_Y^2 - 2Cov(X,Y)$$

Linear Combinations of Random Variables

- A linear combination of two random variables, X and Y, (where a and b are constants) is

$$W = aX + bY$$

- The mean of W is

$$\mu_W = E[W] = E[aX + bY] = a\mu_X + b\mu_Y$$

Linear Combinations of Random Variables

(continued)

- The variance of W is

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Cov}(X,Y)$$

- Or using the correlation,

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\rho(X,Y)\sigma_X\sigma_Y$$

- If both X and Y are joint normally distributed random variables then the linear combination, W, is also normally distributed

Example

- Two tasks must be performed by the same worker.
  - X = minutes to complete task 1; $\mu_x = 20$, $\sigma_x = 5$
  - Y = minutes to complete task 2; $\mu_y = 20$, $\sigma_y = 5$
  - X and Y are normally distributed and independent

- What is the mean and standard deviation of the time to complete both tasks?

Example

(continued)

- X = minutes to complete task 1; $\mu_x = 20$, $\sigma_x = 5$
- Y = minutes to complete task 2; $\mu_y = 30$, $\sigma_y = 8$

- What are the mean and standard deviation for the time to complete both tasks?

$$W = X + Y$$

$$\mu_W = \mu_X + \mu_Y = 20 + 30 = 50$$

- Since X and Y are independent, Cov(X,Y) = 0, so

$$\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 + 2\mathrm{Cov}(X,Y) = (5)^2 + (8)^2 = 89$$

- The standard deviation is

$$\sigma_W = \sqrt{89} = 9.434$$

Financial Investment Portfolios

- A financial portfolio can be viewed as a linear combination of separate financial instruments

$$\begin{pmatrix}\text{Return on}\\ \text{portfolio}\end{pmatrix} = \begin{pmatrix}\text{Proportion of}\\ \text{portfolio value}\\ \text{in stock1}\end{pmatrix} \times \begin{pmatrix}\text{Stock 1}\\ \text{return}\end{pmatrix} + \begin{pmatrix}\text{Proportion of}\\ \text{portfolio value}\\ \text{in stock2}\end{pmatrix} \times \begin{pmatrix}\text{Stock 2}\\ \text{return}\end{pmatrix}$$

$$\cdots + \begin{pmatrix}\text{Proportion of}\\ \text{portfolio value}\\ \text{in stock N}\end{pmatrix} \times \begin{pmatrix}\text{Stock N}\\ \text{return}\end{pmatrix}$$

Portfolio Analysis Example

- Consider two stocks, A and B
  - The price of Stock A is normally distributed with mean 12 and variance 4
  - The price of Stock B is normally distributed with mean 20 and variance 16
  - The stock prices have a positive correlation, $\rho_{AB} = .50$
- Suppose you own
  - 10 shares of Stock A
  - 30 shares of Stock B

Portfolio Analysis Example

(continued)

- The mean and variance of this stock portfolio are: (Let W denote the distribution of portfolio value)

$$\mu_W = 10\mu_A + 20\mu_B = (10)(12) + (30)(20) = 720$$

$$\sigma^2_W = 10^2\sigma^2_A + 30^2\sigma^2_B + (2)(10)(30)\text{Corr}(A,B)\sigma_A\sigma_B$$

$$= 10^2(4)^2 + 30^2(16)^2 + (2)(10)(30)(.50)(4)(16)$$

$$= 251{,}200$$

Portfolio Analysis Example

(continued)

- What is the probability that your portfolio value is less than $500?

$$\mu_W = 720$$

$$\sigma_W = \sqrt{251{,}200} = 501.20$$

- The Z value for 500 is $Z = \dfrac{500 - 720}{501.20} = -0.44$
- P(Z < -0.44) = 0.3300
  - So the probability is 0.33 that your portfolio value is less than $500.

Chapter Summary

- Defined continuous random variables
- Presented key continuous probability distributions and their properties
  - uniform, normal, exponential
- Found probabilities using formulas and tables
- Interpreted normal probability plots
- Examined when to apply different distributions
- Applied the normal approximation to the binomial distribution
- Reviewed properties of jointly distributed continuous random variables