MAT B41C Test 3 Version 1. Solutions

1. Suppose X is a subspace of $\mathbf{R}^6$, that $X \neq \{0\}$ and that $X \neq \mathbf{R}^6$. Which of the following statements are true?

I. X has a spanning set consisting of 6 vectors.
II. X has a linearly independent subset consisting of 6 vectors.
III. $1 \leq \dim X \leq 5$.
IV. X has a basis that spans $\mathbf{R}^6$.
V. For all vectors u, v, w in X, $au + bv + cw = 0$ implies $a = b = c = 0$.

A. III & II
B. I & IV
C. II & IV
D. III & V
E. I & III
F. I & V

- $X \neq \{0\} \Rightarrow \dim X \geq 1$
- $X \neq \mathbb{R}^6 \Rightarrow \dim X \leq 5$

$\therefore$ III is true

- Every subspace ($\neq \{0\}$) has a spanning set with 6 vectors $\therefore$ I is true.

Hence E is correct

(The statement II is be false, since $\dim X < 6$
IV is false since $X \neq \mathbb{R}^6$
V is false : take $u = v = w = 0$, $a = b = c = 1$.

2. Suppose $\{u, v\}$ is a linearly **independent** set in vector space V, and that $w \in V$ is chosen so that $\{u, v, w\}$ is linearly **dependent**. Which of the following statements is **ALWAYS** true?

A. $\{u, w\}$ is linearly dependent.
B. $\{v, w\}$ is linearly dependent.
C. $\{v, u\}$ is linearly dependent.
D. $u \in \text{span}\{v, w\}$.
E. $v \in \text{span}\{u, w\}$.
F. $w \in \text{span}\{u, v\}$.

We know that if $\{u, v\}$ is l.i. and $w \in V$, then $\{u, v, w\}$ is l.i. $\Leftrightarrow w \notin \text{span}\{u, v\}$.
Since we're told $\{u, v, w\}$ is l.d., we conclude $w \in \text{span}\{u, v\}$

3. Each statement below is True or False.

- Every system of 2 equations in 2 unknowns has a unique solution. False: $x+y=0$, $2x+2y=0$
- The set of solutions of the system consisting of the single equation

$$2x - 3y = 0$$

in the three variables x, y and z is a subspace of $\mathbf{R}^3$. True: $\{(x,y,z) \mid 2x-3y=0\}$ is a plane through 0.
- There is a linear system in 2 variables which is inconsistent.

Choose the correct sequence from the possibilities below. True: e.g. $x+y=1$, $2x+2y=1$

A. True, True, False.
B. True, False, True.
~~C. True, False, False.~~
D. False, True, True.
E. False, False, True.
F. False, True, False.

4. Let $v_1 = (-1,1,1,1), v_2 = (1,-1,1,1), v_3 = (1,1,-1,0)$, and let W be the subspace of $\mathbf{R}^4$ defined by

$$W = \text{span}\{v_1, v_2, v_3\}. \quad \bigstar$$

2 a) Show that $\{v_1, v_2, v_3\}$ is a linearly independent set.

$1\frac{1}{2}$ b) Find a basis of W, and hence find $\dim W$.

$1\frac{1}{2}$ c) Is $\{v_1 + v_2, v_1 - v_3, v_2 + v_3\}$ a basis of W?

1 d) Assuming that $v_4 = (1,0,0,0)$ does not belong to W (you do **not** have to check this), explain why $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbf{R}^4$.

a) Suppose $av_1 + bv_2 + cv_3 = (0,0,0,0)$. (*) Then, equating components on both sides of this equation, we have

$$\begin{aligned} -a + b + c &= 0 \\ a - b + c &= 0 \\ a + b - c &= 0 \\ a + b &= 0 \end{aligned}$$

$\Rightarrow 2c = 0$ so $c = 0$

$\Rightarrow 2a = 0$ so $a = 0$

$\Rightarrow$ (since $a = 0$) $b = 0$

Hence (*) $\Rightarrow a = b = c = 0$, So $\{v_1, v_2, v_3\}$ is l.i.

b) The defn of W ($\bigstar$) shows that $\{v_1, v_2, v_3\}$ spans W; part (a) shows $\{v_1, v_2, v_3\}$ is l.i. Hence $\{v_1, v_2, v_3\}$ is a basis for W. Thus $\dim W = 3$

c) No, because $\{v_1 + v_2, v_1 - v_3, v_2 + v_3\}$ is linearly dependent: note that $(v_1 + v_2) + (v_1 - v_3) - (v_2 + v_3) = 0$.

d) Since $v_4 \notin \text{Span}\{v_1, v_2, v_3\}$, and $\{v_1, v_2, v_3\}$ is l.i., we know $\{v_1, v_2, v_3, v_4\}$ is l.i. But this is a set of 4 l.i. vectors in $\mathbb{R}^4$ (and $\dim \mathbb{R}^4 = 4$), so $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$.

5.

a) Find the reduced row-echelon form of the matrix, indicating at each step the row operations you use in the form (for example) "$aR_i + R_j \to R_j$":

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 2 & -5 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{-2R_1+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R_2+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{2R_2+R_1 \to R_1}{\overset{-4R_3+R_1 \to R_1}{\sim}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and this is in RRE form

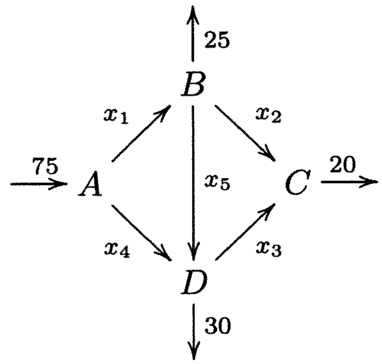
b) If the augmented matrix of a linear system in the variables x_1, x_2, x_3 and x_4 is

$$\begin{array}{cccccc} & s & & t & & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

find the general solution.

$$\begin{aligned} x_1 &= -1 - 2s - 3t \\ x_2 &= s \\ x_3 &= 1 - t \qquad ; \; s,t \in \mathbb{R} \\ x_4 &= t \end{aligned}$$

6. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter A or leave B, C and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



Write down a system of linear equations which describes the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (**Do not solve the linear system.**)

Intersection	Flow in	=	Flow out
A	75	=	$x_1 + x_4$
B	x_1	=	$25 + x_2 + x_5$
C	$x_2 + x_3$	=	20
D	$x_4 + x_5$	=	$30 + x_3$

Constraints

- Since streets are one-way, $x_i \geq 0$, $i = 1, \ldots, 5$
- Since fractional cars are not permitted $x_i \in \mathbb{Z}$, $i = 1, \ldots, 5$ (integer)

7. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit example - with numbers!</u>
- If you say the statement is always true, you must give a clear explanation.

a) If V is a vector space and $\{v_1, v_2\} \subset V$ is linearly independent, then $\dim V = 2$.

Let $V = \mathbb{R}^3$, and $v_1 = (1,0,0)$, $v_2 = (0,1,0)$.

Then $\{v_1, v_2\}$ is l.i. but $\dim V = 3$

ANSWER $\boxed{\text{FALSE}}$

b) If a linear system is consistent, it must be homogeneous.

$$\begin{aligned} x \quad &= 1 \\ y &= 0 \end{aligned}$$

is a consistent linear system which is not homogeneous

ANSWER $\boxed{\text{FALSE}}$

8. [Bonus] Suppose that U and V are subspaces of $\mathbf{R}^{20}$, that $\{u_1, \ldots, u_7\}$ is a basis of U and $\{v_1, \ldots, v_9\}$ is a basis of V.

If there is a non-zero vector $w \in U \cap V = \{w \in \mathbf{R}^{20} \mid w \in U \text{ and } w \in V\}$ prove that $\{u_1, \ldots, u_7, v_1, \ldots, v_9\}$ must be linearly dependent.

(Your proof must work for all subspaces U, V, all choices of bases $\{u_1, \ldots, u_7\}$ of U and $\{v_1, \ldots, v_9\}$ of V, and all vectors w satisfying the condition above: do not choose particular vectors!)

Write $w = a_1u_1 + \cdots + a_7u_7$ and $w = b_1v_1 + \cdots + b_9v_9$.
Since $w \neq 0$, at least one of $a_1, \ldots, a_7$ and at least one of $b_1, \ldots, b_9 \neq 0$.
But $0 = w - w = a_1u_1 + \cdots + a_7u_7 - b_1v_1 - b_2v_2 - \cdots - b_9v_9$ (*)
Since at least 2(!) of $a_1, \ldots, a_7, b_1, \ldots, b_9$ are non-zero,
(*) shows that $\{u_1, \ldots, u_7, v_1, \ldots, v_9\}$ is l. dependent.