MCV4U-A

12

Optimization Problems

Introduction

You have already learned that one of the many uses of the derivative is to help find maximum and minimum values of functions. Problems that involve finding maximum or minimum values of functions are called optimization problems, and arise frequently in a wide variety of real-world situations. In some cases, you're given a mathematical expression for a function you're required to optimize. In other cases, you have to construct a mathematical model for the problem in order to come up with a function to optimize.

There isn't much in the way of new mathematical content in this lesson. Instead, you'll look at how what you've already learned can be applied to a wide variety of optimization problems.

Estimated Hours for Completing This Lesson	
Examples of Optimization Problems	3.5
Key Questions	1.5

What You Will Learn

After completing this lesson, you will be able to

- use your previous knowledge of derivatives to answer optimization problems involving polynomial functions, rational functions, and exponential problems
- solve real-world optimization problems by first constructing a mathematical model for the problem and then applying your knowledge of derivatives

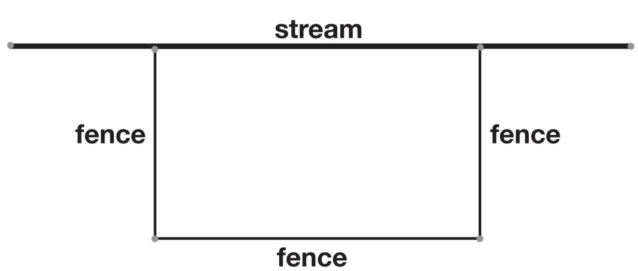
Examples of Optimization Problems

There are many examples of optimization problems. You saw some problems in Lesson 11 that came from the world of business. Specifically, you saw problems where you were interested in maximizing revenue (one involving the sale of tennis rackets and the other involving tickets for a minor-league hockey team). The world of economics and business is one place where optimization problems can arise. Other examples of optimization problems:

- maximizing the volume of a container that must be built out of a certain amount of sheet metal
- maximizing the number of letters per hour that can be sorted by employees at a post office
- minimizing the total expense of laying an underground cable between two points

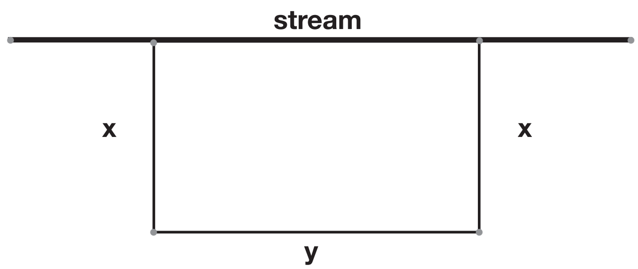
Example

Farmer Lee has 200 m of fence that he wishes to use to enclose a rectangular area. One side of the rectangle will be formed by a stream (which conveniently is straight), so only three sides of the rectangle will be formed by the fence, as shown in the diagram. What is the area of the largest rectangle that Farmer Lee can form?



Solution

The first step is to identify the unknowns and define the variables. In this case, the most obvious unknowns are the lengths of the sides of the rectangle. You can choose any names you want for the variables: call the sides of the rectangle x and y as in the following diagram. You also need to decide how to choose x and y so that the total amount of fence is 200 m and the area of the rectangle is maximized.



Another important step is to identify what you want to optimize. In this case, you wish to optimize the area of the rectangle—specifically, to maximize it. The formula for the area of this rectangle is simply xy.

To find maximum or minimum values of a function, you normally compute the derivative of the function and then determine where the derivative is zero, positive, or negative. The formula for the area of the rectangle, however, involves two variables. Before taking the derivative, you need to rewrite the area as a function of just one variable. In other words, the quantity you're trying to optimize should be written as a function of a single variable.

How do you do this? You need to see if any of the information given in the problem can be used to eliminate one of the variables x or y. One fact you haven't yet used is that the total amount of fence is 200 m, which means $2x + y = 200$, so $y = 200 - 2x$. Substitute this into the expression for the area:

$A = xy$

$A = x(200 - 2x)$

$A = 200x - 2x^2$

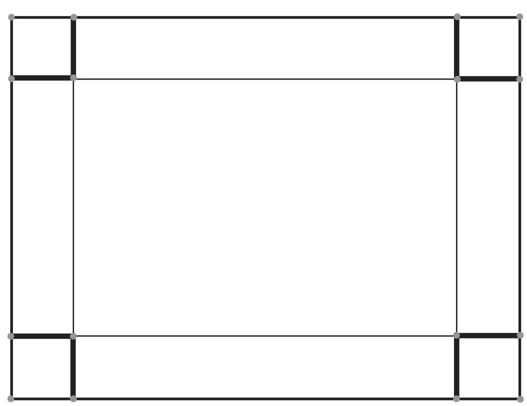
You've successfully expressed what you're trying to optimize as a function of the single variable x. Now take the derivative:

$A'(x) = 200 - 4x$

As a result, $A'(x)$ is zero when $x = 50$, $A'(x)$ is positive when $x < 50$, and $A'(x)$ is negative when $x > 50$. You can conclude that the function $A(x)$ has a maximum when $x = 50$. This means that $y = 100$, and you conclude that Farmer Lee gets the maximum area from a rectangle with sides of length 50 m and 100 m, resulting in an area of 5000 m^2.

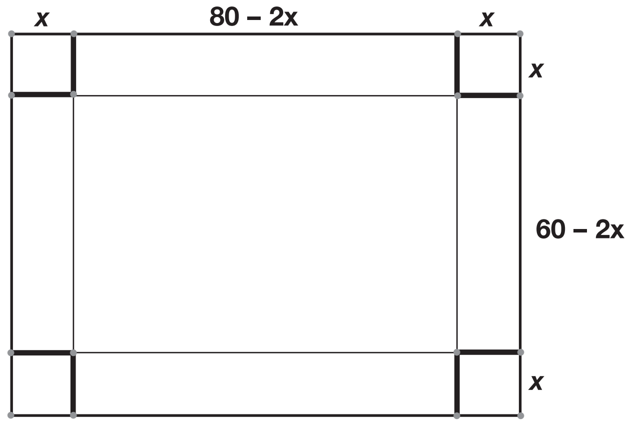
Example

An open-topped box is to be constructed from a rectangular piece of cardboard whose dimensions are 80 cm by 60 cm. The box is formed by cutting away a square from each corner of the cardboard, then folding along the dotted lines to create the sides of the box. What size should the squares you remove from the corners be in order to maximize the volume of the box?



Solution

Start by identifying unknowns and defining variables. The problem is to decide on the size of the squares to be cut out of the corners, so it makes sense to let x be the length of one side of these squares. This means the dimensions of the base of the box are going to be $60 - 2x$ cm by $80 - 2x$ cm.



Notice from the diagram that there are limits on the possible values of x. It is impossible for x to be negative, and it is also impossible for x to be greater than 30. In other words, the domain of the function is $0 \le x \le 30$.

The volume of a rectangular box is the area of the base multiplied by the height. In this case, the area of the base is $(80 - 2x)(60 - 2x)$, and the height of the box is x. The volume is

$V = (80 - 2x)(60 - 2x)x$

$V = (4800 - 160x - 120x + 4x^2)x$

$V = 4800x - 280x^2 + 4x^3$

The volume is expressed as a function of a single variable, so take the derivative to help maximize the volume:

$V' = 4800 - 560x + 12x^2$.

To find all points where V' is zero, you have to solve a quadratic:

$12x^2 - 560x + 4800 = 0$

$4(3x^2 - 140x + 1200) = 0$

$3x^2 - 140x + 1200 = 0$

This does not factor, so use the quadratic formula:

$$\begin{aligned} x &= \frac{-(-140) \pm \sqrt{(-140)^2 - 4(3)(1200)}}{2(3)} \\ &= \frac{140 \pm \sqrt{19\,600 - 14\,400}}{6} \\ &= \frac{140 \pm \sqrt{5200}}{6} \\ &= \frac{140 \pm 72.11}{6} \end{aligned}$$

The solutions are $x = \frac{(140 + 72.11)}{6} = 35.35$ and $x = \frac{(140 - 72.11)}{6} = 11.31$. In other words, the only points where $V' = 0$ are at $x = 11.31$ and $x = 35.35$. If $x = 35.35$, $60 - 2x$ is a negative value. To find out if you have a maximum or a minimum at $x = 11.31$, consider the derivative V' on either side of 11.31. You'll find that if $x < 11.31$ (for example, $x = 0$), then V' is positive, and if $x > 11.31$ (for example, $x = 20$), then V' is negative. You can conclude that the function $V(x)$ has a maximum at $x = 11.31$.

To get a box of the maximum possible volume, remove a square 11.31 cm by 11.31 cm from each corner, resulting in a box whose base has dimensions 37.38 cm by 57.38 cm and whose height is 11.31 cm. The volume of that box is $37.38 \times 57.38 \times 11.31 = 24\,258$ cm^3.

Now consider a financial example similar to those in Lesson 11.

Example

A ferry service between two islands carries 300 cars daily, charging $8 per car. The ferry service can handle up to 400 cars daily. Market research shows that if the price is dropped by $0.25, 10 more cars will take the ferry. The ferry service must make $2200 in fares each day to cover operating costs. What fare should they charge to maximize revenue?

Solution

You are told that each \$0.25 decrease in the fare will result in 10 more cars taking the ferry. In other words, if the fare is reduced to $8 - 0.25x$, then the number of cars taking the ferry will be $300 + 10x$.

Since the ferry can't take more than 400 cars, x can't be larger than 10.

With a fare of $8 - 0.25x$ dollars per car, and the number of cars being $300 + 10x$, the total daily revenue is

$R(x) = (8 - 0.25x)(300 + 10x)$

$R(x) = 2400 + 80x - 75x - 2.5x^2$

$R(x) = 2400 + 5x - 2.5x^2$

This is the function you wish to maximize. Compute the derivative:

$R'(x) = 5 - 5x$

This time, the numbers are nice and manageable. You can see that $R'(x) = 0$ when $x = 1$. Also, $R'(x)$ is positive when $x < 1$, and negative when $x > 1$. You can conclude that the function $R(x)$ has a maximum when $x = 1$.

Therefore, the maximum revenue is attained when the fare is \$7.75 per car. The expected number of cars per day at this rate is 310. This results in a total daily revenue of 310 × \$7.75 = \$2402.50. Fortunately, this is more than the \$2200 required to cover operating costs. (Interestingly, \$2402.50 is only \$2.50 more than the total revenue at the original fare of \$8.00.)

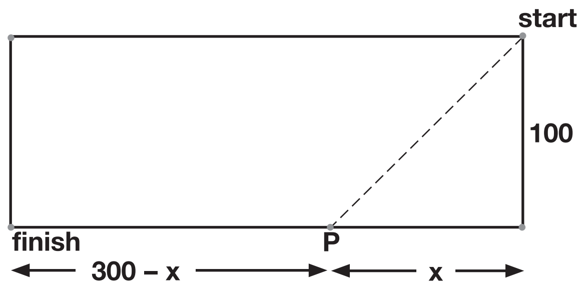
Here's a more complicated example:

Example

Shannon is standing on the northern edge of a muddy field that measures 100 m north to south. She wants to jog to a point on the southern edge of the field located 300 m west of her. She plans to run diagonally across the field to some point on the southern edge, and then jog the rest of the way along the dry ground on the southern edge of the field. Her running speed is 7 m/s on dry ground, and 4 m/s on mud. What route should she take in order to reach her destination in the least amount of time?

Solution

Start by drawing a picture of the scenario and defining the variables. Shannon wants to arrive at a point 300 m west of her current location, on the other side of the field. She has to choose the point on the southern edge of the field she should run toward.



Let the distance x be defined as in the picture. Notice that x must be somewhere between 0 and 300.

The quantity you want to minimize is the total time Shannon takes to run to her destination. The total time is equal to the time she spends running diagonally across the field plus the time it takes her to run the remaining $300 - x$ metres along the southern edge of the field.

In general, you have the relationship speed = distance/time. Rearranging this, you have time = distance/speed.

You need to find the length of Shannon's diagonal path across the field. Using the Pythagorean theorem, the length of this diagonal path is $\sqrt{x^2 + 100^2} = \sqrt{x^2 + 10\,000}$ metres.

Shannon must run a distance of $\sqrt{x^2 + 10\,000}$ metres at a speed of 4 m/s, followed by a distance of $300 - x$ metres at a speed of 7 m/s.

Shannon's total elapsed time is

$$T = \frac{\sqrt{x^2 + 10\,000}}{4} + \frac{300 - x}{7} = \frac{1}{4}(x^2 + 10\,000)^{\frac{1}{2}} + \frac{1}{7}(300 - x).$$

You want to optimize the time T, and you have successfully expressed T as a function of a single variable. Take the derivative:

$$T' = \frac{1}{4} \cdot \frac{1}{2}(x^2 + 10\,000)^{-\frac{1}{2}} \cdot 2x + \frac{1}{7}(-1)$$

$$T' = \frac{x}{4(x^2 + 10\,000)^{\frac{1}{2}}} - \frac{1}{7}$$

Now you want to know where T' is zero, positive, or negative. Solve the equation $T' = 0$:

$$\frac{x}{4(x^2 + 10\,000)^{\frac{1}{2}}} - \frac{1}{7} = 0$$

$$\frac{x}{4(x^2 + 10\,000)^{\frac{1}{2}}} = \frac{1}{7}$$

$$\begin{aligned}
7x &= 4(x^2 + 10\,000)^{\frac{1}{2}} \\
(7x)^2 &= (4(x^2 + 10\,000)^{\frac{1}{2}})^2 \\
49x^2 &= 16(x^2 + 10\,000) \\
49x^2 &= 16x^2 + 160\,000 \\
33x^2 &= 160\,000 \\
x^2 &= \frac{160\,000}{33} = 4848.48 \\
x &= 69.63
\end{aligned}$$

The only point where $T' = 0$ is when $x = 69.63$. To determine if this really gives the minimum time, you have to see whether T' is positive or negative when x is greater or less than 69.63. For example, you can plug in $x = 60$ and $x = 80$ and see whether these give positive or negative values of T':

$$T'(60) = \frac{60}{4(60^2 + 10\,000)^{\frac{1}{2}}} - \frac{1}{7} = -0.0142$$

$$T'(80) = \frac{80}{4(80^2 + 10\,000)^{\frac{1}{2}}} - \frac{1}{7} = 0.0133$$

This verifies that T' is negative when $x < 69.63$ and T' is positive when $x > 69.63$. Therefore, T has a minimum when $x = 69.63$

You can conclude that Shannon should run to a point 69.63 m west of her current location. Her total elapsed time will be $\frac{\sqrt{69.63^2 + 10\,000}}{4} + \frac{300 - 69.63}{7} = 63.4$ seconds.

Support Questions
(do not send in for evaluation)

5. You have 50 cm of wire that you are going to break into two pieces. One of the pieces will be formed into a circle and the other will be formed into a square. How much wire should you form into the circle and how much into the square in order to minimize the total area of the two shapes? What is the minimum area?

6. You are creating a cylindrical container with a total volume of 20π m^3. The material for the top and bottom of the container costs \$10 per m^2, and the material for the side of the container costs \$8 per m^2. What dimensions should you make the container in order to minimize the total cost?

There are Suggested Answers to Support Questions at the end of this unit.

Conclusion

This is the end of the calculus part of the course. The remainder will focus on vectors in two- and three-dimensional space. You will learn how to represent vector quantities, and to represent lines and planes in two- and three-dimensional spaces.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(52 marks)

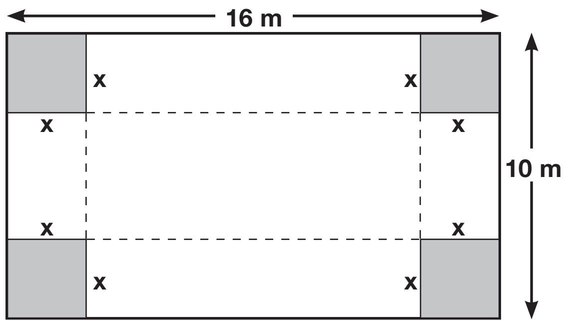
31. The number of bus riders from the suburbs to downtown per day is represented by $1200(1.50 - x)$, where x is the fare in dollars. What fare will maximize the total revenue? **(3 marks)**

32. A study has determined that interactions at a large social party follow the mathematical progression $N(t) = 30t - t^2$, where t is the time in minutes since the party began, and N is the number of separate conversations occurring. At what time in a party do the most conversations occur? What is the maximum number of interactions? **(4 marks)**

33. a) A new cottage is built across the river and 300 m downstream from the nearest telephone relay station. The river is 120 m wide. In order to wire the cottage for phone service, wire will be laid across the river under water, and along the edge of the river above ground. The cost to lay wire under water is \$15 per m and the cost to lay wire above ground is \$10 per m. How much wire should be laid under water to minimize the cost? **(11 marks)**

    b) Currently 2000 people attend performances of *The Sound of Music* if tickets cost \$40. Expenses are \$8 per person in attendance for each performance. For each \$2 decrease in the ticket price, 200 more people attend. Calculate the ticket price that produces maximum profit. **(7 marks)**

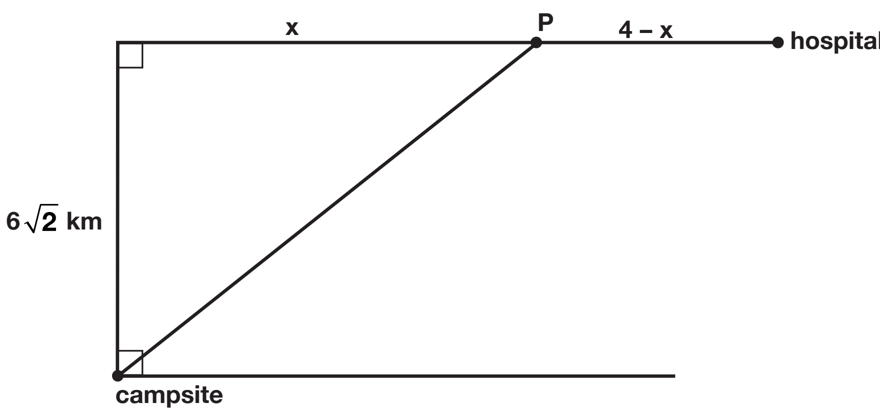
c) Four identical squares are cut from the corners of a rectangular sheet of metal with dimensions as shown. The sides are turned up to form a rectangular box. Find the dimensions of the box that has a maximum volume. **(7 marks)**



d) A large cylindrical soup can has a volume of 3456π cm^3. Find the radius and the height of the can that requires a minimum amount of material to construct it. ($V = \pi r^2 h$ and $S = 2\pi r^2 + 2\pi rh$) **(6 marks)**

e) The length of a cedar chest is twice its width. The cost/dm^2 of the lid is four times the cost/dm^2 of the rest of the cedar chest. If the volume of the cedar chest is 1440 dm^3, find the dimensions so that the cost is a minimum. Begin by drawing a diagram sketch. **(8 marks)**

f) Simon suffers an injury at a campsite on the bank of a canal that is $6\sqrt{2}$ km wide. If he is able to paddle his kayak at 5 km/h and pedal a bicycle at 15 km/h along the opposite bank, where should he land on the opposite bank to reach the hospital in minimum time? **(6 marks)**



Now go on to Lesson 13. Do not submit your coursework to ILC until you have completed Unit 3 (Lessons 11 to 15).