MAT136H1F - Calculus I (B)
Midterm Test — October 30th, 2015

Time: 110 minutes

Please, fill this page with ALL CAPITAL LETTERS:

Last name ....

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Student number ....

Please **clearly mark** your lecture and tutorial section (circle or fill):

Lecture		Tutorial				
L0101	L5101	T0101 (M3)	T0201 (R4)	T0301 (T4)	T5101 (T5)	T5201 (R5)

INSTRUCTIONS: (READ CAREFULLY!)

- This exam booklet contains 13 pages including this one. It consists of 10 questions. The maximum score is 60 points.
- **For each of question 1-4, write your final answer in the box provided**. Write any justifications, explanations, or calculations you need underneath.
- **Always justify your answers.** An incorrect final answer supported by correct calculations and explanations might still receive partial credit.
- **If you need extra space for a question, you may use the back of the pages.** If you do so, clearly indicate it on the corresponding problem page. You can also use the back of the pages for rough work.
- **Organize your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, calculators, cellphones, or any other electronic device.
- **Do not turn this page over until the invigilators instruct you to do so.**

Good luck!

FOR MARKERS ONLY:

Question	Marks	Value
1		4
2		5
3		8
4		3
5		4
6		8
7		4
8		5
9		6
10		13
Total		60

1. *[4 points] (2 points each).*

(a) Compute $\int (x^5 - 5x)\, dx$

Your answer: $\frac{x^6}{6} - \frac{5}{2}x^2 + C$

Solution:
Integrating yields $\int \left(x^5 - 5x\right)\, dx = \frac{x^6}{6} - \frac{5}{2}x^2 + C$.

(b) Suppose that $\int_{-1}^{3} f(x)\, dx = 5$ and $\int_{2}^{3} f(x)\, dx = -1$. Find $\int_{-1}^{2} f(x)\, dx$.

Your answer: 6

Solution:
Using the additivity property of the integral we have: $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$. Applying this to our case we have

$$\int_{-1}^{3} f(x)\, dx = \int_{-1}^{2} f(x)\, dx + \int_{2}^{3} f(x)\, dx.$$

Therefore,

$$\int_{-1}^{2} f(x)\, dx = \int_{-1}^{3} f(x)\, dx - \int_{2}^{3} f(x)\, dx = 5 - (-1) = 6.$$

2. *[5 points]*

(a) *[2 points]* Evaluate $\int_0^{\pi/2} \sin x\, e^{\cos x}\, dx$

Your answer: $e-1$

Solution:

We use substitution rule. Let $u = \cos(x)$. Then $du = -\sin(x)\,dx$. Also, $x = 0$ implies $u = \cos(0) = 1$ and $x = \frac{\pi}{2}$ implies $u = \cos(\frac{\pi}{2}) = 0$. Therefore,

$$\int_0^{\frac{\pi}{2}} \sin(x)e^{cos(x)}\,dx = -\int_1^0 e^u\,du = -\left[e^u\right]_1^0 = -e^0 + e^1 = e-1.$$

(b) *[3 points]* Evaluate $\int_1^2 2x \ln x\, dx$

Your answer: $4\ln(2) - \frac{3}{2}$**.**

Solution:

We use integration by parts.

Let $u = \ln(x)$ and $dv = 2x\,dx$. Then we have $du = \frac{1}{x}\,dx$ and $v = x^2$. Therefore,

$$\begin{aligned}
\int_1^2 2x\ln(x)\,dx &= uv]_1^2 - \int_1^2 v\,du \\
&= x^2\ln(x)\big]_1^2 - \int_1^2 x^2\frac{1}{x}\,dx \\
&= 4\ln(2) - \ln(1) - \int_1^2 x\,dx \\
&= 4\ln(2) - \frac{x^2}{2}\Big]_1^2 \\
&= 4\ln(2) - (2 - \frac{1}{2}) \\
&= 4\ln(2) - \frac{3}{2}.
\end{aligned}$$

3. *[8 points] (4 points each)*

(a) Compute $\int x^2 e^x\,dx$

Your answer: $x^2e^x - 2xe^x + 2e^x + C$

Solution:
Using integration by parts with $u = x^2$ and $dv = e^x\,dx$, we have $du = 2x\,dx$, $v = e^x$, and

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx.$$

Integrating by parts one more time with $\tilde{u} = x$ and $d\tilde{v} = e^x\,dx$, we have $d\tilde{u} = dx$, $\tilde{v} = e^x$, and

$$\int x^2e^x\,dx = x^2e^x - 2\left(xe^x - \int e^x\,dx\right) = x^2e^x - 2xe^x + 2e^x + C.$$

(b) Evaluate $\int_{-\pi}^{\pi} |\sin x|\,dx$

Your answer: 4

Solution 1:
Let $f(x) = |\sin(x)|$. Then $f(-x) = |\sin(-x)| = |-\sin(x)| = |\sin(x)|$. Hence, $f(x)$ is an even function. Therefore,

$$\begin{aligned}
\int_{-\pi}^{\pi} |\sin(x)|\,dx &= 2\int_0^{\pi} |\sin(x)|\,dx \\
&= 2\int_0^{\pi} \sin(x)\,dx, \text{ since } \sin(x) \geq 0 \text{ for all } x \in [0, -\pi] \\
&= 2\left[-\cos(x)\right]_0^{\pi} \\
&= 2(-\cos(-\pi) + \cos(0)) \\
&= 4.
\end{aligned}$$

Solution 2:
Notice that $\sin(x) \geq 0$, for all $x \in [0, \pi]$, and $\sin(x) \leq 0$ for all $x \in [-\pi, 0]$.
Therefore,

$$\begin{aligned}
\int_{-\pi}^{\pi} |\sin(x)|\,dx &= \int_{-\pi}^{0} |\sin(x)|\,dx + \int_0^{\pi} |\sin(x)|\,dx \\
&= \int_{-\pi}^{0} (-\sin(x))\,dx + \int_0^{\pi} \sin(x)\,dx \\
&= [\cos(x)]_{-\pi}^{0} + [-\cos(x)]_0^{\pi} \\
&= \cos(0) - \cos(-\pi) - \cos(\pi) + \cos(0) \\
&= 1 - (-1) - (-1) + 1 \\
&= 4.
\end{aligned}$$

4. *[3 points]*

A particle moves in a straight line with velocity $v(t) = e^{3t-3} + 1$.

Its position at time $t = 1$ is $s(1) = \frac{1}{3}$. Find its initial position $s(0)$.

Your answer: $\frac{1}{3}e^{-3} - 1$

Solution:

We know that $v(t) = s'(t)$. Therefore, to find $s(t)$ we need to find an antiderivative of v.

$$s(t) = \int v(t)\,dt = \int \left(e^{3t-3} + 1\right)\,dt = \frac{1}{3}e^{3t-3} + t + C.$$

At $t = 1$, we have

$$\frac{1}{3} = s(1) = \frac{1}{3}e^{3(1)-3} + 1 + C = \frac{1}{3} + 1 + C.$$

Hence, $C = -1$. Therefore, $s(0) = \frac{1}{3}e^{0-3} + 0 - 1 = \frac{1}{3}e^{-3} - 1$.

5. *[4 points] (2 points each)*

Solution:

Suppose that the temperature of a certain region is modeled by the function $T(t) = 10 - 5\cos\left(\frac{\pi}{12}t\right)$.

(a) Find the average temperature on the time interval $[0, 12]$. The average value is given by

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{12-0}\int_0^{12}\left(10 - 5\cos\left(\frac{\pi}{12}t\right)\right)\,dt \\ &= \frac{1}{12}\left[10t - \frac{60}{\pi}\sin\left(\frac{\pi}{12}t\right)\right]_0^{12} \\ &= \frac{1}{12}\left[120 - \frac{60}{\pi}\sin(\pi) + \frac{60}{\pi}\sin(0)\right] \\ &= 10. \end{aligned}$$

(b) Find a time t_0 that satisfies the Mean Value Theorem for Integrals for the temperature function $T(t) = 10 - 5\cos\left(\frac{\pi}{12}t\right)$ on the interval $[0, 12]$.

Solution:

By the Mean value Theorem for Integrals we know that there exists $t_0 \in [0, 12]$ such that

$$f(t_0) = f_{\text{ave}} = 10.$$

Solving the equation $f(t_0) = 10$ we have

$$\begin{aligned} 10 - 5\cos\left(\frac{\pi}{12}t_0\right) = 10 \quad &\Rightarrow \quad \cos\left(\frac{\pi}{12}t_0\right) = 0 \\ &\Rightarrow \quad \frac{\pi}{12}t_0 = \frac{\pi}{2} \qquad \text{(recall that } t \in [0, 12]\text{)} \\ &\Rightarrow \quad t_0 = 6. \end{aligned}$$

6. *[8 points] (4 points each)*

(a) Evaluate the following limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2}\frac{1}{n}$.

Solution:

First notice that this is a limit of Riemann sums. Therefore, we can rewrite the limit as an integral. The expression suggests that $\Delta x = \frac{1}{n}$. We try then $a = 0$ and $b = 1$. Hence, $x_i = a + i \cdot \Delta x = \frac{i}{n}$. Therefore, our function must be $f(x) = \frac{1}{1+x^2}$.
Hence,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2}\frac{1}{n} &= \int_0^1 \frac{1}{1+x^2}\,dx \\
&= tan^{-1}(x)\Big]_0^1 \\
&= tan^{-1}(1) - tan^{-1}(0) \\
&= \frac{\pi}{4} - 0 \\
&= \frac{\pi}{4}.
\end{aligned}$$

(b) Let $g(x) = \int_x^{e^x} \tan(\sin t)\,dt$. Find $g'(x)$.

Solution:

Using the additivity of the integral, we have

$$\begin{aligned}
g(x) &= \int_x^0 \tan(\sin(t))\,dt + \int_0^{e^x} \tan(\sin(t))\,dt \\
&= -\int_0^x \tan(\sin(t))\,dt + \int_0^{e^x} \tan(\sin(t))\,dt.
\end{aligned}$$

Then by the Fundamental Theorem of Calculus and the chain rule we have

$$\boxed{g'(x) = -\tan(\sin(x)) + \tan(\sin(e^x)) \cdot e^x}$$

Here we used the Chain Rule to find the derivative of the second term. If $u = e^x$, then $\int_0^{e^x} \tan(\sin(t))\,dt = \int_0^u \tan(sin(t))\,dt$. Therefore,

$$\frac{d}{dx}\int_0^{e^x} \tan(\sin(t))\,dt = \frac{d}{du}\int_0^{u} \tan(\sin(t))\,dt \cdot \frac{du}{dx} = \tan(\sin(u)) \cdot \frac{du}{dx} = \tan(\sin(e^x)) \cdot e^x.$$

7. *[4 points]* Compute $\displaystyle\int \frac{3}{x^2 - x - 2}\, dx$.

<u>Solution:</u>

Notice that this is a rational function. We can use the method of partial fractions.

Partial fraction decomposition.

$$\frac{3}{x^2 - x - 2} = \frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}.$$

Therefore, $3 = A(x+1) + B(x-2)$.

- If $x = 2$, then $3 = 3A \Rightarrow A = 1$.
- If $x = -1$, then $3 = -3B \Rightarrow B = -1$.

Hence,

$$\begin{aligned} \int \frac{3}{x^2 - x - 2}\, dx &= \int \left(\frac{1}{x-2} - \frac{1}{x+1} \right) dx \\ &= \ln|x-2| - \ln|x+1| + C \\ &= \ln\left| \frac{x-2}{x+1} \right| + C. \end{aligned}$$

8. *[5 points]* Compute $\int \tan^3 x \ \sec^{3/2} x \, dx$.

<u>Solution:</u>

Notice that

$$\begin{aligned}\tan^3(x)\sec^{3/2}(x) &= \tan^2(x)\sec^{1/2}(x)(\sec(x)\tan(x))\\ &= (\sec^2(x)-1)\sec^{1/2}(x)(\sec(x)\tan(x)).\end{aligned}$$

Therefore, if we let $u = \sec(x)$ then $du = \sec(x)\tan(x)\,dx$. Hence,

$$\begin{aligned}\int \tan^3 x \ \sec^{3/2} x \, dx &= \int (u^2-1)u^{1/2} \, du\\ &= \int (u^{5/2} - u^{1/2}) \, du\\ &= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C\\ &= \frac{2}{7}\sec^{7/2}(x) - \frac{2}{3}\sec^{3/2}(x) + C.\end{aligned}$$

9. *[6 points]* Compute $\int \sqrt{8 + 2x - x^2}\, dx$.

<u>Solution:</u>

This is the square root of a quadratic polynomial, so we would like to use trigonometric substitution. For that, we first complete the square:

$$8 + 2x - x^2 = 9 - 1 + 2x - x^2 = 9 - (x^2 - 2x + 1) = 9 - (x-1)^2.$$

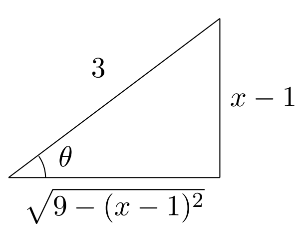
Therefore,

$$\int \sqrt{8 + 2x - x^2}\, dx = \int \sqrt{9 - (x-1)^2}\, dx.$$

Hence, if we let $x - 1 = 3\sin(\theta)$, then we have $dx = 3\cos(\theta)\, d\theta$. Thus,

$$\begin{aligned}
\int \sqrt{8 + 2x - x^2}\, dx &= \int \sqrt{9 - 9\sin^2(\theta)}(3\cos(\theta) d\theta) \\
&= \int 3\sqrt{1 - \sin^2(\theta)}(3\cos(\theta) d\theta) \\
&= \int 3\sqrt{\cos^2(\theta)}(3\cos(\theta) d\theta) \qquad \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos(\theta) \geq 0 \\
&= \int (3\cos(\theta))(3\cos(\theta))\, d\theta \\
&= \int 9\cos^2(\theta)\, d\theta \\
&= 9\int \frac{1 + \cos(2\theta)}{2}\, d\theta \\
&= 9\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + C \\
&= 9\left(\frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta)\right) + C \\
\text{(Back substitude)} \quad &= \frac{9}{2}\sin^{-1}\left(\frac{x-1}{3}\right) + \frac{9}{2}\frac{x-1}{3} \cdot \frac{\sqrt{9 - (x-1)^2}}{3} + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x-1}{3}\right) + \frac{1}{2}(x-1) \cdot \sqrt{9 - (x-1)^2} + C.
\end{aligned}$$

Notice that $x - 1 = 3\sin(\theta) \Rightarrow \sin(\theta) = \frac{x-1}{3}$ and so $\theta = \sin^{-1}\left(\frac{x-1}{3}\right)$.

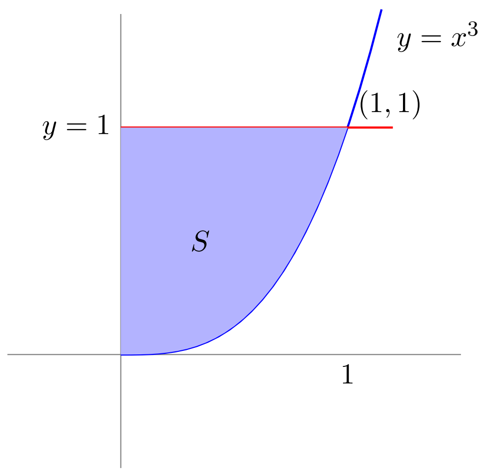


10. *[13 points]* Consider the region $\mathcal{R}$ enclosed by the curves $y = x^3$, $y = 1$ and $x = 0$.

Solution: Intersection points $x^3 = 1 \Leftrightarrow x = 1$

(a) *[2 points]* Sketch the region.

Solution:



(b) *[3 points]* Find the area of the region $\mathcal{R}$.

Solution:

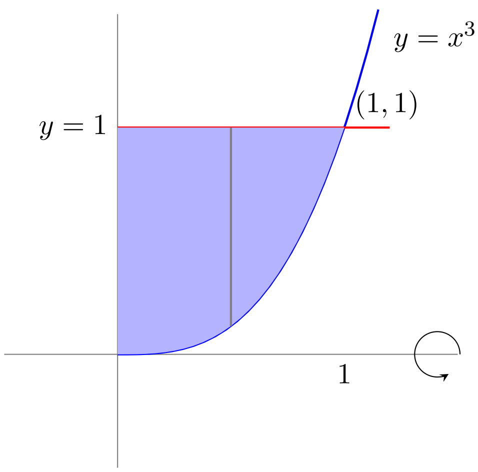
The area of $\mathcal{R}$ is

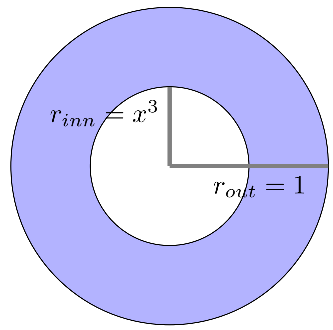
$$A = \int_0^1 (1 - x^3)\, dx = \left[x - \frac{x^4}{4}\right]_0^1 = 1 - \frac{1}{4} = \frac{3}{4}.$$

Find the volume of the solid obtained by rotating the region $\mathcal{R}$ about the x-axis in two different ways:

(c) *[4 points]* Integrating with respect to x.

Solution:



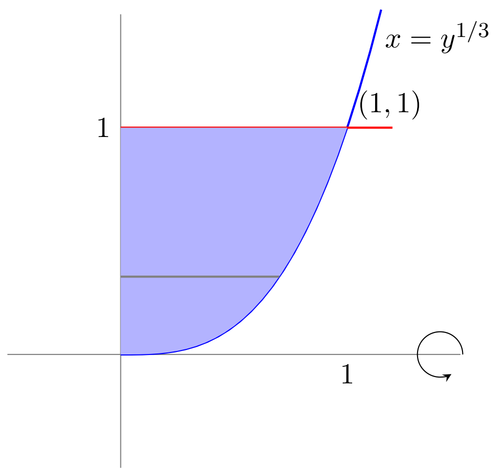


We use washers. The inner radius is $r_{inn} = x^3$ and the outer radius is $r_{out} = 1$. Therefore,

$$\begin{aligned} V &= \int_0^1 \pi(r_{out}^2 - r_{inn}^2) \\ &= \int_0^1 \pi(1 - x^6)\, dx \\ &= \pi \left[x - \frac{x^7}{7} \right]_0^1 \\ &= \pi \left(1 - \frac{1}{7} \right) \\ &= \frac{6}{7}\pi. \end{aligned}$$

(d) *[4 points]* Integrating with respect to y (Hint: Use cylindrical shells.).

<u>Solution:</u>



As the hint suggests, we use cylindrical shells. The radius is y and the height is $y^{1/3}$. Therefore,

$$\begin{aligned} V &= \int_0^1 2\pi \cdot y \cdot y^{1/3}\, dy \\ &= 2\pi \int_0^1 y^{4/3}\, dy \\ &= 2\pi \frac{3}{7} \cdot y^{7/3}\Big]_0^1 \\ &= 2\pi \cdot \frac{3}{7} \\ &= \frac{6}{7}\pi. \end{aligned}$$