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RSM332 **PROBLEM SET #3**

SOLUTIONS

1. (a) The expected returns on each security are all equal to 17%. As an example of how to calculate this,

$$E[R_1] = 0.4(0.3) + 0.4(0.1) + 0.1(0.1) + 0.1(0.0) = 0.17 \text{ or } 17\%.$$

The standard deviations on each security are respectively, $\sigma_1 = 11\%$, $\sigma_2 = 46.05\%$, and $\sigma_3 = 10.3\%$. As an example of how to calculate this,

$$\begin{aligned} \sigma_1 &= \left[0.4(0.3-0.17)^2 + 0.4(0.1-0.17)^2 + 0.1(0.1-0.17)^2 + 0.1(0.0-0.17)^2\right]^{\frac{1}{2}} \\ &= 0.11. \end{aligned}$$

(b) The covariances between the returns are respectively, $\text{Cov}[R_1, R_2] = -0.0479$, $\text{Cov}[R_1, R_3] = -0.0084$, and $\text{Cov}[R_2, R_3] = 0.043$. As an example of how to calculate this,

$$\begin{aligned} \text{Cov}[R_1, R_2] &= 0.4(0.3-0.17)(-0.3-0.17) + 0.4(0.1-0.17)(0.3-0.17) \\ &+ 0.1(0.1-0.17)(0.5-0.17) + 0.1(0.0-0.17)(1.2-0.17) = -0.0479. \end{aligned}$$

The correlations between the returns of different securities are respectively, $\text{Corr}[R_1, R_2] = -0.9456$, $\text{Corr}[R_1, R_3] = -0.7414$ and $\text{Corr}[R_2, R_3] = 0.9195$. As an example of how to calculate this,

$$\text{Corr}[R_1, R_2] \equiv \frac{\text{Cov}[R_1, R_2]}{\sigma_1 \sigma_2} = \frac{-0.0479}{(0.11)(0.4605)} = -0.9456.$$

(c) The expected return and standard deviation of a portfolio with equal proportions in two securities (i and j) is given by

$$\begin{aligned} \mu_p &= 0.5E[R_i] + 0.5E[R_j], \\ \sigma_p &= \sqrt{0.25\sigma_i^2 + 0.25\sigma_j^2 + 0.5\text{Cov}[R_i, R_j]}. \end{aligned}$$

Substituting in the values in parts (a) and (b) above gives us the same mean return on each of the portfolios, i.e. 17%. Substituting in the values in parts (a) and (b), the standard deviations of the portfolios are (i) $\sigma_p = 0.1792$ for the equally weighted portfolio of assets 1 and 2, (ii) $\sigma_p = 0.0384$ for the equally weighted portfolio of assets 1 and 3, and (iii) $\sigma_p = 0.2778$ for the equally weighted portfolio of assets 2 and 3.

(d) The expected return and standard deviation of a portfolio with equal proportions in the three securities is given by

$$\begin{aligned}
\mu_p &= \frac{1}{3}E[R_1] + \frac{1}{3}E[R_2] + \frac{1}{3}E[R_3], \\
\sigma_p &= \left[\frac{1}{9}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{2}{9}(\text{Cov}[R_1, R_2] + \text{Cov}[R_1, R_3] + \text{Cov}[R_2, R_3])\right]^{1/2}.
\end{aligned}$$

Substituting in the values in parts (a) and (b) gives us $\mu_p = 0.17$ and $\sigma_p = 0.1525$.

(e) The covariance between the return of the portfolio in part (d) and the return of an equally weighted portfolio of securities 1 and 2 is given by

$$\begin{aligned}
& \text{Cov}\left[\frac{1}{3}R_1 + \frac{1}{3}R_2 + \frac{1}{3}R_3, \frac{1}{2}R_1 + \frac{1}{2}R_2\right] \\
&= \frac{1}{6}\text{Var}[R_1] + \frac{1}{6}\text{Cov}[R_1, R_2] + \frac{1}{6}\text{Cov}[R_1, R_2] \\
&\quad + \frac{1}{6}\text{Var}[R_2] + \frac{1}{6}\text{Cov}[R_1, R_3] + \frac{1}{6}\text{Cov}[R_2, R_3] \\
&= \frac{(0.11)^2 - 0.0479 - 0.0479 + (0.4605)^2 - 0.0084 + 0.043}{6} \\
&= 0.0272.
\end{aligned}$$

Dividing by their standard deviations, we obtain the correlation between the returns of the two portfolios as $0.0272/(0.1792 \times 0.1525) = 0.994$.

2. We first figure out the tangency portfolio. Let x_1 be the weight on asset 1 and $1 - x_1$ be the weight on asset 2 of a portfolio. The expected excess return on this portfolio is

$$\mu_p - R_f = x_1(0.12 - 0.05) + (1 - x_1)(0.15 - 0.05) = 0.1 - 0.03x_1.$$

The standard deviation of a portfolio is

$$\begin{aligned}
\sigma_p &= \sqrt{x_1^2(0.05)^2 + (1 - x_1)^2(0.1)^2 + 2x_1(1 - x_1)(0.2)(0.05)(0.1)} \\
&= \sqrt{0.0105x_1^2 - 0.018x_1 + 0.01}.
\end{aligned}$$

Therefore, the Sharpe ratio of a portfolio is given by

$$SR_p = \frac{\mu_p - R_f}{\sigma_p} = \frac{0.1 - 0.03x_1}{\sqrt{0.0105x_1^2 - 0.018x_1 + 0.01}}.$$

The tangency portfolio is the portfolio with the maximum Sharpe ratio. Differentiating SR_p with respect to x_1, we obtain

$$\begin{aligned}\frac{\mathrm{d}SR_p}{\mathrm{d}x_1} &= \frac{-0.03\sigma_p - (0.1-0.03x_1)\frac{1}{2\sigma_p}(0.021x_1-0.018)}{\sigma_p^2} \\ &= \frac{-0.06\sigma_p^2 - (0.1-0.03x_1)(0.021x_1-0.018)}{2\sigma_p^3}.\end{aligned}$$

Setting the derivative equal to zero, we have

$$0.00156x_1 - 0.0012 = 0 \Rightarrow x_1^* = 0.76923.$$

Therefore, the tangency portfolio (T) should have a weight of 0.76923 in risky asset 1 and a weight of 0.23077 in risky asset 2.

Another way to solve for the weights of the tangency portfolio is to make use of the first order condition. Let $R_T = x_1^* R_1 + (1-x_1^*)R_2$, the tangency portfolio must satisfy

$$\begin{aligned}\frac{\mu_1 - R_f}{\mathrm{Cov}[R_1, R_T]} &= \frac{\mu_2 - R_f}{\mathrm{Cov}[R_2, R_T]} \\ \Rightarrow \frac{0.12-0.05}{x_1^*\sigma_1^2 + (1-x_1^*)\sigma_{12}} &= \frac{0.15-0.05}{x_1^*\sigma_{12} + (1-x_1^*)\sigma_2^2} \\ \Rightarrow \frac{0.07}{0.001+0.0015x_1^*} &= \frac{0.1}{0.01-0.009x_1^*} \\ \Rightarrow x_1^* &= \frac{10}{13}.\end{aligned}$$

The expected return of the tangency portfolio is $E[R_T] = x_1^*(0.12) + (1-x_1^*)(0.15) = 0.1269$. The optimal portfolio is a linear combination of the risk-free asset and the tangency portfolio. Let w_0 be the weight on the risk-free asset, we should have

$$0.05w_0 + 0.1269(1-w_0) = 0.1 \Rightarrow w_0 = 0.35.$$

Therefore, the investor should invest $\$1000 \times 0.35 = \$350$ in the risk-free asset. For the remaining \$650, he should invest in the tangency portfolio, which means $\$650 \times 0.76923 = \$500$ should be invested in risky asset 1 and $\$650 \times (1-0.76923) = \$150$ should be invested in risky asset 2.

3. Since T is the tangency portfolio, we have the following first order condition

$$\frac{E[R_p] - R_F}{\sigma_{pT}} = \frac{E[R_T] - R_F}{\sigma_T^2}.$$

Multiplying σ_T on both sides and using the fact that $\sigma_{pT} = \rho_{pT}\sigma_p\sigma_T$, we have

$$\begin{aligned}
\frac{E[R_p] - R_F}{\rho_{pT}\sigma_p} &= \frac{E[R_T] - R_F}{\sigma_T} \\
\Rightarrow \quad SR(p) &= \rho_{pT} SR(T) \\
\Rightarrow \quad SR(p) &= -0.75 \times 0.8 \\
\Rightarrow \quad SR(p) &= -0.6.
\end{aligned}$$

4. See `hw3a.xls` for answers.