

## Solutions to Problem Set 1

**Problem 1.** A real number  $r$  is called *sensible* if there exist positive integers  $a$  and  $b$  such that  $\sqrt{a/b} = r$ . For example, setting  $a = 2$  and  $b = 1$  shows that  $\sqrt{2}$  is sensible. Prove that  $\sqrt[3]{2}$  is not sensible. (Consider only positive real roots in this problem)

**Solution.** The proof is by contradiction. Assume for the purpose of contradiction that  $\sqrt[3]{2}$  is sensible. Then there exist positive integers  $a$  and  $b$  such that  $\sqrt{a/b} = \sqrt[3]{2}$ . Squaring both sides of this equation gives  $a/b = \sqrt[3]{4}$ , which implies that  $\sqrt[3]{4}$  is rational.

Since  $\sqrt[3]{4}$  is rational, we can write it as a fraction  $x/y$  in lowest-terms, where  $x$  is an integer and  $y$  is a positive integer. Therefore, we have:

$$\begin{aligned}\sqrt[3]{4} &= x/y \\ 4 &= x^3/y^3 \\ 4y^3 &= x^3\end{aligned}$$

In the last equation, the left side is even, and so the right side must be even. Since  $x^3$  is even,  $x$  itself must be even. This implies that the right side is actually divisible by 8, and so the left side must also be divisible by 8. Therefore,  $y^3$  is even, and so  $y$  itself must be even.

The fact that both  $x$  and  $y$  are even contradicts the fact that  $x/y$  is a fraction in lowest terms. Therefore,  $\sqrt[3]{2}$  is not sensible. ■

**Problem 2.** Translate the following sentence into a predicate formula:

There is a student who has e-mailed exactly two other people in the class, besides possibly herself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are equality and  $E(x, y)$ , meaning that “ $x$  has sent e-mail to  $y$ .”

**Solution.** A good way to begin tackling this problem is by trying to translate parts of the sentence. Begin by trying to assert that student  $x$  has emailed students  $y$  and  $z$ :

$$E(x, y) \wedge E(x, z).$$

Now we want to say that  $y$  and  $z$  not the same student, and neither of them is  $x$  either:

$$x \neq y \wedge x \neq z \wedge y \neq z,$$

where  $x \neq y$  abbreviates  $\neg (x = y)$ .

Now, we must think of a way to say that the only people  $x$  might have e-mailed are  $x$ ,  $y$  and  $z$ :

$$\forall s, E(x, s) \longrightarrow s = x \vee s = y \vee s = z.$$

Finally, we can say there is some student who emailed exactly two other two students by existentially quantifying  $x, y$  and  $z$ . So the complete translation is

$$\exists x \exists y \exists z. E(x, y) \wedge E(x, z) \wedge \quad (1)$$

$$x \neq y \wedge x \neq z \wedge y \neq z \wedge \quad (2)$$

$$\forall s, E(x, s) \longrightarrow s = x \vee s = y \vee s = z. \quad (3)$$

■

**Problem 3.** Express each of the following predicates and propositions in formal logic notation. The domain of discourse is the nonnegative integers,  $\mathbb{N}$ .

In addition to the propositional operators, variables and quantifiers, you may define predicates using addition, multiplication, and equality symbols, but no *constants* (like 0, 1, ...). For example, the proposition “ $n$  is an even number” could be written

$$\exists m. (m + m = n).$$

(a)  $n$  is the sum of three perfect squares.

**Solution.**

$$\exists x \exists y \exists z. (x \cdot x + y \cdot y + z \cdot z = n)$$

■

Since the constant 0 is not allowed to appear explicitly, the predicate “ $x = 0$ ” can’t be written directly, but note that it could be expressed in a simple way as:

$$x + x = x.$$

Then the predicate  $x > y$  could be expressed

$$\exists w. (y + w = x) \wedge (w \neq 0).$$

Note that we’ve used “ $w \neq 0$ ” in this formula, even though it’s technically not allowed. But since “ $w \neq 0$ ” is equivalent to the allowed formula “ $\neg(w + w = w)$ ,” we can use “ $w \neq 0$ ” with the understanding that it abbreviates the real thing. And now that we’ve shown how to express “ $x > y$ ,” it’s ok to use it too.

**(b)**  $x > 1$ .

**Solution.** The straightforward approach is to define  $x = 1$  as  $\forall y. xy = y$  and then express  $x > 1$  as  $\exists y. (y = 1) \wedge (x > y)$ . ■

**(c)**  $n$  is a prime number.

**Solution.**

$$\text{IS-PRIME}(n) ::= (n > 1) \wedge \neg(\exists x \exists y. (x > 1) \wedge (y > 1) \wedge (x \cdot y = n))$$

■

**(d)**  $n$  is a product of two distinct primes.

**Solution.**

$$\exists x \exists y. \neg(x = y) \wedge (n = x \cdot y) \wedge \text{IS-PRIME}(x) \wedge \text{IS-PRIME}(y).$$

■

**(e)** There is no largest prime number.

**Solution.** Of course this is a true statement, so it could be expressed by the logically equivalent formula “ $1 = 1$ ,” but even if we didn’t know this, we could transcribe the statement directly as:

$$\neg(\exists p. \text{IS-PRIME}(p) \wedge (\forall q. \text{IS-PRIME}(q) \longrightarrow p \geq q))$$

■

**(f)** (Goldbach Conjecture) Every even natural number  $n > 2$  can be expressed as the sum of two primes.

**Solution.** We can define  $n > 2$  with the formula  $\exists y. (y = 1) \wedge (x > y + y)$ . Likewise,  $n = 2k$  can be expressed as  $n = k + k$ . Then we can express the Conjecture with:

$$\forall n. ((n > 2 \wedge \exists k. n = 2k) \longrightarrow \exists p \exists q. \text{IS-PRIME}(p) \wedge \text{IS-PRIME}(q) \wedge (n = p + q)))$$

■

**(g)** (Bertrand's Postulate) If  $n > 1$ , then there is always at least one prime  $p$  such that  $n < p < 2n$ .

**Solution.**

$$\forall n. ((n > 1) \longrightarrow (\exists p. \text{IS-PRIME}(p) \wedge (n < p) \wedge (p < 2n)))$$

■

**Problem 4.** If a set,  $A$ , is finite, then  $|A| < 2^{|A|} = |\mathcal{P}(A)|$ , and so there is no surjection from set  $A$  to its powerset. Show that this is still true if  $A$  is infinite. *Hint:* Remember Russell's paradox and consider  $\{x \in A \mid x \notin f(x)\}$  where  $f$  is such a surjection.

**Solution.** We prove there is no surjection by contradiction: suppose there was a surjection  $f : A \rightarrow \mathcal{P}(A)$  for some set  $A$ . Let  $W ::= \{x \in A \mid x \notin f(x)\}$ . So by definition,

$$(x \in W) \longleftrightarrow (x \notin f(x)) \tag{4}$$

for all  $x \in A$ . But  $W \subseteq A$  by definition and hence is a member of  $\mathcal{P}(A)$ . This means  $W = f(a)$  for some  $a \in A$ , since  $f$  is a surjection to  $\mathcal{P}(A)$ . So we have from (4), that

$$(x \in f(a)) \longleftrightarrow (x \notin f(x)) \tag{5}$$

for all  $x \in A$ . Substituting  $a$  for  $x$  in (5) yields a contradiction, proving that there cannot be such an  $f$ . ■

**Problem 5. (a)** Prove that

$$\exists z. [P(z) \wedge Q(z)] \longrightarrow [\exists x. P(x) \wedge \exists y. Q(y)] \tag{6}$$

is valid. (Use the proof in the subsection on Validity in Week 2 Notes as a guide to writing your own proof here.)

**Solution. Proof.** Assume

$$\exists z. [P(z) \wedge Q(z)]. \quad (7)$$

That is,  $P(z) \wedge Q(z)$  holds for some element,  $z$ , of the domain. Let  $c$  be this element; that is, we have  $P(c) \wedge Q(c)$ .

In particular,  $P(c)$  holds by itself. So we conclude (by Existential Generalization)  $\exists x P(x)$ . We conclude  $\exists y Q(y)$  similarly. Hence,

$$\exists x. P(x) \wedge \exists y. Q(y) \quad (8)$$

holds.

This shows that (8) holds in any interpretation in which (7) holds. Therefore, (7) implies (8) in all interpretations, that is, (6) is valid.  $\square$

■

(b) Prove that the converse of (6) is not valid by describing a counter model as in Week 2 Notes.

**Solution. Proof.** We describe a counter model in which, (8) is true and (7) is false. Namely, let the domain,  $D$ , be  $\{\pi, e\}$ ,  $P(x)$  mean “ $x = \pi$ ,” and  $Q(y)$  mean “ $y = e$ .” Then,  $\exists x. P(x)$  is true (let  $x$  be  $\pi$ ) and likewise  $\exists y. Q(y)$  is true (let  $y$  be  $e$ ), so (8) holds.

On the other hand,  $Q(\pi)$  is not true, so  $P(\pi) \wedge Q(\pi)$  is not true. Likewise  $P(e) \wedge Q(e)$  is not true. Since these are the only elements of  $D$ , it is not true that there is an element,  $z$ , of  $D$ , such that  $P(z) \wedge Q(z)$ . That is, (7) is not true.  $\square$

■

**Problem 6. (a)** Give an example where the following result fails:

**False Theorem.** For sets  $A, B, C$ , and  $D$ , let

$$\begin{aligned} L &::= (A \cup C) \times (B \cup D), \\ R &::= (A \times B) \cup (C \times D). \end{aligned}$$

Then  $L = R$ .

**Solution.** If  $A = D = \emptyset$  and  $B$  and  $C$  are both nonempty, then  $L = C \times B \neq \emptyset$ , but  $R = \emptyset$ .  $\blacksquare$

(b) Identify the mistake in the following proof of the False Theorem.

*Bogus proof.* Since  $L$  and  $R$  are both sets of pairs, it's sufficient to prove that  $(x, y) \in L \longleftrightarrow (x, y) \in R$  for all  $x, y$ .

The proof will be a chain of iff implications:

$(x, y) \in L$	iff
$x \in A \cup C$ and $y \in B \cup D$ ,	iff
either $x \in A$ or $x \in C$ , and either $y \in B$ or $y \in D$ ,	iff
$(x \in A$ and $y \in B)$ or else $(x \in C$ and $y \in D)$ ,	iff
$(x, y) \in A \times B$ , or $(x, y) \in C \times D$ ,	iff
$(x, y) \in (A \times B) \cup (C \times D) = R$ .	

□

**Solution.** The mistake is in the third “iff.” If  $[x \in A$  or  $x \in C$ , and either  $y \in B$  or  $y \in D]$ , it does not necessarily follow that  $(x, y) \in (A \times B) \cup (C \times D)$ . It might be that  $(x, y)$  is in  $A \times D$  instead. This happens, for example, if  $A = \{1\}$ ,  $B = \{2\}$ ,  $C = \{3\}$ ,  $D = \{4\}$ , and  $(x, y) = (1, 4)$ . ■

(c) Fix the proof to show that  $R \subseteq L$ .

**Solution.** Replacing the third “iff” by “which will be true when,” yields a correct proof that  $(x, y) \in L$  will be true when  $(x, y) \in R$ , which implies that  $R \subseteq L$ . ■