Problem 1. (20 pts) Construct a truth table for the following compound propositions

a. (6 pts) $p \oplus (p \to q)$.

b. (7 pts) $(p \to q) \vee (\neg p \leftrightarrow q)$.

c. (7 pts) $(p \oplus q) \to (p \wedge \neg q)$.

a)

p	q	$p \to q$ (3pts)	$p \oplus (p \to q)$ (3pts)
T	T	T	F
T	F	F	T
F	T	T	T
F	F	T	T

b)

p	q	$p \to q$ (2pts)	$\neg p$	$\neg p \Leftrightarrow q$ (2pts)	$(p \to q) \vee (\neg p \Leftrightarrow q)$ (3pts)
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	T	F	T

c)

p	q	$p \oplus q$ (2 pts)	$p \wedge \neg q$ (2 pts)	$(p \oplus q) \to (p \wedge \neg q)$ (3 pts)
T	T	F	F	T
T	F	T	T	T
F	T	T	F	F
F	F	F	F	T

Problem 2. (20 pts) Use logical equivalence laws to show that the following compound propositions are tautologies:

a. (6 pts) $p \to (p \lor q)$.

b. (6 pts) $(p \land q) \to (p \to q)$.

c. (8 pts) $(\neg p \land (p \lor q)) \to q$.

Note: You will only get 50% of the mark for this problem if you use truth tables instead of logical equivalence laws.

a) $p \to (p \lor q) \equiv \neg p \lor (p \lor q)$

$\equiv (\neg p \lor p) \lor q$

$\equiv T \lor q \equiv T$.

b) $(p \land q) \to (p \to q) \equiv \neg (p \land q) \lor (p \to q)$

$\equiv (\neg p \lor \neg q) \lor (\neg p \lor q)$

$\equiv \neg p \lor (\neg q \lor q)$

$\equiv \neg p \lor T \equiv T$.

c) $\left(\neg p \wedge (p \vee q)\right) \to q$

$$\equiv \neg\left(\neg p \wedge (p \vee q)\right) \vee q$$

$$\equiv p \vee \neg(p \vee q) \vee q$$

$$\equiv (p \vee q) \vee \left(\neg(p \vee q)\right) \equiv T.$$

Since if $Q \equiv p \vee q$ then the above is

$$Q \vee \neg Q \equiv T.$$

Problem 3. (20 pts) I) (8 pts) Find the negation of the quantified statements

a. (3 pts) $\exists x\ \forall y\ (x^2 < y)$.

b. (5 pts) $\forall x\ \forall y\ \big(((x<0) \wedge (y<0)) \to (xy>0)\big)$.

II) (12 pts) Find a counterexample to show that the following universally quantified statements are false, where the domain of the variables consists of all integers.

c. (6 pt) $\forall x\ \exists y\ (y^2 = x)$.

d. (6 pt) $\forall x\ \forall y\ ((x^2 = y^2) \to (x = y))$.

a) $\forall x\ \exists y\ (x^2 \geq y)$

b) $\exists x\ \exists y\ \neg\Big([(x<0) \wedge (y<0)] \to (xy>0)\Big)$

$\equiv \exists x\ \exists y\ \neg\Big(\neg[(x<0)\wedge(y<0)] \vee (xy>0)\Big)$

$\equiv \exists x\ \exists y\ \Big((x<0)\wedge(y<0) \wedge (xy \leq 0)\Big)$

c) The negation is

$\exists x\ \forall y\ (y^2 \neq x)$

Hence $\boxed{x=-1}$ is a counterexample, since for all integers y, $y^2 \geq 0 > -1$

So $y^2 \neq -1$.

d) The negation is

$$\exists x \, \exists y \left((x^2 = y^2) \wedge (x \neq y) \right)$$

Hence $x = 1$ and $y = -1$ is a Counterexample

since $1^2 = (-1)^2$ but $1 \neq -1$.
(and)

Problem 4. (20 pts)

a. (10 pts) Use a proof by contraposition to show that if n is an integer such that n^3+1 is odd, then n is even.

b. (10 pts) Let x, y, z be real numbers and put

$$s = \frac{x+y+z}{3}.$$

Use a proof by contradiction to show that at least one of the numbers x, y, z is greater than or equal to s.

a) Let p: n^3+1 is odd. q: n is even

Then $\boxed{p \to q \equiv \neg q \to \neg p}$ (2 pts)

Here $\neg q$: n is odd and $\neg p$: n^3+1 is even (8 pts)

Now, if n is odd, then $n = 2k+1$ for some integer k. Hence

$$n^3+1 = (2k+1)^3 + 1 = (2k+1)(4k^2+4k+1)+1$$

$$= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 + 1$$

$$= 2(4k^3 + 6k^2 + 3k + 1) \text{ is even}$$

as desired.

b) Here let p: at least one of x, y, z is greater than or equal to S

Assume by contradiction that $\neg p$ is true. That is: all the numbers x, y, z are $< S$.

Hence $x + y + z < 3S$ since $x < S$, $y < S$ and $z < S$.

but $3S = x + y + z$. So we obtain

$x + y + z < x + y + z$ which is a contradiction.

Problem 5. **(20 pts)**

a. (10 pts) Let A and B be sets such that $|A| = 4$, $|B| = 5$ and $|A \cap B| = 2$. Find the cardinality of the following sets (justify your answer):

$i)$ $A \cup B$, $ii)$ $A \times B$, $iii)$ $\mathcal{P}(A)$ (the power set of A), $iv)$ $A - B$, $v)$ $B - A$.

b. (10 pts) Let A and B be sets. Prove, without using a Venn Diagram, that

$$A - B = A \cap \overline{B}.$$

a) i) (2 pts) by the principle of inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B| = 4 + 5 - 2 = \boxed{7}.$$

(2 pts) ii) $|A \times B| = |A| \cdot |B| = 4 . 5 = \boxed{20.}$

iii) (2 pts) We have:

$$|\mathcal{P}(A)| = 2^{|A|} = 2^4 = \boxed{16}.$$

iv) (2 pts) We have $|A - B| = |A| - |A \cap B|$

$$= 4 - 2 = \boxed{2.}$$

v) (2 pts) $|B - A| = |B| - |A \cap B| = 5 - 2 = \boxed{3}$

b) We are going to show that

$$A - B \subseteq A \cap \overline{B} \quad \text{and} \quad A \cap \overline{B} \subseteq A - B.$$

<u>Part 1</u>, (5 pts). $\left(A - B \subseteq A \cap \overline{B}\right)$.

Let $x \in A - B$, then $x \in A$ and $x \notin B$

Hence $\boxed{x \in A}$ <u>and</u> $\boxed{x \in \overline{B}}$ and so $x \in A \cap \overline{B}$.

<u>Part 2</u>, (5 pts) $\left(A \cap \overline{B} \subseteq A - B\right)$.

Let $x \in A \cap \overline{B}$, then $x \in A$ <u>and</u> $x \in \overline{B}$ so

$x \in A$ and $x \notin B$ and hence $\boxed{x \in A - B}$