Math 135 Algebra

Midterm Examination

June 6, 2005 7:00 – 9:00 p.m.

Instructor: Ashwin Nayak

Instructions

- Please print your name and student identification number.
  - NAME ____________________
  - STUDENT IDENTIFICATION NUMBER ____________
- Make sure you have all **eight** pages (including this cover page).
- Only the Faculty approved calculators with the Pink Tie stickers will be allowed on the midterm.
- **When solving a numerical example, show all of your work.** (You do not have to show how you added, divided, etc., just what you added, divided, etc., and the result.)
- If you require more space to present your solution, please use the back of the **previous** page. Indicate clearly where your solution continues.

Question	Value	Mark		Question	Value	Mark
1	12			2	10	
3	12			4	12	
5	8			6	10	
7	8					
				Total	72	

Question 1. **(a)** [4 marks] Consider the following two statements:

$$\exists x\, \exists y (P(x) \implies Q(y)), \quad \text{and} \quad (\forall x\, P(x)) \implies (\exists y\, Q(y))$$

Give an example of a universe of discourse, and meaning for P and Q for which the **first** statement is **true**. Give an example where the **second** statement is **false**.

(b) [4 marks] Negate the two statements given in part (a) above, and simplify them so that no quantifier or compound statement is negated. Show the steps in the simplification.

(c) [4 marks] Are the two statements given in part (a) above equivalent? Justify your answer appropriately.

Question 2. (a) [4 marks] Factorize the two integers 11571 and 9338 into a product of primes, and compute their greatest common divisor. Explain the method you used to factorize the numbers, and show your steps.

(b) [6 marks] Find all integers x, y such that $\gcd(11571, 9338) = 11571x + 9338y$.

Question 3. **(a)** [6 marks] Prove that $\gcd(a, c) = \gcd(b, c) = 1$ if and only if $\gcd(ab, c) = 1$.

(b) [6 marks] The nickel (5 cent coin) slot of a pay phone will not accept coins. Can a call costing 95 cents be paid for exactly using only dimes (10 cent coins) and quarters (25 cent coins)? If so, in how many ways can it be done?

Question 4. [12 marks] Let $a, b, c \in \mathbb{Z}$. State whether the following statements are true or false, and give a short justification.

(i) If $c = ax + by$ for some $x, y \in \mathbb{Z}$, then $c = \gcd(a, b)$.

(ii) Any common divisor of $a, b \in \mathbb{Z}$ divides $\gcd(a, b)$.

(iii) If $c|(ab)$, then $c|a$ or $c|b$.

(iv) $\gcd(ab, c) = \gcd(a, c) \cdot \gcd(b, c)$ if a and b are prime numbers.

(v) If a, b are distinct primes, then $ax + by = c$ has an integer solution x, y for every c.

(vi) If $a \equiv b \pmod{c}$ for some $c > 0$, then b is the remainder when a is divided by c.

Question 5. (a) [4 marks] Prove that for all $a, b \in \mathbb{Z}$, $\gcd(a, a+b) = \gcd(a, b)$.

(b) [4 marks] Recall from the lectures that the Fibonacci sequence is defined as

$$f_0 = 0, \quad \text{and} \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}, \qquad \text{for all } n \geq 2.$$

Prove that for all $n \geq 0$, $\gcd(f_n, f_{n+1}) = 1$.

Question 6. **(a)** [6 marks] Let $n \geq 1$ be an integer. Recall from the lectures that the n-th harmonic number H_n is defined as

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n}.$$

Using mathematical induction, prove that $H_{2^m} \leq 1 + m$ for all $m \geq 0$. (In H_{2^m}, the subscript 2^m is 2 raised to the power m.)

(b) [4 marks] Define the notation $a \equiv b \pmod{m}$.

Question 7. **(a)** [4 marks] For each integer $x \in X = \{0, 1, 2, 3, \ldots, 15\}$, find the smallest **positive** integer y such that $x^2 \equiv y \pmod{16}$.

(b) [4 marks] Using the properties of congruences, find the smallest **positive** integer congruent to 10^{45} modulo 7. Show the steps in your calculations.