University of Ottawa
Department of Mathematics and Statistics

MAT 1341C: Introduction to Linear Algebra
Instructor: Erhard Neher

Final Exam (April 2009) **Time: 3 hours**

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FIRST NAME (CAPITALS)	______________________
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Student number	______________________

Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. For questions 10–16 you may use the back of pages if necessary, but be sure to indicate to the marker that you have done so. There is an extra page at the end of the exam.
- Questions 1–9 are short answer questions, and no part marks will be given. You must show all the details for questions 10–16, and argue logically. Write legibly.
- Where it is possible to check your work, do so! Read each question carefully – you will save yourself time and unnecessary grief later on.
- **This is a closed book exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.**
- Do not detach the pages of the exam.

Good luck! Bonne chance !

Question	1–4	5–8	9–10	11–12	13–14	15–16	Total
Score							
Max. score	11	11	8	11	11	14	66

(1) (3 pts) For a homogeneous linear system answer the following questions

(a) If the system has 6 equations and 7 variables and the rank of the coefficient matrix is 5, the general solution has how many parameters?

My answer:__________________

Solution: The number of parameters is $n - r$, where n is the number of variables and r is the rank of the coefficient matrix. Hence in our case $n = 7$, $r = 5$, so the general solution depends on 2 parameters.

(b) If the system has more equations than variables, it has infinitely many solutions. Answer with T for true or F for false.

My answer:__________________

Solution: This is false. The number of equations is not important for solvability. For example, the system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_1 + 3x_2 &= 0 \end{aligned}$$

has 3 equations, 2 variables, but it is uniquely solvable.

(c) If the system has only the trivial solution, the coefficient matrix is invertible. Answer with T for true or F for false.

My answer:__________________

Solution: False. The coefficient matrix need not be a square matrix, see the example above.

(2) (2 pts) If $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$, compute the first column of $C = AB$.

Solution: Using the multiplication scheme "Row times Column" one gets that $C = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$, so the answer is $\begin{bmatrix} 12 \\ 8 \end{bmatrix}$

My answer:__________________

(3) (3 pts) Calculate the inverse of the 3×3 matrix: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution: We row-reduce:

$$A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \overset{R_2-2R_1,R_3-2R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\overset{-R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \overset{R_1-R_2,R_2-R_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 4 & -1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right].$$

It follows that A is invertible and $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -1 & -1 \\ -2 & 0 & 1 \end{bmatrix}$.

My answer:__________________

(4) (3 pts) Calculate the determinant of the following matrix: $A = \begin{bmatrix} -3 & -1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$.

Solution: Expand across the first row of A and get

$$\begin{aligned} \det A &= (-3)(-1)^{1+1} \det \begin{bmatrix} -4 & 3 \\ 4 & -2 \end{bmatrix} + (-1)(-1)^{1+2} \det \begin{bmatrix} -2 & 3 \\ 5 & -2 \end{bmatrix} \\ &= (-3)(8-12) + (4-15) = 12 - 11 = 1. \end{aligned}$$

My answer:__________________

(5) (3 pts) Let V be a vector space of dimension 11. Answer the following questions with T for true and F for false.

(a) Every subset of 11 vectors if V is a basis of V.

My answer:__________________

(b) Every spanning set of V can be extended to a basis of V.

My answer:__________________

(c) V contains exactly 3 subspaces of dimension 3.

My answer:__________________

Solution: (a) is false: Only a **spanning** set of 11 vectors is a basis, or a **linearly independent** subset of 11 vectors is a basis of V.

(b) is false: Every **linearly independent** subset can be extended to a basis; every spanning set **contains** a basis.

(c) is false: There are infinitely many subspaces of dimension n for $1 \leq n \leq 10$.

(6) (3 pts) Let $T : \mathbb{R}^4 \to \mathbb{R}^7$ be a linear map whose kernel has the basis $v = \begin{bmatrix} 1 & 0 & 1 & 2 \end{bmatrix}^T$. What is the dimension of the image of T?

My answer:__________________

Solution: The dimension of $ker\, T$ is 1 and the dimension of $\mathbb{R}^4$ is 4. Hence, by the Dimension Theorem $\dim im\, T = 4 - \dim ker\, T = 4 - 1 = 3$.

(7) (3 pts) If $T : \mathbb{P}_2 \to \mathbb{R}^2$ is a linear map with $T(1+x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(x+x^2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $T(x^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $T(1) = ?$

Solution: It is immediate that $1 = (1+x) - (x+x^2) + x^2$. Of course, this can also be seen as follows: We write 1 as a linear combination of the three given polynomials with unknown coefficients s_i:

$$1 = s_1(1+x) + s_2(x+x^2) + s_3x^2 = s_1 1 + (s_1+s_2)x + (s_2+s_3)x^2.$$

Comparing coefficients yields $s_1 = 1$, $s_1 + s_2 = 0$ and $s_2 + s_3 = 0$. Therefore $s_1 = 1 = -s_2 = s_3$. Hence

$$T(1) = T(1+x) - T(x+x^2) + T(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

My answer:________________

(8) (2 pts) Let $Z = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. The map $T : \mathbb{R}^3 \to \mathbb{R}$, $T(X) = X \cdot Z$ is linear. Find its standard matrix.

Solution: If $X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ then $T(X) = x_1 + 2x_2 + 3x_3$. Since the standard matrix A of T is given by $A = [T(E_1) \quad T(E_2) \quad T(E_3)]$ where $E_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $E_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ it follows that $A = [1 \quad 2 \quad 3]$.

My answer:__________________________

(9) (3 pts) For an $n \times n$ matrix A answer the questions (ii), (iii) and (iv) below. As an example to show you what type of answer is expected from you, I have given the answer to question (i),

(i) State a condition on the columns of A which is equivalent to the condition rank $(A) < n$.

Answer : *The columns of A are linearly dependent.*

(ii) State a condition on the determinant of A which is equivalent to the condition rank $(A) < n$.

Answer : ______________________________

(iii) State a condition regarding the homogeneous linear system $AX = 0$ which is equivalent to the condition rank $(A) < n$.

Answer : ______________________________

(iv) State a condition regarding invertibility of A which is equivalent to the condition rank $(A) < n$.

Answer : ______________________________

Solution: (ii) $\det(A) = 0$. (iii) The homogenous linear system $AX = 0$ has a non-trivial solution; equivalently, it has infinitely many solutions. (iv) A is not invertible.

(10) (5 pts) Let U be the subspace of $\mathbb{R}^4$ spanned by $\{v_1, v_2, v_3, v_4, v_5\}$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 11 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 13 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}.$$

Find a basis of U and determine the dimension of U.

Solution: U is the column space of the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 1 & 2 & 4 & 5 & 1 \\ 1 & 3 & 5 & 7 & 2 \\ 3 & 5 & 11 & 13 & 2 \end{bmatrix}$.

The reduced row-echelon form of A is $R = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The leading 1's are in the columns 1, 2 and 5 of R. Hence the columns 1, 2 and 5 of A are a basis of the column space of A.

Answer : $\{v_1, v_2, v_5\}$ is a basis of U, therefore $\dim U = 3$.

Other solution: One can take the vectors $v_1, \ldots, v_5$ as the row vectors of a 5×4-matrix B, and then calculate the (reduced) row echelon form. The rows with leading 1's are a basis of the row space of B, the transpose vectors are a basis of U.

(11) (a) (3 pts) Find all eigenvalues of the matrix A below together with their multiplicities:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) (1 pt) Based on the knowledge of the eigenvalues alone, decide if A is diagonalizable or not. Justify your answer!

(c) (1 pt) Based on the knowledge of the eigenvalues alone, decide if A is invertible or not. Justify your answer!

Solution: We calculate the characteristic polynomial

$$\det(xI_3 - A) = \begin{vmatrix} x & 0 & -1 \\ 3 & x-5 & -1 \\ 0 & 0 & x-3 \end{vmatrix} = (x-3)\begin{vmatrix} x & 0 \\ 3 & x-5 \end{vmatrix}$$
$$= (x-3)x(x-5)$$

The eigenvalues are the roots of $c_A(x)$, which are $0, 3, 5$, all with multiplicity 1.

(b) Since A has 3 distinct eigenvalues, A is diagonalizable.

(c) Since A has eigenvalue 0, it is not invertible.

(12) (6 pts) The eigenvalues of the matrix

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

are 0 and 1. (You do not need to show this.)

(a) (4 pts) For each eigenvalue of A find a basis of the corresponding eigenspace.

(b) (2 pts) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Justify your answer.

Solution: (a) For the eigenvalue $\lambda = 0$ we have to solve the homogeneous linear system with coefficient matrix $-A$:

$$-A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding homogeneous linear system is $x + 2z = 0$, $y - z = 0$, whose general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad (t \text{ a real parameter})$$

Hence a basis of the eigenspace $E_0(A)$ is

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

For the eigenvalue $\lambda = 1$ we have to solve the homogeneous linear system with coefficient matrix $I_3 - A$:

$$I_3 - A = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

The corresponding homogeneous linear system is $x + z = 0$, whose general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad (s \text{ and } t \text{ are real parameter})$$

Hence a basis of the eigenspace $E_0(A)$ is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Since for each eigenvalue the multiplicity equals the dimension of the corresponding eigenspace, the matrix A is diagonalizable, $PAP^{-1} = D$ with P the matrix of basic eigenvectors and D the diagonal matrix of eigenvalues

$$P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(13) For the system of linear equations

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 2x & + & ay & - & 12z & = & -10 \\ -x & - & y & + & (a+2)z & = & 1 \end{array}$$

(a) (5 pts) determine the values of a for which the system has
 (i) no solution,
 (ii) infinitely many solutions,
 (iii) a unique solution.

(b) (2 pts) In case (ii) above describe give all solutions.

Solution: The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & a & -12 & -10 \\ -1 & -1 & a+2 & 1 \end{array}\right]$$

We perform the following operations, where R_i is row i: $R_2 \rightsquigarrow R_2 - 2R_1$, $R_3 \rightsquigarrow R_1 + R_3$; we then switch R_2 and R_3 and replace row R_3 by $R_3 - (a-4)R_2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & a-4 & -18 & -12 \\ 0 & 1 & a+5 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a+5 & 2 \\ 0 & a-4 & -18 & -12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a+5 & 2 \\ 0 & 0 & b & c \end{array}\right] = B$$

where $b = -18 - (a-4)(a+5) = -18 - a^2 - a + 20 = -(a^2 + a - 2) = -(a-1)(a+2)$ and $c = -12 - 2(a-4) = -12 - 2a + 8 = -2(a+2)$. Hence

$$B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a+5 & 2 \\ 0 & 0 & -(a-1)(a+2) & -2(a+2) \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a+5 & 2 \\ 0 & 0 & (a-1)(a+2) & 2(a+2) \end{array}\right] = C$$

We get:

- If $a = 1$, then the last row of C is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 6 \end{array}\right]$. Hence the system is inconsistent.
- If $a = -2$ the last row of C is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$. Hence the system has infinitely many solutions.
- If $a \notin \{-2, 1\}$, then

$$C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a+5 & 2 \\ 0 & 0 & * & * \end{array}\right]$$

  where the stars "$*$" are non-zero numbers. Hence the system is uniquely solvable, because there does not exist a free variable.

The answer to question (a) is therefore:
 (i) The system in inconsistent if $a = 1$.
 (ii) The system has infinitely many solutions if $a = -2$.
 (iii) The system is uniquely solvable if $a \notin \{1, -2\}$.

To answer (b), let $a = -2$ in the matrix C above. Then

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence the corresponding linear system is $x - 3z = -3$ and $y + 3z = 2$ whose general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3+3t \\ 2-3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

hence the set of solutions is

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \qquad (t \in \mathbb{R})}$$

(14) (4 pts) Are the following sets linear independent? Give details!
(a) $\left\{ \frac{1}{x^2-1}, \frac{1}{x^2+2x-3}, \frac{1}{x^2+4x+3} \right\}$ in $\mathbb{F}[-1/2, 1/2]$, where $\mathbb{F}[-1/2, 1/2]$ is the vector spaces of functions $f : [-1/2, 1/2] \to \mathbb{R}$;

(b) $\left\{ \begin{bmatrix} -7 & 13 \\ -9 & 47 \end{bmatrix}, \begin{bmatrix} 0 & 16 \\ 25 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & -6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 3 \end{bmatrix} \right\}$ in $\mathbb{M}_{22}$, where $\mathbb{M}_{22}$ is the vector space of 2×2 matrices.

My answer:__________________

Solution: (a) (3 pts) Observe that $x^2 - 1 = (x-1)(x+1)$, $x^2 + 2x - 3 = (x-1)(x+3)$ and $x^2 + 4x + 3 = (x+1)(x+3)$. Suppose there exists scalars $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$s_1 \frac{1}{x^2-1} + s_2 \frac{1}{x^2+2x-3} + s_3 \frac{1}{x^2+4x+3} = 0$$

in $\mathbb{F}[-1/2, 1/2]$. Then, multiplying with the common denominator $(x-1)(x+1)(x+3) = p$

$$\begin{aligned} 0 &= s_1 \frac{x+3}{p} + s_2 \frac{x+1}{p} + s_3 \frac{x-1}{p} = \frac{1}{p}\big(s_1(x+3) + s_2(x+1) + s_3(x-1)\big) \\ &= \frac{1}{p}\big((s_1+s_2+s_3)x + (3s_1 + s_2 - s_3\big) \end{aligned}$$

Multiplying by p shows that the polynomial $(s_1+s_2+s_3)x + (3s_1+s_2-s_3) = 0$ on $[-1/2, 1/2]$. This forces $s_1 + s_2 + s_3 = 0$ and $3s_1 + s_2 - s_3 = 0$. Since this is a homogeneous linear system with more variables than equations, it has a non-trivial solution, which means that the 3 functions are not linear independent.

Indeed, we have

$$\frac{1}{x^2-1} - 2\frac{1}{x^2+2x-3} + \frac{1}{x^2+4x+3} = 0.$$

(b) (1 pt) No, since $\dim \mathbb{M}_{22} = 4$ and hence any set with more than 4 matrices, like the one given, is linearly independent.

(15) (8 pts) Show that $U = \{p \in \mathbb{P}_3 : 3p(1) = p(0)\}$ is a subspace of $\mathbb{P}_3$, find a basis and determine $\dim U$. (Recall that $\mathbb{P}_3$ is the vector space of polynomials of degree ≤ 3.)

(bonus question: (2 bonus pts) Find a linear transformation $T : \mathbb{P}_3 \to \mathbb{R}$ such that U is the kernel of T.)

Solution: Let $p \in \mathbb{P}_3$, say $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Since $p(0) = a_0$ and $p(1) = a_0 + a_1 + a_2 + a_3$, it follows that $p \in U \iff a_0 = 3(a_0 + a_1 + a_2 + a_3) \iff a_0 = \frac{-3}{2}(a_1 + a_2 + a_3)$. It follows that

$$p(x) = a_1x^1 + a_2x^2 + a_3x^3 - \tfrac{3}{2}a_1 - \tfrac{3}{2}a_2 - \tfrac{3}{2}a_3 = a_3(x^3 - \tfrac{3}{2}) + a_2(x^2 - \tfrac{3}{2}) + a_1(x - \tfrac{3}{2}),$$

so $U = span\{x^3 - \frac{3}{2}, x^2 - \frac{3}{2}, x - \frac{3}{2}\}$. We get that U is a subspace since it is a span! Moreover since the spanning polynomials have distinct degrees, they are linearly independent. So a basis for U is given by $\{x^3 - \frac{3}{2}, x^2 - \frac{3}{2}, x - \frac{3}{2}\}$, and thus the dimension of U is 3.

For the bonus you should notice that $T : \mathbb{P}_3 \to \mathbb{R}$ given by $T(p) = 3p(1) - p(0)$ does the trick. Indeed, $ker\, T = \{p \in \mathbb{P}_3 : T(p) = 0\} = \{p \in \mathbb{P}_3 : 3p(1) - p(0) = 0\} = \{p \in \mathbb{P}_3 : 3p(1) = p(0)\} = U$. But is T a linear transformation? It is: $T(p + q) = 3(p + q)(1) - (p + q)(0) = 3p(1) + 3q(1) - p(0) - q(0) = 3p(1) - p(0) + 3q(1) - q(0) = T(p) + T(q)$ for any p and q in $\mathbb{P}_3$, and if c is a scalar and p in $\mathbb{P}_3$, then $T(cp) = 3(cp(1)) - (cp)(0) = 3cp(1) - cp(0) = c\{3p(1) - p(0)\} = cT(p)$.

(16) (6 pts) Let U be the subspace of $\mathbb{R}^4$ spanned by $\{u_1, u_2\}$ where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}.$$

(a) (3 pts) Find a basis of $U^\perp$ where $U^\perp$ is the orthogonal complement of U:

$$\begin{aligned} U^\perp &= \{v \in \mathbb{R}^4 : \ v \text{ is orthogonal to every element of } U \ \} \\ &= \{v \in \mathbb{R}^4 : u \cdot v = 0 \text{ for every } u \in U \ \}. \end{aligned}$$

(b) (3 pts) Find the orthogonal projection of $\begin{bmatrix} -3 & -4 & 1 & 1 \end{bmatrix}^T$ onto U. (Hint: The result will have integer coefficients.)

(c) (0 pts) (practice problem – not part of the exam) Find the orthogonal projection of $\begin{bmatrix} -3 & -4 & 1 & 1 \end{bmatrix}^T$ onto $U^\perp$. (Hint: The result will have integer coefficients.)

Solution: (a) A vector $X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ is in $U^\perp$ if

$$x_1 + x_2 + x_3 + 3x_4 = 0 \quad \text{and} \quad x_1 + 2x_2 + 3x_3 + 5x_4 = 0.$$

Solving this system we get that the general solution is

$$X = t \begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix}^T + s \begin{bmatrix} -1 & -2 & 0 & 1 \end{bmatrix}^T,$$

where t and s are scalars. Hence

$$U^\perp = span\{\begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -2 & 0 & 1 \end{bmatrix}\},$$

and a basis is given by the set: $\{\begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -2 & 0 & 1 \end{bmatrix}\}$.

(b) Recall that $proj_U(X) = \frac{X \cdot F_1}{||F_1||^2} F_1 + \frac{X \cdot F_2}{||F_2||^2} F_2$, where $\{F_1, F_2\}$ is an orthogonal basis of U. We get such a basis by applying the Gram-Schmidt orthogonalization process to a basis of U. Since u_1 and u_2 are not multiples of each other, it follows that they form a basis of U. We therefore first apply the Gram-Schmidt algorithm to $\{u_1, u_2\}$. We get

$$\begin{aligned} F_1 &= u_1 = \begin{bmatrix} 1 & 1 & 1 & 3 \end{bmatrix} \\ ||F_1||^2 &= F_1 \cdot F_1 = 1 + 1 + 1 + 9 = 12, \\ u_2 \cdot F_1 &= [1 \quad 2 \quad 3 \quad 5] \cdot [1 \quad 1 \quad 1 \quad 3] = 1 + 2 + 3 + 15 = 21 \\ F_2 &= u_2 - \frac{u_2 \cdot F_1}{F_1 \cdot F_1} F_1 = \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix} - \frac{21}{12} \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix} - \frac{7}{4} \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -3 & 1 & 5 & -1 \end{bmatrix} \end{aligned}$$

We can now calculate $proj_U(X) = \frac{X \cdot F_1}{||F_1||^2} F_1 + \frac{X \cdot F_2}{||F_2||^2} F_2$:

$$\begin{aligned} X \cdot F_1 &= \begin{bmatrix} -3 & -4 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 3 \end{bmatrix} = -3 - 4 + 1 + 3 = -3 \\ X \cdot F_2 &= \frac{1}{4} \begin{bmatrix} -3 & -4 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 1 & 5 & -1 \end{bmatrix} \\ &= \frac{1}{4}(9 - 4 + 5 - 1) = \frac{9}{4} \\ F_2 \cdot F_2 &= \frac{1}{16}(9 + 1 + 25 + 1) = \frac{36}{16} = \frac{9}{4} \\ proj_U(X) &= \frac{-3}{12} F_1 + \frac{9/4}{9/4} F_2 = -\frac{1}{4} F_1 + F_2 \\ &= -\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 3 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -3 & 1 & 5 & -1 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} -4 & 0 & 4 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & -1 \end{bmatrix}. \end{aligned}$$

(c) We proceed as in (b): $proj_U(X) = \frac{X\cdot G_1}{||G_1||^2}G_1 + \frac{X\cdot G_2}{||G_2||^2}G_2$, where $\{G_1, G_2\}$ is an orthogonal basis of $U^{\perp}$. We get such a basis by applying the Gram-Schmidt orthogonalization process to the basis $\{X_1, X_2\}$ we found in (a). We have

$$\begin{aligned}
G_1 &= X_1 = \begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix} \\
||G_1||^2 &= G_1 \cdot G_1 = 1 + 4 + 1 = 6, \\
X_2 \cdot G_1 &= [-1 \quad -2 \quad 0 \quad 1] \cdot [1 \quad -2 \quad 1 \quad 0] = -1 + 4 = 3 \\
G_2 &= X_2 - \frac{X_2 \cdot G_1}{G_1 \cdot G_1} G_1 = \begin{bmatrix} -1 & -2 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3}{2} & -1 & -\frac{1}{2} & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & -2 & -1 & 2 \end{bmatrix}
\end{aligned}$$

We can now calculate $proj_U(X) = \frac{X\cdot G_1}{||G_1||^2}G_1 + \frac{X\cdot G_2}{||G_2||^2}G_2$:

$$\begin{aligned}
X \cdot G_1 &= \begin{bmatrix} -3 & -4 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix} = -3 + 8 + 1 = 6 \\
X \cdot G_2 &= \frac{1}{2} \begin{bmatrix} -3 & -4 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & -2 & -1 & 2 \end{bmatrix} \\
&= \frac{1}{2}(9 + 8 - 1 + 2) = \frac{1}{2}18 = 9 \\
G_2 \cdot G_2 &= \frac{1}{4}(9 + 4 + 1 + 4) = \frac{18}{4} = \frac{9}{2} \\
proj_U(X) &= \frac{6}{6}G_1 + \frac{9}{9/2}G_2 = G_1 + 2G_2 \\
&= \begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 0 & 2 \end{bmatrix}.
\end{aligned}$$