

MATHEMATICS FOR EDUCATION STUDENTS  
MATH 111, SECTION 1

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# Chapter 1

## Numeration systems and sets

“There are only 10 types of people in the world: those who understand binary, and those who don’t.”

“Why do mathematicians always confuse Halloween and Christmas?  
Because 31 Oct = 25 Dec.”

---

Mathematical Joke, Wikipedia.

### 1.1 Numeration systems

**Definition 1.1 – Power.** If  $a$  is any number and  $n$  is any natural number, then  $a$  to the power  $n$  is

$$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors}} \quad (1.1)$$

#### Examples 1.2

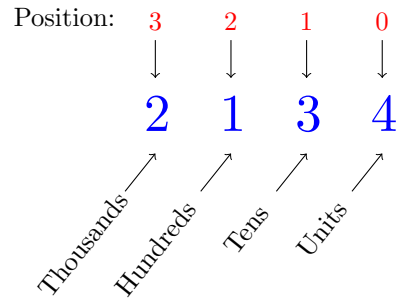
- $2^1 = 2$ ;     $2^2 = 4$ ;     $2^3 = 8$ ;     $2^4 = 16$ ;     $2^5 = 32$ ;     $2^6 = 64$ .
- $5^1 = 5$ ;     $5^2 = 25$ ;     $5^3 = 125$ ;     $5^4 = 625$ ;     $5^5 = 3125$ ;     $5^6 = 15625$ .
- $10^1 = 10$ ;     $10^2 = 100$ ;     $10^3 = 1,000$ ;     $10^4 = 10,000$ ;     $10^5 = 100,000$ ;     $10^6 = 1,000,000$ .

**Remark 1.3** Any nonzero number  $a$  to the power 0 is 1 :  $a^0 = 1$ .

#### 1.1.1 Hindu-Arabic numeration system

In the Hindu-Arabic numeration system, all numerals are constructed from 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Place value is based on powers of 10, the number base of the system. The Hindu-Arabic numeration system is also called the decimal numeration system, or base ten.

Lets consider 2134: units 4, tens 3, hundreds 1, and thousands 2.



Then 2134 can be written in expounded form as:

$$2134 = 2 \cdot 10^3 + 1 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 \quad (1.2)$$

Another equivalent notation is :

$$2134 = 2 \times 10^3 + 1 \times 10^2 + 3 \times 10^1 + 4 \times 10^0 \quad (1.3)$$

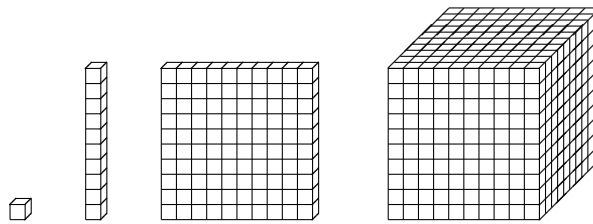


Figure 1.1: Base-ten unit, long, flat and block.

1 unit; 10 units = 1 long; 100 =  $10^2$  units = 1 flat; 1000 =  $10^3$  units = 1 block. One can deduce that 1 flat = 10 longs; 1 block = 10 flats; 1 block = 100 longs. This is summarized in the following table:

	Units	Longs	Flats	Blocks
1 Unit =	1			
1 Long =	10	1		
1 Flat =	100	10	1	
1 Block =	1000	100	10	1

Table 1.1: Units, longs, flats and blocks for the decimal numeration system.

Using units, longs, flats, and blocks, the number 2134 represents 4 units, 3 longs, 1 flat and 2 blocks.

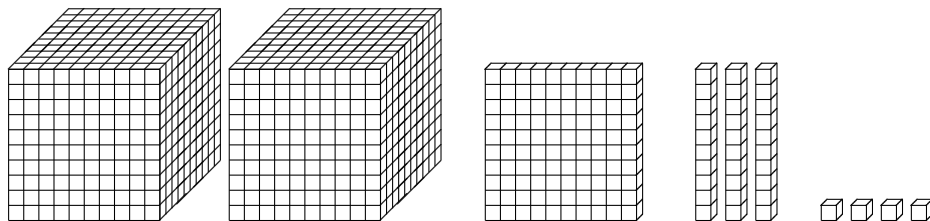


Figure 1.2: The number 2134

**Remark 1.4 – Connection with money.** If 1 unit represents one cent coin (1¢), 1 long represents a dime coin (10¢), 1 flat represent one dollar coin (1 CAD), and 1 block represents a ten dollars (10 CAD), then it is easy to find table 1.1. Note also that 10 bills of ten dollars is a hundred dollar bill.

**Exercise 1.5** If you have 12 flats, 15 longs and 26 units, how many units do you have in total? (The question may be the following: In your piggy bank, you have 12 one-dollar coins, 15 dime coins, and 26 one-cent coins. How many cents do your piggy bank contain?)

**Solution.** 12 flats are equal to 2 flats and 1 block. 15 longs are equal to 5 longs and 1 flat. 26 units are equal to 6 units and 2 longs. Then, you have in total 6 units, 7 longs, 3 flats and 1 block. Which is  $1 \cdot 10^3 + 3 \cdot 10^2 + 7 \cdot 10 + 6 \cdot 1 = 1376$  units.

$$\begin{array}{r|l}
 12 \text{ flats} = & 1 \text{ block} + 2 \text{ flats} \\
 15 \text{ longs} = & \phantom{1 \text{ block}} 1 \text{ flat} + 5 \text{ longs} \\
 26 \text{ units} = & \phantom{1 \text{ block}} \phantom{1 \text{ flat}} 2 \text{ longs} + 6 \text{ units} \\
 \hline
 \text{Total} = & 1 \text{ block} + 3 \text{ flats} + 7 \text{ longs} + 6 \text{ units}
 \end{array}$$

**Remark 1.6** Note that if you go to the bank and you give them 1376 coins of 1 cent, and you ask the equivalent amount of money with the minimum number of change, than, they should give you 1 ten-dollars bill, 3 one-dollar coins, 7 dime-coins and 6 one-cent coins.

### 1.1.2 Other number base systems

#### Base five

The Luo people of Kenya use a *quinary*, or base-five system. This system can be modeled by counting only with one hand. The digits available are 0, 1, 2, 3 and 4. After 4 there is 10 and 10 represent one hand and no fingers. 11 is one hand and one finger, 12 is one hand and two fingers, and so on.

**Examples 1.7 – Base-five numerals.**  $0_{\text{five}}, 1_{\text{five}}, 2_{\text{five}}, 3_{\text{five}}, 4_{\text{five}}, 10_{\text{five}}, 11_{\text{five}}, 12_{\text{five}}, 13_{\text{five}}, 14_{\text{five}}, 20_{\text{five}}, 21_{\text{five}}$  are respectively equal in base ten to 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.

**Remark 1.8** If we do not precise the base for a numeral, then it is in the base-ten numeration system.

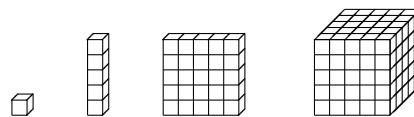


Figure 1.3: Base-five unit, long, flat and block.

Counting in base five is similar to counting in base ten. 5 units = 1 base-five long; 5 base-five longs = 1 base-five flat; 5 base-five flats = 1 base-five block. Consequently, 1 base-five flat = 25 units, 1 base-five block = 25 base-five longs = 125 units. This can be summarized in the following table:

	base-five units	base-five longs	base-five flats	base-five blocks
1 base-five unit =	1			
1 base-five long =	5	1		
1 base-five flat =	25	5	1	
1 base-five block =	125	25	5	1

Table 1.2: Units, longs, flats and blocks for the base-five numeration system.

Base Five	$0_{\text{five}}$	$1_{\text{five}}$	$2_{\text{five}}$	$3_{\text{five}}$	$4_{\text{five}}$	$10_{\text{five}}$	$11_{\text{five}}$	$12_{\text{five}}$	$13_{\text{five}}$	$14_{\text{five}}$	$20_{\text{five}}$
Picture		▣	▣▣	▣▣▣	▣▣▣▣	▣▣▣▣▣	▣▣▣▣▣▣	▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣▣▣
Base Ten	0	1	2	3	4	5	6	7	8	9	10

Base Five	$21_{\text{five}}$	$22_{\text{five}}$	$23_{\text{five}}$	$24_{\text{five}}$	$30_{\text{five}}$	$31_{\text{five}}$	$32_{\text{five}}$	...
Picture	▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣▣▣▣▣	▣▣▣▣▣▣▣▣▣▣▣▣▣	...
Base Ten	11	12	13	14	15	16	17	...

Table 1.3: Base-five numbers

**Example 1.9 – Connection with money.** If 1 unit represents one cent coin (1¢), then 1 base-five long represents a nickel coin (5¢), 1 base-five flat represents one quarter of a dollar coin (25¢). Unfortunately, there is no 1.25 dollars value coin. If this was the case, then this 1.25 dollar coin will represent a base-five block.

**Examples 1.10**

- $13_{\text{five}}$  is 1 base-five long and 3 units. Hence,  $13_{\text{five}} = 1 \cdot 5 + 3 \cdot 1 = 8$ .

We have then  $13_{\text{five}} = \begin{array}{|c|} \hline \square \\ \hline \square \square \square \\ \hline \end{array} = 8$ .

- $234_{\text{five}}$  is 2 base-five flats, 3 base-five longs and 4 units. Hence  $234_{\text{five}} = 2 \cdot 5^2 + 3 \cdot 5 + 4 \cdot 1 = 50 + 15 + 4 = 69$ .

We have then  $234_{\text{five}} = \begin{array}{|c|c|} \hline \square \square & \square \square \square \\ \hline \square \square \square & \square \square \square \square \\ \hline \end{array} = 69$ .

- $4201_{\text{five}}$  is 4 base-five blocks, 2 base-five flats, no base-five long and 1 unit. Hence,  $4201_{\text{five}} = 4 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 1 = 4 \cdot 125 + 2 \cdot 25 + 4 = 500 + 50 + 1 = 551$ .

We have then  $4201_{\text{five}} = \begin{array}{|c|c|c|c|} \hline \square \square \square & \square \square \square & \square \square \square & \square \square \square \\ \hline \square \square \square & \square \square \square & & \square \square \square \square \\ \hline \end{array} = 551$ .

**Exercise 1.11** Convert  $11234_{\text{five}}$  to base ten.

**Solution.**  $11234_{\text{five}} = 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 3 \cdot 5 + 4 \cdot 1 = 1 \cdot 625 + 1 \cdot 125 + 2 \cdot 25 + 3 \cdot 5 + 4 \cdot 1 = 819$ .

**Exercise 1.12** Convert 2897 to base five.

**Solution.** To convert a base-ten number, for example 2897, to a base-five number, we divide that number (2897) by the successive powers of 5, starting with the greatest power of 5 less than or equal to 2897. In examples 1.2, one can notice that  $5^4 = 625$  is less than 2897, but the next power, i.e.,  $5^5 = 3125$ , is greater than 1297. Hence, we start by dividing our number 2897 by 625. The result is 4 and the remainder is 397. Then we divide this remainder 397 by 125 ( $5^3 = 125$ ). The result is 3 and the remainder is 22. We divide now this remainder 22 by 25 ( $5^2 = 25$ ), and we notice that the result is 0 and the remainder is 22. So, we divide the remainder 22 by 5, and the result is 4 and the remainder is 2. Finally, we divide the remainder 2 by 1, and the result is obviously 2 and the remainder is 0. Therefore, we conclude that  $2897 = 4 \cdot 625 + 3 \cdot 125 + 0 \cdot 25 + 4 \cdot 5 + 2 = 4 \cdot 5^4 + 3 \cdot 5^3 + 0 \cdot 5^2 + 4 \cdot 5 + 2 \cdot 1$ , and hence  $2897 = 43042_{\text{five}}$ .

$$5^0 = 1; \quad 5^1 = 5; \quad 5^2 = 25; \quad 5^3 = 125; \quad 5^4 = 625; \quad 5^5 = 3125.$$

- The European Way:

$$\begin{array}{r} 2897 \quad \overline{)625} \quad 397 \quad \overline{)125} \quad 22 \quad \overline{)25} \quad 22 \quad \overline{)5} \quad 2 \quad \overline{)1} \\ - 2500 \quad \underline{4} \quad - 375 \quad \underline{3} \quad - 00 \quad \underline{0} \quad - 20 \quad \underline{4} \quad - 2 \quad \underline{2} \\ = 0397 \quad \quad = 022 \quad \quad = 22 \quad \quad = 02 \quad \quad = 0 \end{array} . \text{ Therefore, } 2897 = 43042_{\text{five}}.$$

- The American Way:

$$625 \overline{)2897} \quad 125 \overline{)397} \quad 25 \overline{)22} \quad 5 \overline{)22} \quad 1 \overline{)2} \\ \underline{- 2500} \quad \underline{- 375} \quad \underline{- 00} \quad \underline{- 20} \quad \underline{- 2} \\ = 0397 \quad = 022 \quad = 22 \quad = 02 \quad = 0 . \text{ Therefore, } 2897 = 43042_{\text{five}}.$$

**Exercise 1.13** Ask a student to give us a number less than 1000, to convert to base five.

**Base two**

Base two is also called the binary system because numbers in base-two are made only with 0 and 1 (2 different digits). Most of electronic devices uses binary system, in particular computer. It works in the same way then base ten and base five.

Base-ten	0	1	2	3	4	5	6	7	8	9	10
Base-two	$0_{\text{two}}$	$1_{\text{two}}$	$10_{\text{two}}$	$11_{\text{two}}$	$100_{\text{two}}$	$101_{\text{two}}$	$110_{\text{two}}$	$111_{\text{two}}$	$1000_{\text{two}}$	$1001_{\text{two}}$	$1010_{\text{two}}$

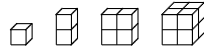


Figure 1.4: Base-two unit, long, flat and block.

It works the same way to convert a base-two number to base-ten, and conversely, a base-ten number to base-two. It is even easier when you memorize, let's say, the first six powers of 2.

**Exercise 1.14** Convert  $110101_{\text{two}}$  to base ten.

**Solution.**

$$\begin{aligned} 110101_{\text{two}} &= 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1 \cdot 1 = 1 \cdot 32 + 1 \cdot 16 + 0 \cdot 8 + 0 \cdot 4 + 0 \cdot 2 + 1 \cdot 1 \\ &= 32 + 16 + 0 + 4 + 0 + 1 = 53 \end{aligned}$$

**Exercise 1.15** Convert 37 to base two.

**Solution.**

- To convert a base-ten number, for instance 37, to a base-two number, we divide that number (37) by the successive powers of 2, starting with the greatest power of 2 less than or equal to 37. In examples 1.2, one can notice that  $2^5 = 32$  is less than 37, but the next power, i.e.,  $2^6 = 64$ , is greater than 37. Hence, we start by dividing our number 37 by 32. The result is 1 and the remainder is 5. We divide then the remainder 5 by 16. The result is 0 and the remainder is 5. We divide then the remainder 5 by 8. The result is 0 and the remainder is 5. We divide then the remainder 5 by 4. The result is 1 and the remainder is 1. We divide then the remainder 1 by 2. The result is 0 and the remainder is 1. Finally, we divide then the remainder 1 by 1. The result is 1 and the remainder is 0. Consequently,  $37 = 100101_{\text{two}}$ .

$$2^0 = 1; \quad 2^1 = 2; \quad 2^2 = 4; \quad 2^3 = 8; \quad 2^4 = 16; \quad 2^5 = 32; \quad 2^6 = 64.$$

- The European Way:

$$\begin{array}{r} 37 \overline{) 32} \\ - 32 \\ \hline 05 \end{array} \begin{array}{r} 5 \overline{) 16} \\ - 16 \\ \hline 00 \end{array}, \begin{array}{r} 5 \overline{) 8} \\ - 8 \\ \hline 00 \end{array}, \begin{array}{r} 5 \overline{) 4} \\ - 4 \\ \hline 01 \end{array}, \begin{array}{r} 1 \overline{) 2} \\ - 2 \\ \hline 00 \end{array}, \begin{array}{r} 1 \overline{) 1} \\ - 1 \\ \hline 00 \end{array}. \text{ Therefore, } 37 = 100101_{\text{two}}.$$

- The American Way:

$$\begin{array}{r} 37 \overline{) 37} \\ - 32 \\ \hline 05 \end{array}, \begin{array}{r} 16 \overline{) 5} \\ - 0 \\ \hline 05 \end{array}, \begin{array}{r} 8 \overline{) 5} \\ - 0 \\ \hline 05 \end{array}, \begin{array}{r} 4 \overline{) 5} \\ - 4 \\ \hline 01 \end{array}, \begin{array}{r} 2 \overline{) 1} \\ - 0 \\ \hline 01 \end{array}, \begin{array}{r} 1 \overline{) 1} \\ - 1 \\ \hline 00 \end{array}. \text{ Therefore, } 37 = 100101_{\text{two}}.$$

- There is another way. Since we deal with small numbers, we can proceed as follows: let try to write 37 as a sum of powers of 2 (using 1, 2, 4, 8, 16, 32, 64, 128, ...). Therefore, we have  $37 = 32 + 4 + 1 = 2^5 + 2^2 + 1$ . Consequently,  $37 = 100101_{\text{two}}$ .

**Exercise 1.16** Ask a student to give us a number less than 100, to convert it to base two.

## 1.2 Describing sets

### 1.2.1 The language of sets

A set is a any collection of objects. Individual objects in a set are elements of the set.

**Examples 1.17**

- The set of lowercase letter of the English alphabet can be written in set notation as follows :  
 $A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$ .

2. The set of digits used in the base-five system can be written in set notation as follows:  $Q = \{0, 1, 2, 3, 4\}$ .

The order in which the elements are written make no difference, and each element is listed only once.

**Remark 1.18** In examples 1.17,  $Q$  can also be written  $Q = \{3, 4, 2, 0, 1\}$  or  $Q = \{4, 3, 2, 1, 0\}$ . Both expressions represent the same set  $Q$ . However, the set  $Q$  **can not** be written  $\{0, 0, 1, 2, 3, 4\}$  nor  $\{0, 1, 2, 2, 3, 4, 4\}$ .

**Notation 1.19** We symbolize an element belonging to a set by the symbol  $\in$ . If a given element does not belong to a set, then we use the symbol  $\notin$ .

**Examples 1.20** The sets  $A$  and  $Q$  are the ones of examples 1.17.  $z \in A$  means the element  $z$  belongs to the set  $A$ . However, the element  $\acute{e}$  does not belong to the set  $A$ . We write then  $\acute{e} \notin A$ . The expression  $3 \in Q$ , means that the element 3 belongs to the set  $Q$ , however the element 7 does not belong to the set  $Q$ . We write then  $7 \notin Q$ .

The set of natural numbers is defined by the following:  $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$  where  $\dots$  means the the sequence continue in the same way. There are two methods of describing sets: the listing method and the set-builder notation.

**Example 1.21** Let  $O$  be the set of the digits used in a base-eight numeration system (octal numeration system) except 0. Then  $O$  can be described as follows:

- Listing method:  $O = \{1, 2, 3, 4, 5, 6, 7\}$ .
- Set-builder notation:  $O = \{x \mid x \in \mathbb{N} \text{ where } x < 8\}$  and is read:  $O$ : Set  $O$ ,  $=$ : is equal to,  $\{$ : the set of,  $x$ : all elements  $x$ ,  $|$ : such that,  $x \in \mathbb{N}$ :  $x$  is a natural number, where: where,  $x < 8$ :  $x$  is less than 8.

**Exercise 1.22** Write the following sets using set-builder notation:

$A = \{2, 4, 6, 8, 10, \dots\}$ ,  $B = \{1, 3, 5, 7, 9, \dots\}$ ,  $C = \{3, 6, 9, 12, 15, \dots\}$  and  $D = \{5, 10, 15, 20, 25, \dots\}$ .

**Solution.**  $A = \{x \mid x = 2n \text{ where } n \in \mathbb{N}\}$ ,  $B = \{x \mid x = 2n - 1 \text{ where } n \in \mathbb{N}\}$ ,  $C = \{x \mid x = 3n \text{ where } n \in \mathbb{N}\}$  and  $D = \{x \mid x = 5n \text{ where } n \in \mathbb{N}\}$ .

**Exercise 1.23** Write the following sets using the listing method:

$E = \{2k+1 \mid k = 2, 3, 4\}$ ,  $F = \{5p-1 \mid p = 2, 3, 4\}$ ,  $G = \{a^2 \mid a = 2, 3, 4\}$  and  $H = \{a^2+b^2 \mid a = 2 \text{ or } 3, \text{ and } b = 2, 3 \text{ or } 4\}$ .

**Solution.**  $E = \{5, 7, 9\}$ ,  $F = \{9, 14, 19\}$ ,  $G = \{4, 9, 16\}$ , and  $H = \{8, 13, 18, 20, 25\}$

## 1.2.2 One-to-one correspondence

As noted earlier, with the listing method, the order in which elements are listed does not matter.

**Definition 1.24 – Equal sets.** Two sets  $A$  and  $B$  are **equal** if, and only if, they contain exactly the same elements. We write then  $A = B$ , otherwise,  $A \neq B$ .

### Examples 1.25

- The set  $A = \{a, b, c\}$  is equal to the set  $B = \{c, b, a\}$ , since they contain exactly the same elements. We write then  $A = B$ .
- The set  $C = \{1, 2, \text{shirt}, \text{yellow}, \star\}$  is not equal to the  $D = \{\text{shirt}, \star, \text{yellow}, 2\}$ , since  $1 \in C$  but is  $1 \notin D$ . We write then  $C \neq D$ . You may notice that the sets  $C$  and  $D$  do not have the same number of elements (5 elements for the set  $C$  and 4 elements for the set  $D$ ). We may then conclude that if two sets do not have the same number of elements, they can not be equal. This condition is necessary but not sufficient as shown in the next example.
- The set  $E = \{1, 2, \text{shirt}, \text{red}, \star\}$  is not equal to the  $F = \{\text{shirt}, \star, \text{yellow}, 2, 1\}$ . The sets  $E$  and  $F$  have the same number of elements. However  $E$  and  $F$  are not equal, since  $\text{red} \notin F$  (and  $\text{yellow} \notin E$ ). We write then  $E \neq F$ .

**Definition 1.26 – One-to-one correspondence.** If the elements of sets  $P$  and  $S$  can be paired so that for each element of  $P$  there is exactly one element of  $S$  and for each element of  $S$  there is exactly one element of  $P$ , then the two sets  $P$  and  $S$  are said to be in **One-to-one correspondence**

### Examples 1.27

- The sets  $P = \{a, b\}$  and  $S = \{1, 2\}$  are in one-to-one correspondence since elements of  $P$  can be paired with element of  $S$  as stated in definition 1.26 as follows: either  $(a, 1)$  and  $(b, 2)$  or  $(a, 2)$  and  $(b, 1)$ .

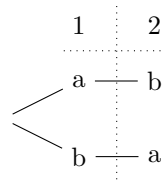


Figure 1.5: One-to-one correspondences between the sets  $\{a, b\}$  and  $\{1, 2\}$  using tree diagram

- The set  $G = \{\text{shirt, pants, shoes}\}$  is in one-to-one correspondence with the set  $H = \{\text{red, blue, yellow}\}$ , since elements of  $E$  can be paired with elements of  $F$  as stated in definition 1.26. One possible one-to-one correspondence between the sets  $E$  and  $F$  is the following:  $(\text{shirt, red})$ ,  $(\text{pants, blue})$  and  $(\text{shoes, yellow})$ .
- The set of students of this classroom is not in one-to-one correspondence with the set  $\{x \mid x \in \mathbb{N} \text{ with } x < 8\}$ , since these two sets don't have the same number of elements.

**Remark 1.28** In examples 1.27, There are several other one-to-one correspondences between the sets  $G$  and  $H$ . Actually, there are exactly six, as shown in the following table:

One-to-one Correspondence $N^\circ$	The Pairing
1	$(\text{shirt, red})$ , $(\text{pants, blue})$ and $(\text{shoes, yellow})$
2	$(\text{shirt, red})$ , $(\text{shoes, blue})$ and $(\text{pants, yellow})$
3	$(\text{pants, red})$ , $(\text{shirt, blue})$ and $(\text{shoes, yellow})$
4	$(\text{pants, red})$ , $(\text{shoes, blue})$ and $(\text{shirt, yellow})$
5	$(\text{shoes, red})$ , $(\text{shirt, blue})$ and $(\text{pants, yellow})$
6	$(\text{shoes, red})$ , $(\text{pants, blue})$ and $(\text{shirt, yellow})$

Table 1.4: One-to-one correspondences between the sets  $E$  and  $F$

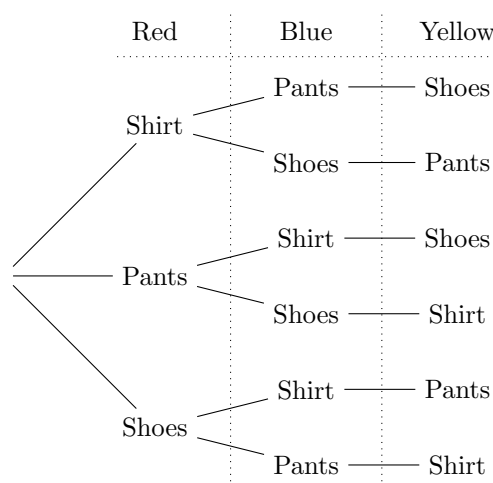


Figure 1.6: One-to-one correspondences between the sets  $E$  and  $F$  using tree diagram

**Theorem 1.29 – Fundamental counting principle.** If event  $M$  can occur in  $m$  ways and, after it has occurred, event  $N$  can occur in  $n$  ways, then event  $M$  followed by event  $N$  can occur in  $mn$  ways.

### 1.2.3 Equivalent sets

**Definition 1.30 – Equivalent sets.** Two sets  $A$  and  $B$  are **equivalent**, written  $A \sim B$ , if and only if there exists a one-to-one correspondence between the sets.

If a set  $A$  is not equivalent to set  $B$ , we write  $A \not\sim B$ . The term *equivalent* should not be confused with the term *equal*. For instance, the sets  $G$  and  $H$  of examples 1.27 are equivalent, since they are in a one-to-one correspondence, but not equal since they have not the same elements. We write then  $E \sim F$ , and  $E \neq F$ .

**Example 1.31** Let consider  $I = \{1, 2, 3, 4\}$ ,  $J = \{\text{red, blue, yellow}\}$ ,  $K = \{3, 2, 1\}$  and  $L = \{\text{yellow, blue, red}\}$ . Then

- $I$  and  $J$  are not equal ( $I \neq J$ ) and they are not equivalent ( $I \not\sim J$ ).
- $I$  and  $K$  are not equal ( $I \neq K$ ) and they are not equivalent ( $I \not\sim K$ ).
- $I$  and  $L$  are not equal ( $I \neq L$ ) and they are not equivalent ( $I \not\sim L$ ).
- $J$  and  $K$  are not equal ( $J \neq K$ ) but they are equivalent ( $J \sim K$ ).
- $J$  and  $L$  are equal ( $J = L$ ) and hence they are equivalent ( $J \sim L$ ).
- $K$  and  $L$  are not equal ( $K \neq L$ ) but they are equivalent ( $K \sim L$ ).

### 1.2.4 Cardinal numbers

**Definition 1.32 – Cardinal number of a set.** The cardinal number  $n(S)$  of a set  $S$  is the number of elements in the set. The cardinal number of an empty set is  $n(\emptyset) = 0$ .

**Notation 1.33** The empty set  $\emptyset$  may also be denoted  $\{\}$ .

**Remark\* 1.34**  $\{\{\}\}$  and  $\{\emptyset\}$  are not empty sets. They are sets that contain each one element, and that element is the empty set.

**Examples 1.35** In examples 1.31,  $n(I) = 4$  and  $n(J) = n(K) = n(L) = 3$ .

**Remark 1.36** Two sets with the same cardinal number are equivalent.

### 1.2.5 More about sets

The **universal set**, or **universe**, denoted  $U$ , is the set of all elements being considered in a giving discussion. For example, suppose that  $U = \{x \mid x \text{ is a student in this classroom}\}$ , and  $F = \{x \mid x \text{ is a female student in this classroom}\}$ . Using a Venn diagram the set of elements in the universal set that are not in  $F$ , denoted by  $\bar{F}$  is the set of male students in this classroom. This is the **complement** of  $F$

**Definition 1.37 – Set complement.** The **complement** of a set  $F$ , written  $\bar{F}$ , is the set of all elements in the universal set  $U$  that are not in  $F$ ; that is

$$\bar{F} = \{x \mid x \in U \text{ and } x \notin F\} \tag{1.4}$$

#### Examples 1.38

- If  $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$ ,  $A = \{0\}$  and  $B = \{5, 7\}$  then:  $\bar{A} = \{1, 2, 3, 4, 5, 6, 7\}$  and  $\bar{B} = \{0, 1, 2, 3, 4, 6\}$ .
- For all  $U$ ,  $\bar{U} = \emptyset$  and  $\bar{\emptyset} = U$ .

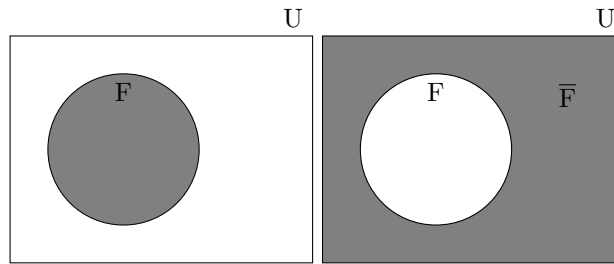


Figure 1.7: The set  $F$  and the set  $\bar{F}$  (Complement of  $F$ )

## 1.2.6 Subsets

**Definition 1.39 – Subset.**  $B$  is a **subset** of  $A$ , written  $B \subseteq A$ , if and only if, every element of  $B$  is an element of  $A$ .

If a set  $B$  is not a subset of  $A$ , we write  $B \not\subseteq A$ .

**Definition 1.40 – Proper subset.**  $B$  is a **proper subset** of  $A$ , written  $B \subset A$ , if and only if, every element of  $B \subseteq A$  and there exists at least one element of  $A$  that is not in  $B$ .

If a set  $B$  is not a proper subset of  $A$ , we write  $B \not\subset A$ .

### Remarks 1.41

- If a set  $B$  is a proper subset of a set  $A$ , i.e.,  $B \subset A$ , then in particular  $B$  is a subset of  $A$ , i.e.,  $B \subseteq A$ . The converse is not always true, as shown in the following examples.
- If two sets are equal, then each set is a subset of the other. In particular, any set is a subset of itself.
- The empty set  $\emptyset$  is a subset of any set.

### Examples 1.42

- Let  $A = \{1, 2, 3, 4, 5\}$  and  $B = \{1, 2\}$ . Since all elements of  $B$  belong to the set  $A$ , then  $B$  is a subset of  $A$ , and we write  $B \subseteq A$ . Moreover, the element 5 belongs to  $A$  but does not belong to  $B$ . Therefore,  $B$  is a proper subset of  $A$  and we write then  $B \subset A$ .
- The set of even natural numbers is a proper subset of the set of natural numbers, since all even natural numbers are natural numbers and there exists a natural number, for example 3, that belongs to the set of natural numbers but does not belong to the set of even natural numbers. The set of odd natural numbers is also a proper subset of the set of natural numbers.

## 1.2.7 Number of sets of a set

How many subsets can be made from a set  $S$  containing  $n$  elements? We already know that the empty set  $\emptyset$  and the set  $S$  itself are two subsets of the set  $S$ . We add all the subsets that contain only one element of  $S$ , then all the subsets that contain only two elements of  $S$ , and so on, and finally, we add all the subsets that contain  $(n - 1)$  elements.

### Examples 1.43

- All subsets of  $A = \{a, b\}$  are:  $\emptyset, \{a\}, \{b\}$  and  $\{a, b\}$ . There are four subsets ( $4 = 2^2$ ).
- All subsets of  $B = \{a, b, c\}$  are:  $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$  and  $\{a, b, c\}$ . There are eight subsets ( $8 = 2^3$ ).
- All subsets of  $C = \{1, 2, 3, 4\}$  are:  $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$  and  $\{1, 2, 3, 4\}$ . There are sixteen subsets ( $16 = 2^4$ ).

**Remark 1.44** If the cardinal number of a set  $S$  is  $n$ , then  $S$  has  $2^n$  subsets.

## 1.3 Other set operations and their properties

### 1.3.1 Set intersection

**Definition 1.45 – Set intersection.** The **intersection** of two sets  $A$  and  $B$ , written  $A \cap B$ , is the set of all elements common to both  $A$  and  $B$ .

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} \quad (1.5)$$

#### Examples 1.46

- If  $A = \{a, b, c, d\}$  and  $B = \{c, d, e, f\}$ , then  $A \cap B = \{c, d\}$ .
- If  $C = \{a, b, c, d, e, f\}$  and  $D = \{u, v, w, x, y, z\}$ , then  $C \cap D = \emptyset$ .
- If  $E = \{a, b, c, d, e, f, g, h\}$  and  $F = \{a, b\}$ , then  $E \cap F = F$ .

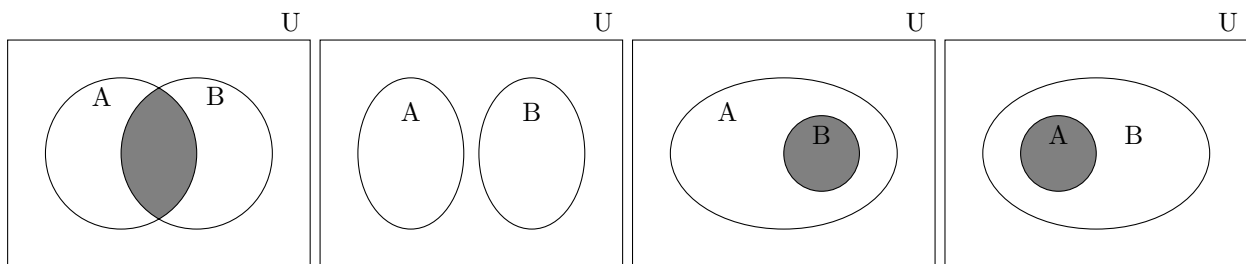


Figure 1.8: Set Intersection:  $A \cap B$

### 1.3.2 Set Union

**Definition 1.47 – Set union.** The **union** of two sets  $A$  and  $B$ , written  $A \cup B$ , is the set of all elements in  $A$  or in  $B$ .

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \quad (1.6)$$

#### Examples 1.48

- If  $A = \{a, b, c, d\}$  and  $B = \{c, d, e, f\}$ , then  $A \cup B = \{a, b, c, d, e, f\}$ .
- If  $C = \{a, b, c, d, e, f\}$  and  $D = \{u, v, w, x, y, z\}$ , then  $C \cup D = \{a, b, c, d, e, f, u, v, w, x, y, z\}$ .
- If  $E = \{a, b, c, d, e, f, g, h\}$  and  $F = \{a, b\}$ , then  $E \cup F = E$ .

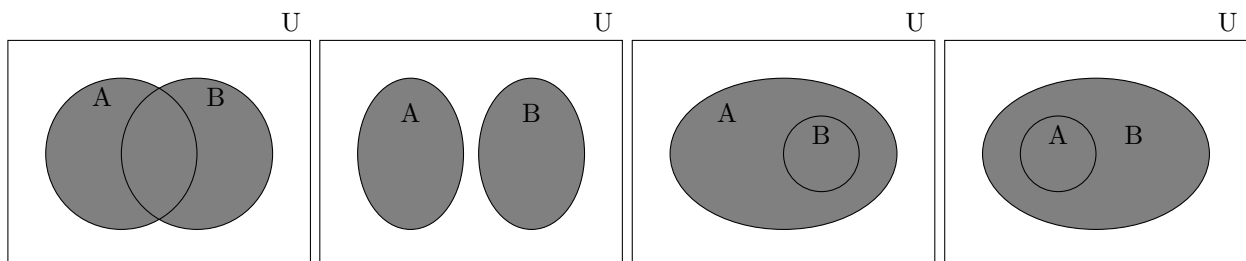


Figure 1.9: Set union:  $A \cup B$

### 1.3.3 Set difference

**Definition 1.49 – Set difference.** The **set difference** of  $A$  and  $B$ , written  $A - B$ , is the set of all elements in  $A$  that are not in  $B$ .

$$A - B = \{x \mid x \in A \text{ and } x \notin B\} \quad (1.7)$$

**Examples 1.50** If  $A = \{1, 2, 3\}$ ,  $B = \{1, 2, 3, 4, 5, 6\}$  and  $C = \{4, 5, 6\}$ , then:

- $A - B = \emptyset$ .
- $B - C = \{1, 2, 3\} = A$ .
- $A - C = A$ .
- $B - A = \{4, 5, 6\} = C$ .
- $C - B = \emptyset$ .
- $C - A = C$ .

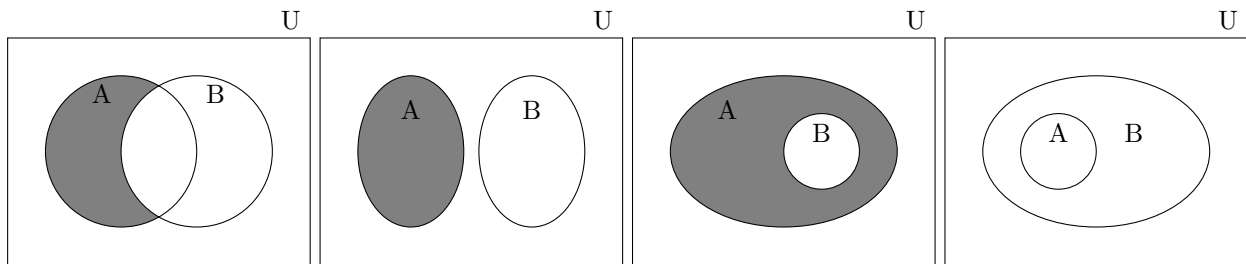


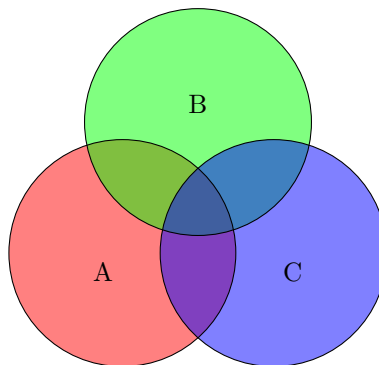
Figure 1.10: Set difference :  $A - B$

### 1.3.4 Properties of set operations

**Theorem 1.51 – Associativity of set intersection and set union.** The property  $A \cap (B \cap C) = (A \cap B) \cap C$  is the **associative property** of set intersection. Similarly,  $A \cup (B \cup C) = (A \cup B) \cup C$  is the **associative property** of set union.

**Theorem 1.52 – Distributivity of set intersection over union.** For all sets  $A, B$  and  $C$ ,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \quad (1.8)$$



### 1.3.5 Cartesian products

**Definition 1.53 – Cartesian product.** For any sets  $A$  and  $B$ , the **cartesian product**  $A$  of  $B$ , written  $A \times B$ , is the set of all ordered pairs such that the first component of each pair is an element of  $A$  and the second component of each pair is an element of  $B$ .

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}. \quad (1.9)$$

|

**Examples 1.54** If  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$ , then:

- $A \times B = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$ .
- $B \times A = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$ .
- $A \times A = \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)\}$ .
- $B \times B = \{(a, a), (b, a), (a, b), (b, b)\}$ .

# Chapter 2

## Whole numbers and their operations

When zero is added to the set of natural numbers,  $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$ , we have the set of whole numbers, denoted  $\mathbb{W} = \{0, 1, 2, 3, 4, 5, 6, \dots\}$ .

### 2.1 Addition and subtraction of whole numbers

#### 2.1.1 Addition of whole numbers

A set model is one way to represent addition of whole number.

**Definition 2.1 – Addition of whole number.** Let  $A$  and  $B$  be two disjoint finite sets. If  $n(A) = a$  and  $n(B) = b$ , then  $a + b = n(A \cup B)$ .

Addition of whole number may also be represented by number-line (measurement) model.

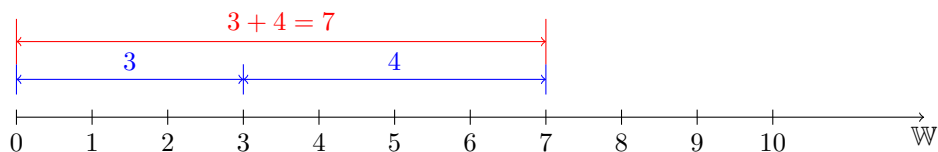


Figure 2.1:  $3 + 4 = 7$  using the line-number.

**Definition 2.2 – Less than.** For any whole number  $a$  and  $b$ ,  $a$  is less than  $b$ , written  $a < b$ , if, and only if, there exists a natural number  $k$  such that  $a + k = b$ .

Equality can also be combined with inequalities greater than, denoted  $>$ , and less than, denoted  $<$ , to give the relations greater than or equal, denoted  $\geq$ , and less than or equal, denoted  $\leq$ . Hence,  $a \geq b$  means  $a > b$  or  $a = b$ , and  $a \leq b$  means  $a < b$  or  $a = b$ .

**Examples 2.3** The statement  $3 \leq 5$  is true because  $3 < 5$ . The statement  $5 \leq 5$  is true because  $5 = 5$ . The statement  $13 \leq 7$  is false because neither  $13 = 7$  nor  $13 < 7$ . Actually, the correct statement for the latter is either  $13 \geq 7$  or  $13 > 7$ .

#### 2.1.2 Whole-number addition properties

**Theorem 2.4 – Closure property of addition of whole numbers.** If  $a$  and  $b$  are whole numbers, then  $a + b$  is a whole number.

The closure property implies that the sum of two whole numbers exists and that the sum is a unique whole number. For example,  $4+3$  is a unique whole number and we identify that number as 7.

**Theorem 2.5 – Commutative property of addition of whole numbers.** If  $a$  and  $b$  are any whole numbers, then  $a + b = b + a$ .

**Theorem 2.6 – Associative property of addition of whole numbers.** If  $a, b$  and  $c$  are whole numbers, then  $(a + b) + c = a + (b + c)$ .

We want to compute  $27+8+12$ . The associativity property implies that we can compute the sum of three numbers by grouping 27 and 8 together:  $(27 + 8) + 12 = 35 + 12 = 47$ . We might also recognize that it is easier to group 8 and 12:  $27 + (8+12) = 27+20=47$ .

**Theorem 2.7 – Identity property of addition of whole numbers.** There is a unique whole number 0, the **additive identity**, such that for any whole number  $a$ ,  $a + 0 = a = 0 + a$ .

**Exercise 2.8** Which properties justify the following?

- |   |                                    |
|---|------------------------------------|
| i. $13 + 78 = 78 + 13$ .                  | iii. $(6 + 9) + 3 = (9 + 6) + 3$ . |
| ii. $2011 + 60$ is a unique whole number. | iv. $(5 + 11) + 7 = 11 + (7 + 5)$  |

**Solution.**

- |                                      |   |
|--------------------------------------|---|
| i. Commutative property of addition. | iii. Commutative property of addition.                  |
| ii. Closure property of addition.    | iv. Commutative and associative properties of addition. |

### 2.1.3 Subtraction of whole numbers

- Take away model.
- Comparison model,
- Number-line model.

**Definition 2.9 – Subtraction of whole numbers.** For any whole numbers  $a$  and  $b$  such that  $a \geq b$ ,  $a - b$  is the unique whole number  $c$  such that  $b + c = a$ .

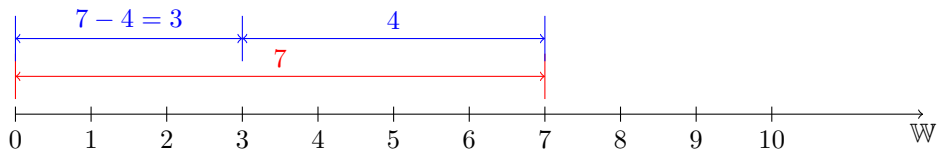


Figure 2.2:  $7 - 4$  using the line-number.

### 2.1.4 Properties of subtraction

Using the definition,  $4 - 7$  is the whole number  $c$ , if and only if,  $c + 7 = 4$ . Since there is no whole number  $c$  that satisfies the equation,  $4 - 7$  is not meaningful in the set of whole numbers. Therefore, subtraction is not closed on the set of whole number.

Since  $7 - 4 = 3$  and  $4 - 7$  is not meaningful, subtraction of whole number is not commutative.

### 2.1.5 Introductory algebra using whole-number addition and subtraction

Sentences such as  $9 + 13 = x$  and  $y - 6 = 7$  can be true or false depending on the value of  $x$  and  $y$ . For example, if  $x = 20$ , then  $9 + 13 = x$  is false. If  $y = 13$ , then  $y - 6 = 7$  is true. If a value that is used makes the equation true, it is a **solution** to the equation.

**Exercise 2.10** Find solutions for each of the following where  $x$  is a whole number.

- $x + 7 = 19$ ,
- $20 - x = 13$ ,
- $x - 6 = 5$ ,
- $x > 5$  and  $x \leq 7$ .

**Solution.**

- $x + 7 = 19$ . Since  $12+7=19$ , then  $x = 12$  is a solution. We can also find the solution as follows: since  $x + 7 = 19$ , then according to the definition of subtraction of whole numbers,  $x$  is the whole number  $19 - 7$  and hence  $x = 12$  is the solution.
- $20 - x = 13$ : If we subtract 7 from 20, the result will be 13. Therefore,  $x = 13$  is the solution.
- $x - 6 = 5$ . Form the definition of subtraction,  $x - 6$  is the unique whole number 5 such that  $x = 5 + 6$ . Therefore,  $x = 11$  is the solution.
- $x > 5$  and  $x \leq 7$ : The only possible value for  $x$  are 6 and 7. The solutions are  $x = 6$  and  $x = 7$ .

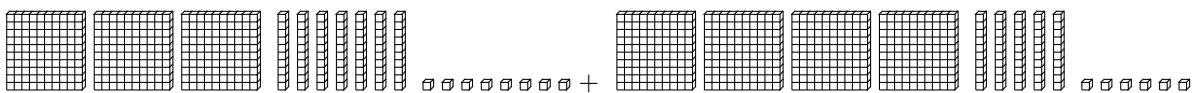
## 2.2 Algorithm for whole-number addition and subtraction

### 2.2.1 Addition algorithms

We want to add 12 and 35. We can use to illustrate that using base-ten units, longs, flats an blocks. To compute  $12 + 35$ , we add base-ten units  $2 + 5 = 7$ , then we add base-ten longs  $10 + 30 = 40$ . Hence, by adding base-ten units and longs, we find 47.

Expanded algorithm 
$$\begin{array}{r} 12 \\ + 35 \\ \hline + 40 \\ \hline = 47 \end{array}$$
 and the standard algorithm 
$$\begin{array}{r} 12 \\ + 35 \\ \hline = 47 \end{array}$$

• **Concrete model vs standard algorithm**  $378 + 456$ .

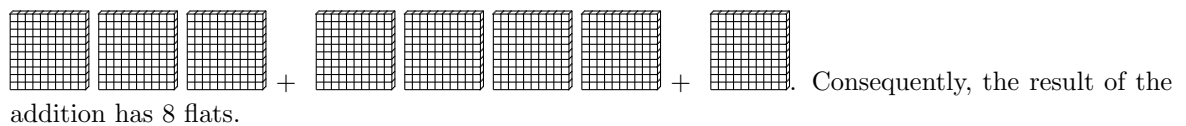


1. We add the units :  $8 + 6 = 14$ .  $\text{units} + \text{units} = \text{long} + \text{units}$ . Consequently, the result of the addition has 4 units.

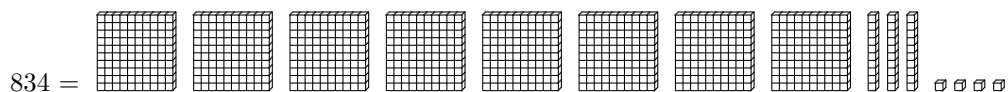
2. We add the longs:  $7 + 5$  and the one long coming from the previous step :



3. We add the flats:  $3 + 4$  and the one flat coming from the previous step :

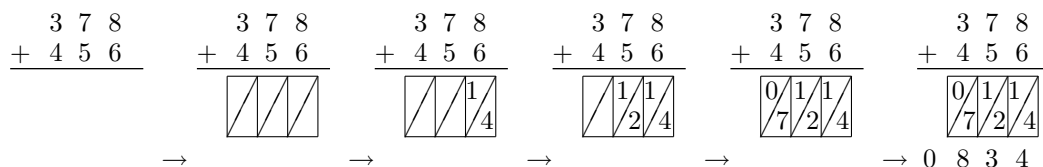


Finally,  $378 + 456 = 834$



• **Lattice algorithm for addition**

◇  $378 + 456 = 834$ .





- $44_{\text{five}} + 2_{\text{five}} = 101_{\text{five}}$ . Because,  $44_{\text{five}}$  is 4 base-five longs and 4 base-five units, and if we add 2 base-five units, we have 4 base-five longs and 6 base-five units. The 6 base-five units are equal to 1 base-five long and 1 unit, and hence the result is 5 base-five longs and 1 base-five unit. Moreover, 5 base-five longs is 1 base-five flat. Therefore, we have 1 base-five flat, no base-five longs and 1 base-five unit, which is  $101_{\text{five}}$ .

$$44_{\text{five}} + 2_{\text{five}} = \begin{array}{c} \text{||||} \\ \text{||||} \end{array} \text{||||} + \text{||} = \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \end{array} \text{||} = \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \end{array} = 101_{\text{five}}$$

- $14_{\text{five}} + 43_{\text{five}} = 112_{\text{five}}$ .

$$14_{\text{five}} + 43_{\text{five}} = \begin{array}{c} \text{||} \\ \text{||||} \end{array} + \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \end{array} = \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||} \end{array} = \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||} \end{array} = 112_{\text{five}}$$

- $444_{\text{five}} + 11_{\text{five}} = 1010_{\text{five}}$ .

$$444_{\text{five}} + 11_{\text{five}} = \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \end{array} \text{||||} + \begin{array}{c} \text{||} \\ \text{||} \end{array} = \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||} \\ \text{||} \end{array} = \begin{array}{c} \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||||} \\ \text{||} \\ \text{||} \end{array} = 1010_{\text{five}}$$

## 2.3 Multiplication and division of whole numbers

### 2.3.1 Multiplication of whole numbers

**Definition 2.13 – Multiplication of whole numbers.** For any whole number  $a$  and  $n \neq 0$ ,

$$n \cdot a = \underbrace{a + a + a + \cdots + a}_{n \text{ terms}} \tag{2.1}$$

If  $n = 0$ , then  $0 \cdot a = 0$ .

**Example 2.14**  $\underbrace{4 + 4 + 4 + 4 + 4}_{\text{five } 4\text{s}} = 5 \cdot 4 = 20$

- Repeated-addition model.

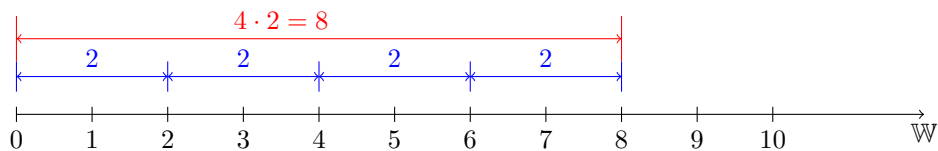
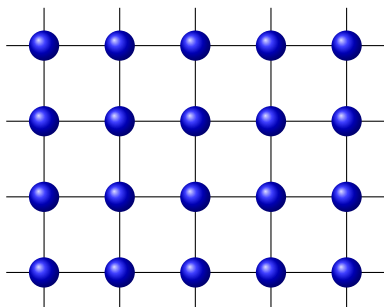
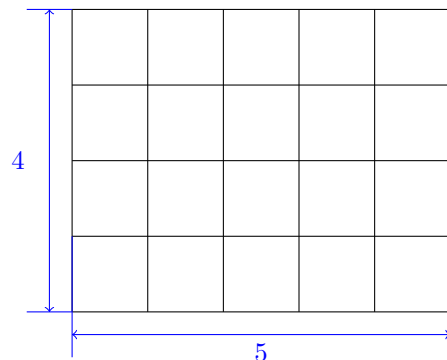


Figure 2.3:  $4 \cdot 2$  using repeated-addition model on the line-number.

- The array model  $5 \cdot 4 = 20$ .



- The area model  $5 \cdot 4 = 20$ .



**Definition 2.15 – Multiplication of whole numbers (alternative).** For finite sets  $A$  and  $B$ , if  $n(A) = a$  and  $n(B) = b$ , then  $a \cdot b = n(A \times B)$ .

In examples 1.54 of the previous chapter,  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$ . Since  $n(A) = 3$  and  $n(B) = 2$ , then

- $n(A \times B) = n(B \times A) = 6$ .
- $n(A \times A) = 9$ .
- $n(B \times B) = 4$ .

## 2.3.2 Properties of whole-number multiplication

**Theorem 2.16 – Properties of multiplication of whole numbers.**

1. **Closure property:** For whole numbers  $a$  and  $b$ ,  $a \cdot b$  is a unique whole number.
2. **Commutative property:** For whole numbers  $a$  and  $b$ ,  $a \cdot b = b \cdot a$ .
3. **Associative property:** For whole numbers  $a, b$  and  $c$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
4. **Identity property:** There is a unique whole number 1 such that for any whole number  $a$ ,  $a \cdot 1 = a = 1 \cdot a$ .
5. **Zero multiplication property:** For any whole number  $a$ ,  $a \cdot 0 = 0 = 0 \cdot a$ .

**Examples 2.17**

- $3 \cdot 4 = 4 \cdot 3 = 12$ .
- On one hand,  $(3 \cdot 4) \cdot 5 = 12 \cdot 5 = 60$ . On the other hand,  $3 \cdot (4 \cdot 5) = 3 \cdot 20 = 60$ . Hence  $(3 \cdot 4) \cdot 5 = 3 \cdot (4 \cdot 5) = 60$ .
- On one hand,  $4 \cdot 1 = 1 + 1 + 1 + 1 = 4$ . On the other hand,  $1 \cdot 4 = 4$ . Hence,  $4 \cdot 1 = 1 \cdot 4 = 4$ .

**Theorem 2.18 – Distributive property of multiplication over addition for whole numbers.** For any whole numbers  $a, b$  and  $c$

$$a(b + c) = ab + ac. \quad (2.2)$$

**Example 2.19** On one hand,  $3(5 + 6) = \underbrace{(5 + 6) + (5 + 6) + (5 + 6)}_{3 \text{ terms}} = \underbrace{11 + 11 + 11}_{3 \text{ terms}} = 3 \cdot 11 = 33$ , and, on the other  $3 \cdot 5 + 3 \cdot 6 = \underbrace{5 + 5 + 5}_{3 \text{ terms}} + \underbrace{6 + 6 + 6}_{3 \text{ terms}} = 15 + 18 = 33$ . We have then  $3(5 + 6) = 3 \cdot 5 + 3 \cdot 6$ .

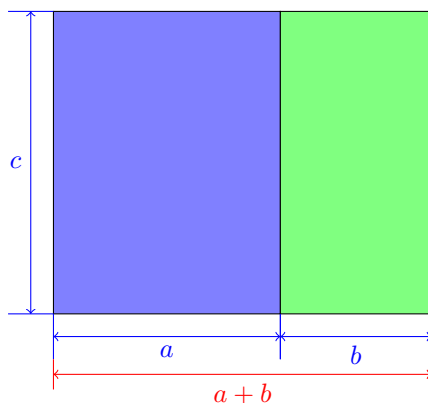


Figure 2.4: Distributive property of multiplication over addition for whole numbers

**Remark 2.20** You may use the distributive property of multiplication over addition of whole numbers to simplify a given multiplication. For example,  $27 \cdot 6 = (20 + 7) \cdot 6 = 20 \cdot 6 + 7 \cdot 6 = 120 + 42 = 162$ .

**Theorem 2.21 – Distributive property of multiplication over subtraction for whole numbers.** For any whole numbers  $a, b$  and  $c$  with  $b > c$ ,

$$a(b - c) = ab - ac. \quad (2.3)$$

**Example 2.22** On one hand,  $3(7 - 5) = \underbrace{(7 - 5) + (7 - 5) + (7 - 5)}_{3 \text{ terms}} = \underbrace{2 + 2 + 2}_{3 \text{ terms}} = 3 \cdot 2 = 6$ , and, on the other hand  $3 \cdot 7 - 3 \cdot 5 = \underbrace{(7 + 7 + 7)}_{3 \text{ terms}} - \underbrace{(5 + 5 + 5)}_{3 \text{ terms}} = 21 - 15 = 6$ . We have then  $3(7 - 5) = 3 \cdot 7 - 3 \cdot 5$ .

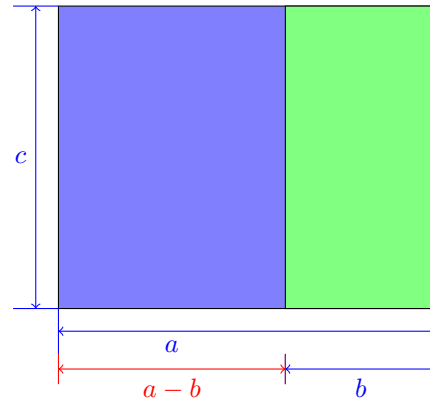


Figure 2.5: Distributive property of multiplication over subtraction for whole numbers

**Remark 2.23** You may use the distributive property of multiplication over subtraction of whole numbers to simplify a given multiplication. For example,  $27 \cdot 6 = (30 - 3) \cdot 6 = 30 \cdot 6 - 3 \cdot 6 = 180 - 18 = 162$ .

**Exercise 2.24** Use the area model to show that  $(a + b)(c + d) = ac + ad + bc + bd$ .

**Remark 2.25** The distributive property of multiplication over addition and subtraction of whole numbers is important in the study of algebra and in developing algorithms for arithmetic operations. For example,  $4x + 5x = (4 + 5)x = 9x$ , or  $4ab + 3b = (4a + 3)b$ .

### 2.3.3 Division of whole numbers

**Example 2.26 – Set (partition) model.** Suppose we have 21 candies and we want to give an equal number of candies for each of three friends. How many candies each friend should receive? 7 candies each.

**Definition 2.27 – Division of whole numbers.** For any whole numbers  $a$  and  $b$ , with  $b \neq 0$ ,  $a \div b = c$  if, and only if,  $c$  is the unique whole number such that  $b \cdot c = a$ .

**Example 2.28 – Missing factor model.** To compute  $42 \div 7$ , we look for a whole number  $c$  such that  $7 \cdot c = 42$ . Since  $7 \cdot 6 = 42$ , then  $42 \div 7 = 6$ .

**Example 2.29 – Repeated-subtraction model.** To compute  $24 \div 6$ , we subtract repeatedly 6 until we have zero.  $24 - 6 - 6 - 6 - 6 = 0$ . Hence,  $24 \div 6 = 4$  since we subtract from 24 four times 6 to obtain zero.

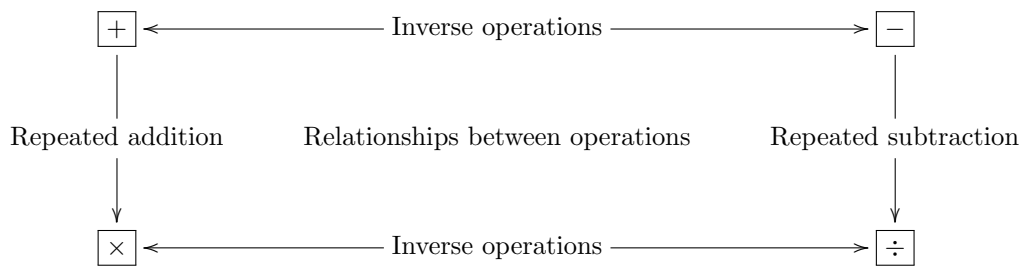
### 2.3.4 The division algorithm

**Definition 2.30 – Division algorithm.** Given any whole numbers  $a$  and  $b$  with  $b \neq 0$ , there exist unique whole numbers  $q$  (quotient) and  $r$  (remainder) such that

$$a = bq + r \quad \text{with } 0 \leq r < b. \quad (2.4)$$

**Example 2.31** We want to compute  $32 \div 5$ . Then  $1 \cdot 5 = 5$ ,  $2 \cdot 5 = 10$ ,  $3 \cdot 5 = 15$ ,  $4 \cdot 5 = 20$ ,  $5 \cdot 5 = 25$ ,  $6 \cdot 5 = 30$  and  $7 \cdot 5 = 35$ . Hence  $32 \div 5 = 6 \cdot 5 + 2$ .

### 2.3.5 Relating multiplication and division as inverse operations



### 2.3.6 Division by 0 and 1

#### Example 2.32

- $3 \div 0$ : Find a whole number  $q$  such that  $0 \cdot q = 3$ ? For all  $q$ ,  $0 \cdot q = 0$ . Consequently,  $3 \div 0$  does not exist. We say then  $3 \div 0$  is undefined.
- $0 \div 0$ : Find a whole number  $q$  such that  $0 \cdot q = 0$ ? For all  $q$ ,  $0 \cdot q = 0$ . Consequently, every whole number  $q$  is the quotient  $3 \div 0$ . Since the result of a division is unique, there is no unique  $q$  such that  $0 \div 0 = q$ . We say then that  $0 \div 0$  is undefined.
- $0 \div 3 = 0$ , since  $3 \cdot 0 = 0$ .
- For all whole number  $a$ ,  $a \div 1 = a$ .

### 2.3.7 Order of operations

To avoid confusion, mathematicians agree that when no parentheses are present, multiplications and divisions are performed before additions and subtractions. The multiplications and divisions are performed in the order they occur.

**Example 2.33**  $8 - 9 \cdot 2 \div 6 + 4 = 8 - 18 \div 6 + 4 = 8 - 3 + 4 = 5 + 4 = 9$ . If we put parentheses, it will be equivalent to do:  $\left(8 - \left((9 \cdot 2) \div 6\right)\right) + 4$ .

$$8 - \underbrace{9 \cdot 2}_{\text{first}} \div 6 + 4 = 8 - 18 \div 6 + 4 = 8 - \underbrace{18 \div 6}_{\text{second}} + 4 = 8 - 3 + 4 = \underbrace{8 - 3}_{\text{third}} + 4 = 5 + 4 = 9.$$

## 2.4 Algorithm for whole-number multiplication and division

### 2.4.1 Multiplication algorithms

- $5 \cdot 13 = 5 \cdot (10 + 3) = 5 \cdot 10 + 5 \cdot 3 = 50 + 15 = 65$ .
- $3(x + 5) = 3x + 3 \cdot 5 = 3x + 15$ .
- $15 \cdot 26 = (10 + 5)(20 + 6) = 10 \cdot 20 + 10 \cdot 6 + 5 \cdot 20 + 5 \cdot 6 = 200 + 60 + 100 + 30 = 390$ .
- $(2x + 5)(x + 3) = 2x \cdot x + 2x \cdot 3 + 5 \cdot x + 5 \cdot 3 = 2x^2 + 6x + 5x + 15 = 2x^2 + 11x + 15$ .
- $13 \times 10^3 = 13,000$ ;  $182 \times 10^5 = 18,200,000$ ;  $56 \times 10^4 = 560,000$ .
- $10^2 \cdot 10^4 = 1,000,000 = 10^6$ ;  $10^3 \times 10^5 = 100,000,000 = 10^8$ ;  $10^4 \times 10 = 10^4 \times 10^1 = 100,000$ .

**Remark 2.34**  $a^m \times a^n = \underbrace{(a \cdot a \cdot \dots \cdot a)}_{m \text{ factors}} \cdot \underbrace{(a \cdot a \cdot \dots \cdot a)}_{n \text{ factors}} = \underbrace{(a \cdot a \cdot \dots \cdot a)}_{m+n \text{ factors}} = a^{m+n}$ .

#### Examples 2.35

- $10^5 \times 10^6 = 10^{5+6} = 10^{11}$ .

- $2^3 \times 2^4 = 2^{3+4} = 2^7$ .
- $5^3 \times 5^2 = 5^{3+2} = 5^5$ .

**Example 2.36 – Multiplication with one-digit factors.**

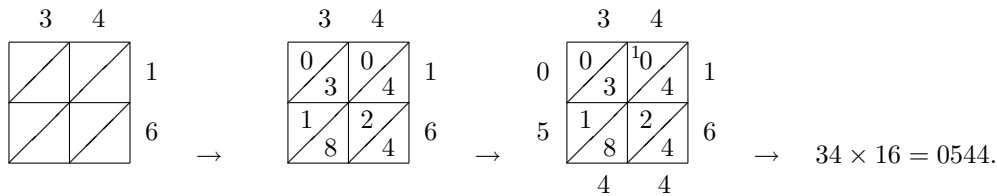
$$\begin{aligned}
 3 \cdot 469 &= 3(4 \cdot 10^2 + 6 \cdot 10 + 9) \\
 &= 3(4 \cdot 10^2) + 3(6 \cdot 10) + 3 \cdot 9 \\
 &= (3 \cdot 4)10^2 + (3 \cdot 6)10 + 3 \cdot 9 \\
 &= 12 \cdot 10^2 + 18 \cdot 10 + 3 \cdot 9 \\
 &= 1200 + 180 + 27 \\
 &= 1407.
 \end{aligned}
 \qquad \longrightarrow \qquad
 \begin{array}{r}
 469 \\
 \times 3 \\
 \hline
 27 \\
 180 \\
 1200 \\
 \hline
 = 1407
 \end{array}$$

**Example 2.37 – Multiplication with two-digits factors.**

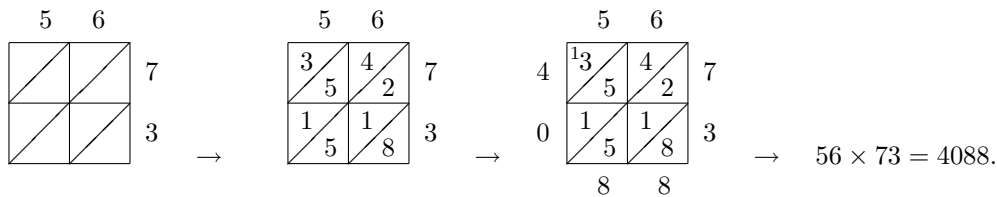
$$\begin{aligned}
 16 \cdot 34 &= (10 + 6) \cdot 34 = 10 \cdot 34 + 6 \cdot 34 \\
 &= 340 + 204 = 544
 \end{aligned}
 \qquad \longrightarrow \qquad
 \begin{array}{r}
 34 \\
 \times 16 \\
 \hline
 204 \\
 340 \\
 \hline
 = 544
 \end{array}
 \qquad \text{or} \qquad
 \begin{array}{r}
 34 \\
 \times 16 \\
 \hline
 204 \\
 34 \\
 \hline
 = 544
 \end{array}$$

**Examples 2.38 – Lattice multiplication.**

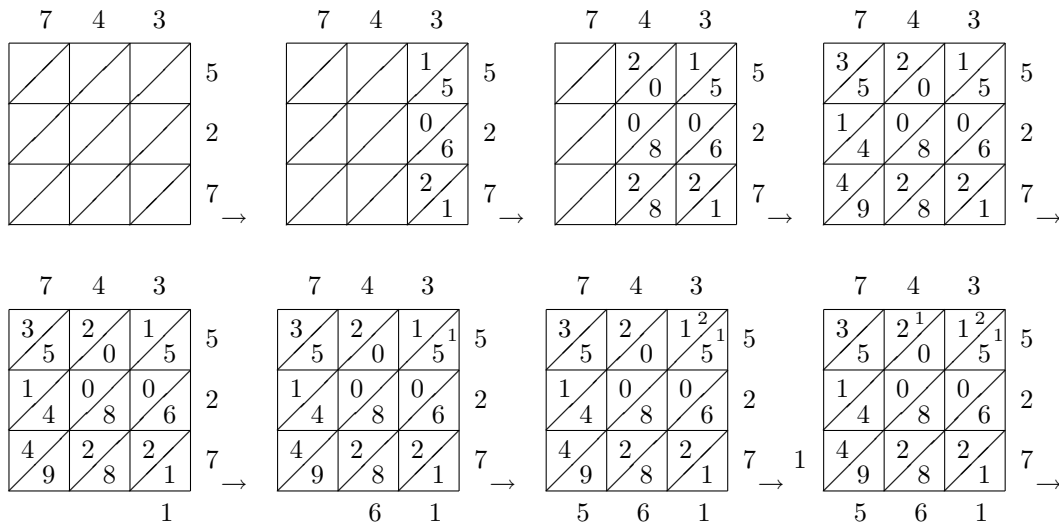
- $34 \times 16$ .

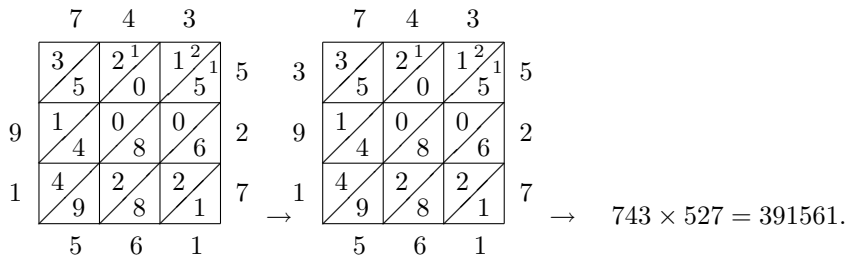


- $56 \times 73$



- $743 \times 527$





2.4.2 Division algorithms

- Using repeated subtraction to develop the standard division algorithm  $861 \div 7 = 123$ .

– The European Way:

$$\begin{array}{r}
 861 \overline{) 7} \\
 \underline{-700} \quad 100 \text{ sevens} \\
 161 \\
 \underline{-70} \quad 10 \text{ sevens} \\
 91 \\
 \underline{-70} \quad 10 \text{ sevens} \\
 21 \\
 \underline{-7} \quad 1 \text{ seven} \\
 14 \\
 \underline{-7} \quad 1 \text{ seven} \\
 7 \\
 \underline{-7} \quad 1 \text{ seven} \\
 0 \quad 123 \text{ sevens}
 \end{array}$$

or

$$\begin{array}{r}
 861 \overline{) 7} \\
 \underline{-700} \quad 100 \text{ sevens} \\
 161 \\
 \underline{-140} \quad 20 \text{ sevens} \\
 21 \\
 \underline{-21} \quad 3 \text{ sevens} \\
 0 \quad 123 \text{ sevens}
 \end{array}$$

or

$$\begin{array}{r}
 861 \overline{) 7} \\
 \underline{-700} \quad 100 \\
 161 \quad 20 \\
 \underline{-140} \quad 3 \\
 21 \quad 123 \\
 \underline{-21} \\
 0
 \end{array}$$

or

$$\begin{array}{r}
 861 \overline{) 7} \\
 \underline{-7} \quad 123 \\
 16 \\
 \underline{-14} \\
 21 \\
 \underline{-21} \\
 0
 \end{array}$$

– The American Way:

$$\begin{array}{r}
 7 \overline{) 861} \\
 \underline{-700} \quad 100 \text{ sevens} \\
 161 \\
 \underline{-70} \quad 10 \text{ sevens} \\
 91 \\
 \underline{-70} \quad 10 \text{ sevens} \\
 21 \\
 \underline{-7} \quad 1 \text{ seven} \\
 14 \\
 \underline{-7} \quad 1 \text{ seven} \\
 7 \\
 \underline{-7} \quad 1 \text{ seven} \\
 0 \quad 123 \text{ sevens}
 \end{array}$$

or

$$\begin{array}{r}
 7 \overline{) 861} \\
 \underline{-700} \quad 100 \text{ sevens} \\
 161 \\
 \underline{-140} \quad 20 \text{ sevens} \\
 21 \\
 \underline{-21} \quad 3 \text{ sevens} \\
 0 \quad 123 \text{ sevens}
 \end{array}$$

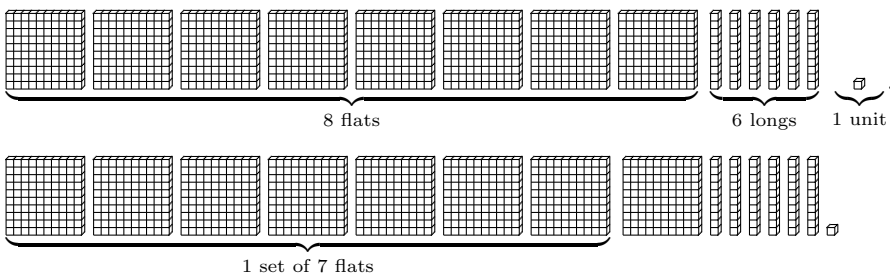
or

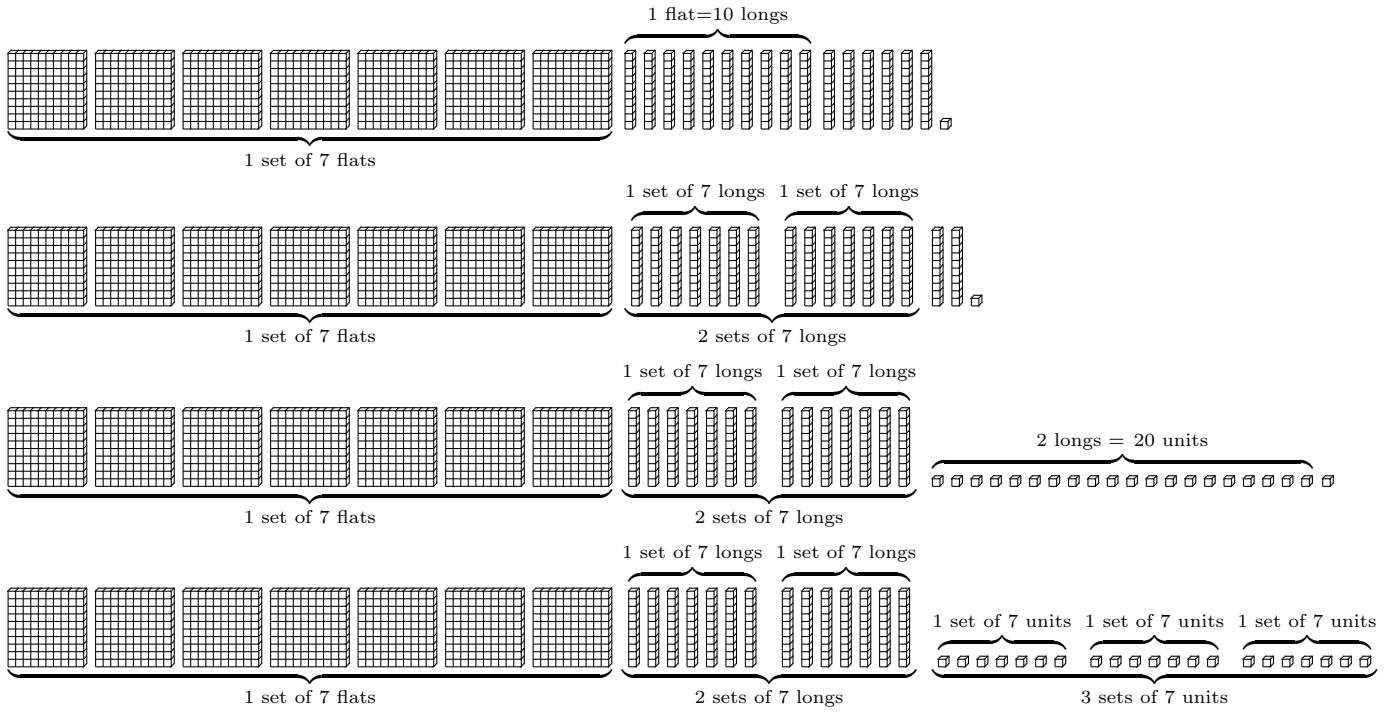
$$\begin{array}{r}
 123 \\
 \underline{3} \\
 20 \\
 100 \\
 7 \overline{) 861} \\
 \underline{-700} \\
 161 \\
 \underline{-140} \\
 21 \\
 \underline{-21} \\
 0
 \end{array}$$

or

$$\begin{array}{r}
 123 \\
 \underline{-7} \\
 16 \\
 \underline{-14} \\
 21 \\
 \underline{-21} \\
 0
 \end{array}$$

- Using base-ten blocks to develop the standard division algorithm  $861 \div 7 = 123$ . The number 861 can be represented by 8 base-ten flats, 6 base-ten longs and 3 base-ten units.





2.4.3 Division by a two-digit divisor

• The European Way:

$$\begin{array}{r}
 3748 \quad | \quad 47 \\
 - 3290 \\
 \hline
 0458 \quad 9 \\
 - 423 \\
 \hline
 035
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{r}
 3748 \quad | \quad 47 \\
 - 3290 \\
 \hline
 0458 \quad 79 \quad R \ 35 \\
 - 423 \\
 \hline
 035
 \end{array}$$

• The American Way:

$$\begin{array}{r}
 79 \\
 \hline
 9 \\
 70 \\
 47 \overline{) 3748} \\
 \underline{- 3290} \\
 0458 \\
 \underline{- 423} \\
 035
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{r}
 79 \quad R35 \\
 47 \overline{) 3748} \\
 \underline{- 329} \\
 0458 \\
 \underline{- 423} \\
 035
 \end{array}$$

Hence,  $3748 \div 47 = 79 \text{ R } 35$



# Chapter 3

## Algebraic thinking

### 3.1 Variables

A *variable* can mean several things in mathematics:

1. A missing element, or an unknown, as in  $x + 7 = 11$ . We can replace  $x$  with any number, but there is exactly only one value of  $x$  that makes the sentence true. Here  $x = 4$ .
2. In this class,  $h$  can represent your height,  $a$  can represent your age,  $g$  can represent your gender, and so on. The value of variable  $h$  can change from one student to another. The same thing holds for  $a$  and  $g$ .
3. An element in a set, as seen in the previous chapter.  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ .

Among other things, variables are needed to solve algebra problems. For that purpose, we translate given information to mathematical expression involving variables designated by letters or words.

**Exercise 3.1** Write each of the following statements in algebraic form:

- a. One more than a number.
- b. Two more than a number.
- c. Three less than a number.
- d. Two times a number.
- e. Five times a number.
- f. A number times itself.
- i. The cost of renting a car for any number of days if the charge per day is \$50.
- j. The distance a car traveled at a constant speed of 70 km/h for any number of hours.

**Solution.**

- a.  $n + 1$ .
- b.  $n + 2$ .
- c.  $n - 3$ .
- d.  $2 \cdot n$  or  $2n$ .
- e.  $5 \cdot n$  or  $5n$ .
- f.  $n \times n$  or  $n^2$ .
- i. If  $n$  is the number of days, the cost of renting a car for  $n$  days is  $\$50 \cdot n$  or  $\$50n$ .

- j. If  $h$  is the number of hours traveled by a car at a constant speed of 70 km/h, then the total distance traveled in  $h$  hours is  $70 \cdot h$  km or  $70h$  km.

To apply algebra in solving problems, we frequently need to translate information into a mathematical expression involving variables designated by letters. In such problems we may name the variables as we want.

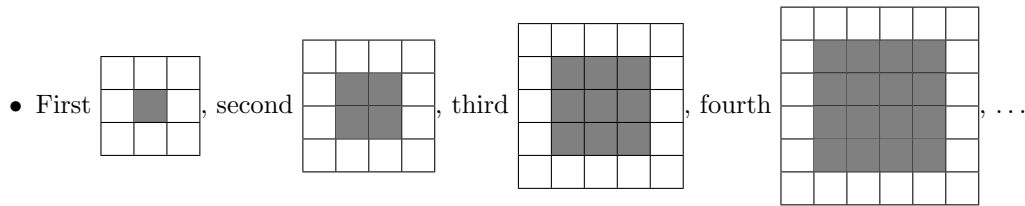
### Examples 3.2

- On a weekend, a store sold twice as many books as newspapers, and 25 fewer cartoons than books. If the store sold  $n$  newspapers, how many books and cartoons did it sell? Twice as many books as newspapers implies  $2n$  books sold. 25 fewer cartoons than books implies  $2n - 25$  cartoons sold.
- French fries have about 12 calories apiece. A hamburger has about 600 calories. Achilles is on a diet of 2000 calories per day. If he ate  $f$  french fries and one hamburger, how many more calories can he consume? 1 french fry is 12 calories. Hence  $f$  french fries is  $12f$  calories. A hamburger and  $f$  french fries is then  $600 + 12f$  calories. Consequently, the number of calories left for the day is  $2000 - (600 + 12f)$ , or  $2000 - 600 - 12f$  or  $1400 - 12f$ .
- A teacher says: take any natural number, add 3 to it, multiply the answer by 4, subtract 6 from the answer, then divide by 2, and finally, subtract 2. Tell me your answer and I will the original number.

Instructions	Symbols
Take any number	$n$
Add 3 to it	$n + 3$
Multiply the answer by 4	$4(n + 3)$
Subtract 6 from the answer	$4(n + 3) - 6$
Divide the answer by 2	$[4(n + 3) - 6] \div 2$
Subtract 2 from the answer	$\left([4(n + 3) - 6] \div 2\right) - 2$

$$\begin{aligned}
 & \left([4(n + 3) - 6] \div 2\right) - 2 \quad \leftarrow \text{Expand } 4(n + 3) \\
 &= \left([4n + 12 - 6] \div 2\right) - 2 \quad \leftarrow \text{Simplify } 4n + 12 - 6 \\
 &= \left([4n + 6] \div 2\right) - 2 \quad \leftarrow \text{Divide } 4n + 6 \text{ by } 2 \\
 &= (2n + 3) - 2 \quad \leftarrow \text{Get rid off the parentheses } (2n + 3) \\
 &= 2n + 3 - 2 \quad \leftarrow \text{Simplify } 2n + 3 - 2 \\
 &= 2n + 1
 \end{aligned}$$

$\left([4(n + 3) - 6] \div 2\right) - 2 = 2n + 1$ . So if you tell me: the answer is 17, then the original number is 8. If you tell me: the answer is 27, then the original number is 13, and so on. If you tell me: the answer is 18, then the teacher will tell you that it is impossible, and that you must have did a mistake in your computations.



We want to know how many shaded tiles there are in the  $n$ th figure. In the first figure,  $n = 1$ , there is 1 shaded tile. In the second figure,  $n = 2$ , there are  $4 = 2^2$  shaded tiles, in the third figure,  $n = 3$ , there are  $9 = 3^2$  shaded tiles, and in the fourth figure,  $n = 4$ , there are  $16 = 4^2$  shaded tiles. Therefore, we conclude that in the  $n$ th figure, there are  $n^2$  shaded tiles.

Now, we want to know how many tiles there are in the  $n$ th figure. In the first figure,  $n = 1$ , there is  $3 \times 3 = 9$  tiles. In the second figure,  $n = 2$ , there are  $4 \times 4 = 16$  tiles, in the third figure,  $n = 3$ , there are  $5 \times 5 = 25$  tiles, and in the fourth figure,  $n = 4$ , there are  $6 \times 6 = 36$  tiles. Therefore, we conclude that in the  $n$ th figure, there are  $(n + 2)^2$  tiles.

From these two results, we can conclude directly how many non-shaded tiles there are in the  $n$ th figure. Indeed, the number of non-shaded tiles in the  $n$ th figure is the number of tiles in the  $n$ th figure minus the number of shaded tiles in the  $n$ th figure, and hence the result is  $(n + 2)^2 - n^2 = n^2 + 4n + 4 - n^2 = 4n + 4$  tiles.

**Exercise 3.3** At the grocery store, three purchases were made: the first purchase is made for \$7 for an apple and a banana, the second purchase is made for \$8 for a banana and a kiwi, and the third purchase is made for \$9 for an apple and a kiwi. What is the cost of each fruit?

**Solution.** Let  $a$  be the price of an apple,  $b$  the price of a banana, and  $k$  is the price of a kiwi. Then  $a + b = 7$ ,  $b + k = 8$  and  $a + k = 9$ . If we group the two first purchases together, i.e., an apple, two bananas and a kiwi, then the purchase would have been made for the price of  $a + 2b + k = 7 + 8 = 15$ . To have the price of two bananas, we subtract the price of the third purchases. Then  $2b = 15 - 9 = 6$  and hence the price of a banana is \$3. Knowing the price of a banana, the price of an apple is the price of the first purchase minus the price of a banana,  $a = 7 - 3 = 4$ . Also, the price of a kiwi is the price of the second purchase minus the price of a banana, and hence  $k = 8 - 3 = 5$ . Finally, the cost of an apple is \$4, the price of a banana is \$3 and the price of a kiwi is \$5.

**Remark 3.4** To make sure our computations were correct in the above exercise, we can check our answers as follows: we found  $a = 4$ ,  $b = 3$  and  $k = 5$ . The cost of the first purchase is  $a + b = 4 + 3 = 7$ . The cost of the second purchase is  $b + k = 3 + 5 = 8$ . The cost of the purchase is  $a + k = 4 + 5 = 9$ . The prices we found satisfy our problem's data.

## 3.2 Equations

**Theorem 3.5 – Addition and multiplication property of equality.**

1. For any numbers  $a, b$  and  $c$ , if  $a = b$ , then  $a + c = b + c$ .
2. For any numbers  $a, b$  and  $c$ , if  $a = b$ , then  $ac = bc$ .

**Examples 3.6**

- If  $a = b$ , then  $a + 3 = b + 3$ .
- If  $a = b$ , then  $2a = 2b$ .
- If  $a = b$ , then  $a + 5 = b + 5$ .
- If  $a = b$ , then  $7a = 7b$ .

**Theorem 3.7 – Cancellation properties of equality.**

1. For any numbers  $a, b$  and  $c$ , if  $a + c = b + c$ , then  $a = b$ .
2. For any numbers  $a, b$  and  $c$ , with  $c \neq 0$ , if  $ac = bc$ , then  $a = b$ .

**Examples 3.8**

- If  $a + 3 = b + 3$ , then  $a = b$ .
- If  $a + 5 = b + 5$ , then  $a = b$ .
- If  $2a = 2b$ , then  $a = b$ .
- If  $7a = 7b$ , then  $a = b$ .

**Theorem 3.9 — Addition and subtraction property of equations.** For any numbers  $a, b, c$  and  $d$ , If  $a = b$  and  $c = d$ , then

$$a + c = b + d \quad (3.1)$$

$$a - c = b - d. \quad (3.2)$$

**Theorem 3.10 — Commutative properties of addition and multiplication.** For any numbers  $a$  and  $b$ ,

$$a + b = b + a \quad (3.3)$$

$$ab = ba. \quad (3.4)$$

**Theorem 3.11 — Associative properties of addition and multiplication.** For any numbers  $a, b$  and  $c$ ,

$$(a + b) + c = a + (b + c) \quad (3.5)$$

$$(ab)c = a(bc). \quad (3.6)$$

**Theorem 3.12 — Distributive property of multiplication over addition.** For all numbers  $a, b$  and  $c$ ,

$$a(b + c) = ab + ac. \quad (3.7)$$

**Theorem 3.13 — Distributive property of multiplication over subtraction.** For all numbers  $a, b$  and  $c$ ,

$$a(b - c) = ab - ac. \quad (3.8)$$

### 3.2.1 Solving equations

To solve an equation, we need to use the above theorems and properties to obtain the value of our variable.

#### Examples 3.14

- $x + 5 = 12$ . This equation remains the same if we subtract to both sides of the equation 5. The equation becomes  $x + 5 - 5 = 12 - 5$ , and after simplification we have  $x = 7$ . The solution of the equation is  $x = 7$ .

$$\begin{array}{l} x + 5 = 12 \quad \boxed{\hspace{10em}} \text{ Subtract 5 from both sides of the equation.} \\ x + 5 - 5 = 12 - 5 \quad \leftarrow \boxed{\hspace{10em}} \\ x = 7 \quad \leftarrow \boxed{\hspace{10em}} \text{ After simplification, the result is:} \end{array}$$

- $5x = x + 20$  This equation remains the same if we subtract to both sides of the equation  $x$ . The equation becomes  $5x - x = x + 20 - x$ , and after simplification we have  $4x = 20$ . The equation remains the same if we divide both sides of the equation by 4. The equation becomes  $x = 5$ . The solution is  $x = 5$ .

$$\begin{array}{l} 5x = x + 20 \quad \boxed{\hspace{10em}} \text{ Subtract } x \text{ from both sides of the equation.} \\ 5x - x = x + 20 - x \quad \leftarrow \boxed{\hspace{10em}} \text{ Simplify} \\ 4x = 20 \quad \leftarrow \boxed{\hspace{10em}} \\ x = 5 \quad \leftarrow \boxed{\hspace{10em}} \text{ Divide both sides of the equation by 4} \end{array}$$

**Exercise 3.15** Solve each of the following equations for  $x$ :

a.  $3x + 4x = 77$ .

b.  $3x + 2 = x + 10$ .

c.  $2(x - 1) + 3(x + 1) = 11$ .

**Solution.**

a.

$$\begin{array}{l} 3x + 4x = 77 \quad \boxed{\hspace{2cm}} \text{Simplify} \\ 7x = 77 \quad \leftarrow \boxed{\hspace{2cm}} \text{Divide both sides of the equation by 7} \\ x = 11 \quad \leftarrow \boxed{\hspace{2cm}} \end{array}$$

b.

$$\begin{array}{l} 3x + 2 = x + 10 \quad \boxed{\hspace{2cm}} \text{Subtract } x \text{ from both sides of the equation.} \\ 3x + 2 - x = x + 10 - x \quad \leftarrow \boxed{\hspace{2cm}} \text{Simplify} \\ 2x + 2 = 10 \quad \leftarrow \boxed{\hspace{2cm}} \text{Subtract 2 from both sides of the equation} \\ 2x + 2 - 2 = 10 - 2 \quad \leftarrow \boxed{\hspace{2cm}} \text{Simplify} \\ 2x = 8 \quad \leftarrow \boxed{\hspace{2cm}} \text{Divide both sides of the equation by 2} \\ x = 4 \quad \leftarrow \boxed{\hspace{2cm}} \end{array}$$

c.

$$\begin{array}{l} 2(x - 1) + 3(x + 1) = 11 \quad \boxed{\hspace{2cm}} \text{Expand} \\ 2x - 2 + 3x + 3 = 11 \quad \leftarrow \boxed{\hspace{2cm}} \text{Simplify} \\ 5x + 1 = 11 \quad \leftarrow \boxed{\hspace{2cm}} \text{Subtract 1 from both sides of the equation} \\ 5x + 1 - 1 = 11 - 1 \quad \leftarrow \boxed{\hspace{2cm}} \text{Simplify} \\ 5x = 10 \quad \leftarrow \boxed{\hspace{2cm}} \text{Divide both sides of the equation by 5} \\ x = 2 \quad \leftarrow \boxed{\hspace{2cm}} \end{array}$$

### 3.2.2 Application problem

Application problem  $\rightarrow$  Mathematical model  $\rightarrow$  Mathematical solution  $\rightarrow$  Original problem interpretation

**Problem 3.16 – Coins in a piggy bank.** A piggy bank contains 67 coins, only dimes and nickels. The amount of money in the box is \$4.20. How many dimes and how many nickels are in the piggy bank?

**Solution.**

**Understanding the problem** In the piggy bank, there are 67 coins. There are only dimes and nickels. Therefore, the number of dimes plus the number of nickels is equal to 67. The amount of money is \$4.20.

**Devising a plan** Let  $d$  be the number of dimes in the piggy bank, and  $n$  be the number of nickels in the piggy bank. There are 67 coins in the piggy bank. Consequently,

$$d + n = 67. \tag{3.9}$$

A dime is 10¢, and a nickel is 5¢. So  $d$  dimes and  $n$  nickels have a value of  $10d + 5n$  ¢. However, in the piggy bank, the amount of money is \$4.20, or equivalently, 420¢. Therefore,

$$10d + 5n = 420. \tag{3.10}$$

**Carrying out the plan** From (3.9), we may notice that  $n = 67 - d$ . We replace than the value of  $n$  in (3.10) and hence

$$\begin{array}{l}
 10d + 5n = 420 \quad \leftarrow \boxed{\hspace{10em}} \text{Substitute } n \text{ by } 67 - d \\
 10d + 5(67 - d) = 420 \quad \leftarrow \boxed{\hspace{10em}} \text{Expand } 5(67 - d) \\
 10d + 335 - 5d = 420 \quad \leftarrow \boxed{\hspace{10em}} \text{Simplify} \\
 5d + 335 = 420 \quad \leftarrow \boxed{\hspace{10em}} \text{Subtract } 335 \text{ from both sides of the equation} \\
 5d + 335 - 335 = 420 - 335 \quad \leftarrow \boxed{\hspace{10em}} \text{Simplify} \\
 5d = 85 \quad \leftarrow \boxed{\hspace{10em}} \text{Divide both sides of the equation by } 5 \\
 d = 17 \quad \leftarrow \boxed{\hspace{10em}}
 \end{array}$$

So, there are 17 dime coins in the piggy bank. The number of nickels is 67 minus the number of dimes. Therefore,  $n = 67 - 17 = 50$ . There 50 nickel coins in the piggy bank.

**Looking back** To check the answer, lets compute the amount of money with 17 dime coins and 50 nickel coins. 17 dime coins worth  $17 \times 10\text{¢} = 170\text{¢}$ . 50 nickel coins worth  $50 \times 5\text{¢} = 250\text{¢}$ . Hence, the total amount of money is  $170 + 250 = 420\text{ ¢}$ , which match our problem data. Moreover, there are 67 coins in the piggy bank with 17 dimes coins and 50 nickel coins. So, our solutions are OK.

# Chapter 4

## Integers and number theory

Negative numbers are useful in everyday life: This mountain is 1200 m above sea level, this abyss is 500 m below sea level, the temperature is 5 degree below freezing ( $-5^{\circ}\text{C}$ ), when you withdraw money, you can see in your bank statement  $-\$20$ , ...

### 4.1 Integers and operations of addition and subtraction

#### 4.1.1 Representations of integers

The set of integers is  $\mathbb{I} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$ . We can also write  $\mathbb{Z}$  instead of  $\mathbb{I}$ .

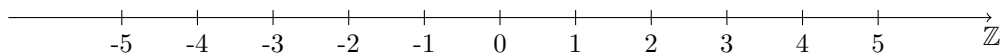


Figure 4.1: Integers on the line-number.

The negative integers are **opposites** of the positives integers, and conversely. For example, the opposite of 2 is  $-2$  and the opposite of  $-3$  is 3. The opposite of 0 is 0. When you add a number  $a$  with its opposite  $-a$ , you find zero.

**Exercise 4.1** Find the opposite of  $x$  for each of the following:

- a.  $x = 6$ ;                      b.  $x = -5$ ;                      c.  $x = -7$ ;                      d.  $x = 0$ .

**Solution.**

- a.  $-x = -6$ .                      b.  $-x = 5$ .                      c.  $-x = 7$ .                      d.  $-x = 0$ .

Using the number-line model, we can understand why  $-3$  is the opposite of 3. An integer and its opposite are symmetric with respect to zero.

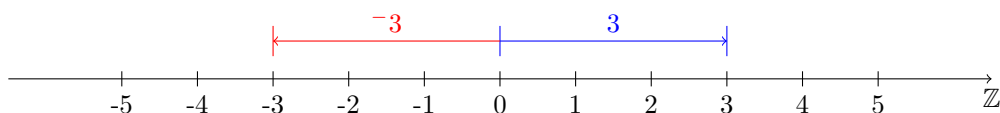


Figure 4.2: The opposite of 3 using the line-number.

### 4.1.2 Integer addition

### 4.1.3 Number-line model

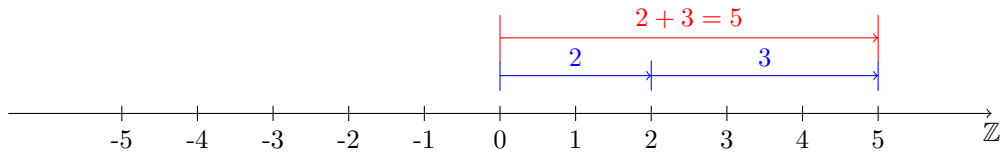


Figure 4.3:  $2 + 3 = 5$  using the number-line

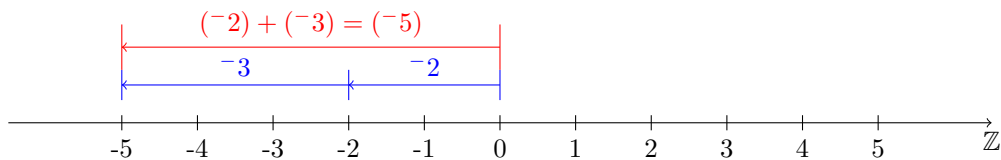


Figure 4.4:  $(-2) + (-3) = -5$  using the number-line

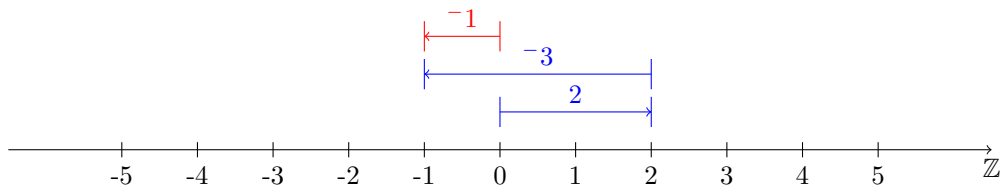


Figure 4.5:  $2 + (-3) = -1$  using the number-line

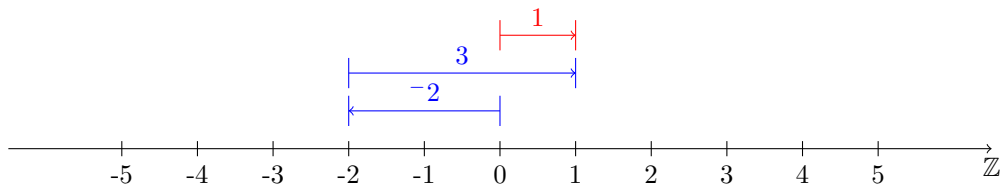


Figure 4.6:  $(-2) + 3 = 1$  using the number-line

### 4.1.4 Charged-field model for addition

A field has 0 charges if it has the same number of positive positive charges and negatives charges.  $5 + ^- 3 = 2$ : we have 5 positive charges and 3 negative charges. So, in total the field has 2 positive charges.  $^- 5 + 2 = ^- 3$ : we have 5 negative charges and 2 positive charges. So, in total the field has 3 negative charges.



Figure 4.7:  $5 + (-2)$  and  $(-5) + 3$  using the charged-field model

**Exercise 4.2** Use either the number-line or the charged-field models to compute each of the following:

- a.  $(-2) + 5$ ;                      b.  $(-3) + (-1)$ ;                      c.  $6 + (-5) + (-2)$ ;                      d.  $(-4) + 1 + 3$ .

**Solution.**

- a. 3;                                      b.  $-4$ ;                                      c.  $-1$ ;                                      d. 0.

### 4.1.5 Absolute value

The distance is always a positive number or zero. The distance between the point corresponding to an integer and 0 is the **absolute value** of the integer. Hence, we can define the absolute value of an integer as follows:

**Definition 4.3 – Absolute value.** The absolute value of an integer  $x$ , written  $|x|$ , is equal to  $x$  if  $x \geq 0$ , and is equal to  $-x$  if  $x < 0$ .

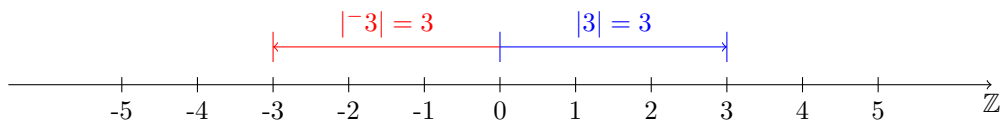


Figure 4.8: The absolute value of 3 and  $-3$ .

#### Examples 4.4

- $|4| = 4$ ;                                      •  $|-3| = 3$ ;                                      •  $|0| = 0$ ;                                      •  $|(-3) + 2| = 1$ ;
- $|-4| = 4$ ;                                      •  $|-5| = 5$ ;                                      •  $|-4| + |-2| = 6$ ;                                      •  $|(-2)| + |2| = 4$ .

### 4.1.6 Properties of integer addition

**Theorem 4.5 – Properties of integer addition.** Given integers  $a, b$  and  $c$ :

1. **Closure property of addition of integers**  $a + b$  is a unique integer.
2. **Commutative property:**  $a + b = b + a$ .
3. **Associative property:**  $(a + b) + c = a + (b + c)$ .
4. **Additive identity element:** 0 is the unique integer such that, for all integer  $a$ ,  $0 + a = a = a + 0$ .

**Theorem 4.6 – Uniqueness of the additive inverse.** For every integer  $a$ , there is a unique  $-a$ , the additive inverse of  $a$ , such that  $a + (-a) = 0 = (-a) + a$ .

**Theorem 4.7 – Additive inverse properties.** For any integers  $a$  and  $b$ :

1.  $-(-a) = a$ .
2.  $-a + (-b) = -(a + b)$ .

**Exercise 4.8** Find the additive inverse of each of the following:

- a.  $-(x + 5)$ ;                                      b.  $a + (-2)$ ;                                      c.  $-2 + (-b)$ ;                                      d.  $3 + (-x)$ .

**Solution.**

a.  $x + 5$ ;

b.  $-a + 2$ ;

c.  $2 + b$ ;

d.  $-3 + x$ .

### 4.1.7 Integer subtraction

**Definition 4.9 – Subtraction.** For integers  $a$  and  $b$ ,  $a - b$  is the unique integer  $n$  such that  $a = b + n$ .

**Theorem 4.10** For all integers  $a$  and  $b$ ,  $a - b = a +^{-} b$ .

**Exercise 4.11** Find  $^{-}3 -^{-}5$  using:

- Number-line model.
- Charged-field model for subtraction.

**Solution.**

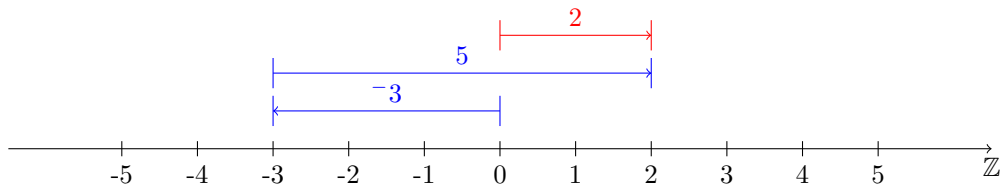


Figure 4.9:  $^{-}3 -^{-}5 =^{-}3 + 5 = 2$  using the number-line

- Using the fact that  $^{-}3 -^{-}5 =^{-}3 + 5$ , we have:

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} - \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} + \begin{array}{c} +++ \\ +++ \\ +++ \end{array} = \begin{array}{c} ++ \\ ++ \end{array}$$

- If we think of subtraction as "taking away", we have :

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} - \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} = \begin{array}{c} \text{---} \\ +++ \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} = \begin{array}{c} ++ \\ ++ \end{array}$$

**Exercise 4.12** Compute the following:

a.  $3 - 10$ ;

c.  $^{-}5 -^{-}4$ ;

e.  $3 - 8$ ;

g.  $^{-}5 -^{-}3$ ;

b.  $^{-}2 -^{-}4$ ;

d.  $3 - 4$ ;

f.  $2 -^{-}4$ ;

h.  $^{-}15 - 5$ .

**Solution.**

a.  $^{-}7$ ;

c.  $^{-}1$ ;

e.  $^{-}5$ ;

g.  $^{-}2$ ;

b.  $2$ ;

d.  $^{-}1$ ;

f.  $6$ ;

h.  $^{-}20$ .

**Exercise 4.13** Simplify each of the following:

a.  $3 - (4 - x)$ ;

b.  $5 - (3x - 2)$ ;

c.  $^{-}(2x - y) - x$ ;

d.  $x - (2x - 3)$ .

**Solution.**

a.  $3 - (4 - x) = 3 - 4 + x =^{-}1 + x$ ;

b.  $5 - (3x - 2) = 5 - 3x + 2 = 7 - 3x$ ;

c.  $-(2x - y) - x = -2x + y - x = -3x + y$ ;

d.  $x - (2x - 3) = x - 2x + 3 = -x + 3$ .

**Exercise 4.14** Compute the following:

a.  $3 - 7 - 7$ ;

b.  $2 - 8 + 5$ ;

c.  $8 - 5 - 6$ ;

d.  $5 - (6 - 9)$ .

**Solution.**

a.  $3 - 7 - 7 = -4 - 7 = -11$ ;

b.  $2 - 8 + 5 = -6 + 5 = -1$ ;

c.  $8 - 5 - 6 = 3 - 6 = -3$ ;

d.  $5 - (6 - 9) = 5 - (-3) = 5 + 3 = 8$ .

## 4.2 Multiplication and division of integers

**Theorem 4.15** For any whole number  $a$  and  $b$ , the following hold:

1.  $(-a)(-b) = ab$ .
2.  $(-a)b = b(-a) = -(ab)$ .

### 4.2.1 Properties of integer multiplication

**Theorem 4.16 – Properties of integer multiplication.** The set of integers  $\mathbb{I}$  satisfies the following properties for all integers  $a, b, c \in \mathbb{I}$ :

1. **Closure property:**  $ab$  is a unique integer.
2. **Commutative property:**  $ab = ba$ .
3. **Associative property:**  $(ab)c = a(bc)$ .
4. **Multiplicative identity element:** 1 is the unique integer such that for all integers  $a$ ,  $1 \cdot a = a = a \cdot 1$ .
5. **Distributive properties of multiplication over addition for integers:**  $a(b + c) = ab + bc$ .
6. **Zero multiplication property:** 0 is the unique integer such that for all integers  $a$ ,  $a \cdot 0 = 0 = 0 \cdot a$ .

**Theorem 4.17** For every integer  $a$ ,  $(-1)a = -a$ .

**Theorem 4.18** For all integers  $a$  and  $b$ ,

$$(-a)b = -(ab) \quad \text{and} \quad (-a)(-b) = ab. \quad (4.1)$$

**Theorem 4.19 – Distributive property of multiplication over subtraction of integers.** For any integers  $a, b$ , and  $c$ ,

$$a(b - c) = ab - ac \quad \text{and} \quad (b - c)a = ba - ca. \quad (4.2)$$

**Exercise 4.20** Simplify each of the following so that there are no parentheses in the final answer:

a.  $(-5)(x-3)$ ;                      b.  $(a-b)(a+b)$ ;                      c.  $(a+b)^2$ ;                      d.  $(a-b)^2$ .

**Solution.**

a.  $(-5)(x-3) = (-5)x - (-5)(3) = -5x - (-15) = -5x + 15$ ;

b.  $(a-b)(a+b) = (a-b)a + (a-b)b = a^2 - ba + ab - b^2 = a^2 - b^2$ ;

c.  $(a+b)^2 = (a+b)(a+b) = (a+b)a + (a+b)b = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$ ;

d.  $(a-b)^2 = (a-b)(a-b) = (a-b)a - (a-b)b = a^2 - ab - (ba - b^2) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$ .

**Exercise 4.21** Using the result  $(a+b)(a-b) = a^2 - b^2$  of the previous exercise to simplify the following:

a.  $(3-x)(3+x)$ ;                      b.  $(-5+a)(-5-a)$ ;                      c.  $(x+2)^2 - (x-2)^2$ .

**Solution.**

a.  $(3-x)(3+x) = 3^2 - x^2 = 9 - x^2$ ;

b.  $(-5+a)(-5-a) = (-5)^2 - a^2 = 25 - a^2$ ;

c.  $(x+2)^2 - (x-2)^2 = ((x+2) + (x-2))((x+2) - (x-2)) = (x+2+x-2)(x+2-x+2) = (2x)(4) = 8x$ .

**Exercise 4.22** Using the result  $(a+b)(a-b) = a^2 - b^2$ , factor each of the following:

a.  $x^2 - 16$ ;                      b.  $(a+b)^2 - c^2$ ;                      c.  $(x+y)^2 - (z+t)^2$ .

**Solution.**

a.  $x^2 - 16 = x^2 - 4^2 = (x+4)(x-4)$ ;

b.  $(a+b)^2 - c^2 = ((a+b) + c)((a+b) - c) = (a+b+c)(a+b-c)$ ;

c.  $(x+y)^2 - (z+t)^2 = ((x+y) + (z+t))((x+y) - (z+t)) = (x+y+z+t)(x+y-z-t)$ .

## 4.2.2 Integer division

**Definition 4.23 — Integer division.** If  $a$  and  $b$  are any integers,  $a \div b$  is the unique integer  $c$ , if it exists, such that  $a = bc$ .

**Exercise 4.24** When possible, use the definition of the integer division to evaluate each of the following:

a.  $15 \div (-3)$ ;                      c.  $-15 \div (-3)$ ;                      e.  $(xy) \div y$ , where  $y \neq 0$ ;

b.  $-15 \div 3$ ;                      d.  $15 \div 4$ ;                      f.  $(xy) \div x$ , where  $x \neq 0$ .

**Solution.**

a.  $15 \div (-3) = -5$  because  $(-3) \times (-5) = 15$ .

b.  $-15 \div 3 = -5$  because  $3 \times (-5) = -15$ .

c.  $-15 \div (-3) = 5$  because  $(-3) \times 5 = -15$ .

d.  $15 \div 4$  is undefined over the set of integers, because there exists no integers  $c$  such that  $4c = 15$ .

e.  $(xy) \div y = x$ , where  $y \neq 0$ , because  $x \times y = xy$ .

f.  $(xy) \div x = y$ , where  $x \neq 0$ , because,  $y \times x = yx = xy$ .

### 4.2.3 Order of operation on integers

When addition, subtraction, multiplication, division and exponentiation appear without parentheses, exponentiation is done first in order from right to left, then multiplications and divisions in the order of their appearance from left to right, and then additions and subtractions in the order of their appearance from left to right. Arithmetic operations that appear inside parentheses must be done first.

**Exercise 4.25** Evaluate each of the following:

- a.  $3 - 2 \cdot 5 + 4$ ;                      c.  $3 - 2 \cdot 5 + 3 \cdot 4 - 7 + 5$ ;    e.  $-2^3$ ;                                      g.  $-2^4$ ;  
 b.  $(3 - 4)5 + 2$ ;                      d.  $5 + 21 \div 7 \cdot 3 - 8$ ;              f.  $(-2)^3$ ;                                      h.  $(-2)^4$ .

**Solution.**

- a.  $3 - 2 \cdot 5 + 4 = 3 - 10 + 4 = -7 + 4 = -3$ ;  
 b.  $(3 - 4)5 + 2 = (-1) \cdot 5 + 2 = -5 + 2 = -3$ ;  
 c.  $3 - 2 \cdot 5 + 3 \cdot 4 - 7 + 5 = 3 - 10 + 12 - 7 + 5 = -7 + 12 - 7 + 5 = 5 - 7 + 5 = -2 + 5 = 3$ ;  
 d.  $5 + 21 \div 7 \cdot 3 - 8 = 5 + 3 \cdot 3 - 8 = 5 + 9 - 8 = 14 - 8 = 6$ .  
 e.  $-2^3 = -2 \times 2 \times 2 = -8$ ;  
 f.  $(-2)^3 = (-2) \times (-2) \times (-2) = -8$ ;  
 e.  $-2^3 = -2 \times 2 \times 2 = -16$ ;  
 f.  $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$ .

### 4.2.4 Ordering integers

**Definition 4.26 – Less than for integers.** For any integers  $a$  and  $b$ ,  $a$  is less than  $b$ , written  $a < b$ , if, and only if, there exists a positive integer  $k$  such that  $a + k = b$ .

**Theorem 4.27**  $a < b$  (or equivalently,  $b > a$ ) if, and only if,  $b - a$  is equal to positive integer, that is  $b - a$  is greater than 0.

**Exercise 4.28** Fill in the gap with either  $<$  or  $>$  in the following:

- a.  $2 \dots 5$ ;                                      c.  $-2 \dots 5$ ;                                      e.  $0 \dots 2$ ;  
 b.  $-2 \dots -5$ ;                                      d.  $2 \dots -5$ ;                                      f.  $0 \dots -2$ .

**Solution.**

- a.  $2 < 5$ ;                                      c.  $-2 < 5$ ;                                      e.  $0 < 2$ ;  
 b.  $-2 > -5$ ;                                      d.  $2 > -5$ ;                                      f.  $0 > -2$ .

**Theorem 4.29**

1. If  $x < y$  and  $n$  is an integer, then  $x + n < y + n$ .
2. If  $x < y$ , then  $-x > -y$ .
3. If  $x < y$  and  $n > 0$ , then  $nx < ny$ .
4. If  $x < y$  and  $n < 0$ , then  $nx > ny$ .

**Exercise 4.30** Use the above theorem to find all integers  $x$  that satisfy the following:

- a.  $x + 4 < -5$ .  
 b.  $-x - 5 < 7$ .  
 c. If  $x \leq -1$ , find the values of  $4 - 3x$ .

**Solution.**

- a.  $x + 4 < -5$  is equivalent to  $x + 4 - 4 < -5 - 4$  and hence  $x < -9$ .  
 b.  $-x - 5 < 7$  is equivalent to  $-x - 5 + 5 < 7 + 5$ , which is the same as  $-x < 12$ , and hence  $x > -12$ .  
 c. If  $x \leq -1$ , then by multiplying by  $-3$  both sides of the inequality, we obtain  $-3x \geq 3$ . By adding 4 to both sides of the inequality, we obtain  $4 - 3x \geq 7$

### 4.3 Divisibility

**Definition 4.31 — "Divides".** If  $a$  and  $b$  are any integers, then  $b$  divides  $a$ , written  $b|a$ , if, and only if, there is a unique integer  $q$  such that  $a = bq$ .

There are many ways to express the fact that a given integer  $b$  divides an integer  $a$ :

1.  $b$  is a divisor of  $a$ ;
2.  $b$  is a factor of  $a$ ;
3.  $a$  is divisible by  $b$ ;
4.  $a$  is a multiple of  $b$ .

**Remark 4.32**  $b$  does not divide  $a$  is denoted  $b \nmid a$ .

**Theorem 4.33** For any integers  $a$  and  $d$ , if  $d|a$  and  $n$  is any integer, then  $d|na$ .

**Exercise 4.34** Classify each of the following as true or false. Explain your answers

- a.  $-3|15$ ;  
 b.  $0|5$ ;  
 c. 0 is even;  
 d.  $7 \nmid 3$ ;  
 e. For all integers  $a$ ,  $1|a$ ;  
 f.  $0|0$ ;  
 g.  $4|8n$ ;  
 h.  $(a - b)|(a^2 - b^2)$  if  $a$  and  $b$  are integers and  $a \neq b$ .

**Solution.**

- a.  $-3|15$  is true because  $15 = (-3) \times (-5)$ ;  
 b.  $0|5$  is false because there is no integer  $c$  such that  $5 = 0 \cdot c$ ;  
 c. 0 is even is true because  $0 = 2 \times 0$ ;  
 d.  $7 \nmid 3$  is true because there is no integer  $c$  such that  $3 = 7 \cdot c$ ;  
 e. For all integers  $a$ ,  $1|a$  is true because  $a = 1 \cdot a$ ;  
 f.  $0|0$  is false because  $0 = 0 \cdot c$  for all integers  $c$ , so  $c$  is not unique;

g.  $4|8n$  is true because  $8n = 4n \cdot 2$ ;

h.  $(a - b)|(a^2 - b^2)$  if  $a$  and  $b$  are integers and  $a \neq b$  is true because  $(a^2 - b^2) = (a - b)(a + b)$  and  $a \neq b$ .

**Theorem 4.35** For any integers  $a, b$  and  $d$  with  $d \neq 0$ ,

1. If  $d|a$  and  $d|b$ , then  $d|(a + b)$ .
2. If  $d|a$  and  $d \nmid b$ , then  $d \nmid (a + b)$ .
3. If  $d|a$  and  $d|b$ , then  $d|(a - b)$ .
4. If  $d|a$  and  $d \nmid b$ , then  $d \nmid (a - b)$ .

### 4.3.1 Divisibility rules

**Theorem 4.36 – Divisibility test for 2.** An integer is divisible by 2, if, and only if, its units digit is divisible by 2; that is, if, and only if, the units digit is 0, 2, 4, 6 or 8.

**Examples 4.37** 1578 is divisible by 2 because the units digit of 1578 is 8. However, 1577 is not divisible by 2 because the units digit of 1577 is 7.

**Theorem 4.38 – Divisibility test for 5.** An integer is divisible by 5, if, and only if, its units digit is divisible by 5; that is, if, and only if, the units digit is 0 or 5.

**Examples 4.39** 1275 is divisible by 5 because the units digit of 1275 is 5. However, 1577 is not divisible by 5 because the units digit of 1577 is 7.

**Theorem 4.40 – Divisibility test for 10.** An integer is divisible by 10, if, and only if, its units digit is divisible by 10; that is, if, and only if, the units digit is 0.

**Examples 4.41** 1570 is divisible by 10 because the units digit of 1570 is 0. However, 1577 is not divisible by 10 because the units digit of 1577 is not 0.

**Theorem 4.42 – Divisibility test for 4.** An integer is divisible by 4, if, and only if, the last two digits of the integer represent a number divisible by 4.

**Examples 4.43** 1536 is divisible by 4 because the last two digits represent the number 36 that is divisible by 4. However, 1538 is not divisible by 4 because the last two digits represent the number 38 that is not divisible by 4.

**Theorem 4.44 – Divisibility test for 8.** An integer is divisible by 8, if, and only if, the last three digits of the integer represent a number divisible by 8.

**Examples 4.45** 5136 is divisible by 8 because the last three digits represent the number 136 that is divisible by 8 (Indeed,  $136 = 8 \times 17$ ). However, 5138 is not divisible by 8 because the last three digits represent the number 138 that is not divisible by 8.

**Theorem 4.46 – Divisibility test for 3.** An integer is divisible by 3, if, and only if, the sum of its digits is divisible by 3.

**Examples 4.47** 5136 is divisible by 3 because the sum of its digits ( $5 + 1 + 3 + 6$ ) is 15 and is divisible by 3. However, 5138 is not divisible by 3 because the sum of its digits ( $5 + 1 + 3 + 8$ ) is 17 and is not divisible by 3.

**Theorem 4.48 – Divisibility test for 9.** An integer is divisible by 9, if, and only if, the sum of its digits is divisible by 9.

**Examples 4.49** 8136 is divisible by 3 because the sum of its digits ( $8 + 1 + 3 + 6$ ) is 18 and is divisible by 3. However, 5138 is not divisible by 9 because the sum of its digits ( $5 + 1 + 3 + 8$ ) is 17 and is not divisible by 9.

**Theorem 4.50 – Divisibility test for 11.** An integer is divisible by 11, if, and only if, the sum of the digits in the places that are even powers of 10 minus the sum of the digits in the places that are odd powers of 10 is divisible by 11.

**Examples 4.51** 121 is divisible by 11 because the sum of the digits in the places that are even powers of 10 (here  $1+1$ ) minus the sum of the digits in the places that are odd powers of 10 (here 2) is  $2 - 2 = 0$  and 0 is divisible by 11. However, 5121 is not divisible by 11 because the sum of the digits in the places that are even powers of 10 (here  $1+1$ ) minus the sum of the digits in the places that are odd powers of 10 (here  $5+2$ ) is  $2 - 5 = -3$  and  $-3$  is not divisible by 11

**Theorem 4.52 – Divisibility test for 6.** An integer is divisible by 6, if, and only if, the integer is divisible by both 2 and 3.

**Examples 4.53** 5124 is divisible by 6 because 5124 is divisible by both 2 and 3. However, 5632 is not divisible by 6 because 5632 is divisible by 2 but is not divisible by 3.

**Exercise 4.54** The number 58,218,396,643 has too many digits for most calculator displays. Determine whether it is divisible by each of the following:

- a. 2;      b. 3;      c. 4;      d. 5;      e. 6;      f. 8;      g. 9;      h. 10;      i. 11.

**Solution.**

- a. No, the units digit, 3, is not divisible by 2;  
 b. No, the sum of the digits is 55, which is not divisible by 3;  
 c. No, the last two digits, 43, is not divisible by 4;  
 d. No, the units digit, 3, is neither 0 nor 5;  
 e. No, the number is divisible neither by 3 nor by 2;  
 f. No, the last three digits, 643, is not divisible by 8;  
 g. No, the sum of the digits is 55, which is not divisible by 9;  
 h. No, the units digit, 3, is not 0;  
 i. Yes,  $(5+2+8+9+6+3) - (8+1+3+6+4) = 33 - 22 = 11$  and 11 is divisible by 11.

The following two divisibility tests (divisibility test for 7 and for 12) are not in the syllabus of this course. It is included in this lecture's notes for the curious students. Therefore, you should be able to decide whether a given integers is divisible by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

**Theorem 4.55 – Divisibility test for 7.** An integer is divisible by 7, if the number obtained by subtracting the double of the units digit from the rest of the number, is divisible by 7.

**Examples 4.56** 672 is divisible by 7 because  $67 - 2 \cdot 2 = 63$ , and 63 is divisible by 7. Note that we can apply the same test for 63. Indeed,  $6 - 2 \cdot 3 = 0$  and 0 is divisible by 7. However, 904 is not divisible by 7 because  $90 - 2 \cdot 4 = 72$ , and 72 is not divisible by 7. Note that we can apply the same test for 72. Indeed,  $7 - 2 \cdot 2 = 3$  and 3 is not divisible by 7, and hence 72 is not divisible by 7.

**Theorem 4.57 – Divisibility test for 12.** An integer is divisible by 12, if, and only if, the integer is divisible by both 3 and 4.

**Examples 4.58** 648 is divisible by 12 because 648 is divisible by both 3 and 4. However, 916 is not divisible by 12 because 916 is divisible by 4 but is not divisible by 3.

## 4.4 Prime and composite numbers

Any integer  $a$ , where  $a \neq 0$  and  $a \neq 1$  has at least 2 positive divisors: 1 and itself.

**Definition 4.59 – Prime and composite numbers.** A integer is greater than 1 is prime number if, and only if, it has only two positive divisors: 1 and itself. Otherwise, the number is a composite number.

**Examples 4.60** 2, 3, 5, 7, 11, 13, 17, 19 are prime numbers. 4, 6, 8, 9, 10, 12, 14, 15, 16, 18 are composite numbers.

**Exercise 4.61** Show that the following numbers are composite:

- a. 1234;                      b. 3627;                      c. 1221;                      d.  $1287 \cdot 98745 + 1$ .

**Solution.**

- a. 1234 is divisible by 2, and hence 1234 is composite;  
 b. 3627 is divisible by 3, and hence 3627 is composite;  
 c. 121 is divisible by 11, and hence 121 is composite;  
 d.  $1287 \cdot 98745 + 1$  is an even number, because the product of an odd number by an odd number ( $1287 \cdot 98745$ ) is odd, and when you add 1 to an odd number it becomes even. Consequently,  $1287 \cdot 98745 + 1$  is composite.

### 4.4.1 Prime factorization

**Theorem 4.62 – Fundamental theorem of arithmetic.** Each composite number can be written as a product of primes in one, and only one, way except for the order of the prime factors in the product.

**Examples 4.63**

- $12 = 4 \cdot 3 = 2^2 \cdot 3$ .
- $48 = 6 \cdot 8 = (2 \cdot 3) \cdot 2^3 = 2 \cdot 3 \cdot 2^3 = 2^4 \cdot 3$ .
- $63 = 9 \cdot 7 = 3^2 \cdot 7$ .
- $140 = 10 \cdot 14 = (2 \cdot 5) \cdot (2 \cdot 7) = 2 \cdot 5 \cdot 2 \cdot 7 = 2^2 \cdot 5 \cdot 7$ .

### 4.4.2 Number of divisors

**Theorem 4.64** If  $p$  and  $q$  are different primes, then  $p^n q^m$  has  $(n + 1)(m + 1)$  positive divisors. In general, if  $p_1, p_2, \dots, p_k$  are primes and  $n_1, n_2, \dots, n_k$  are whole numbers, then  $p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_k^{n_k}$  has  $(n_1 + 1)(n_2 + 1) \cdot \dots \cdot (n_k + 1)$ .

**Examples 4.65**

- $12 = 2^2 \cdot 3$ . Consequently, 12 has 6 positive divisors ( $6 = (2 + 1) \cdot (1 + 1)$ ). Indeed, the positive divisors of 12 are: 1, 2, 3, 4, 6, 12.
- $48 = 2^4 \cdot 3$ . Consequently, 48 has 10 positive divisors ( $10 = (4 + 1) \cdot (1 + 1)$ ). Indeed, the positive divisors of 48 are: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48.
- $63 = 3^2 \cdot 7$ . Consequently, 63 has 6 positive divisors ( $6 = (2 + 1) \cdot (1 + 1)$ ).
- $140 = 2^2 \cdot 5 \cdot 7$ . Consequently, 140 has 12 positive divisors ( $12 = (2 + 1) \cdot (1 + 1) \cdot (1 + 1)$ ).

**Exercise 4.66** Find the number of positive divisors of the following:

- a.  $6^5$ .                                      b. 1,000,000.                                      c.  $30^{15}$ .

**Solution.**

- a.  $6^5 = (2 \cdot 3)^5 = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = 2^5 \cdot 3^5$ . Therefore,  $6^5$  has  $(5 + 1)(5 + 1) = 36$  positive divisors.  
 b.  $1,000,000 = 10^6 = (2 \cdot 5)^6 = (2 \cdot 5)(2 \cdot 5)(2 \cdot 5)(2 \cdot 5)(2 \cdot 5)(2 \cdot 5) = 2^6 \cdot 5^6$ . Hence, 1,000,000 has  $(6 + 1)(6 + 1) = 49$  positive divisors.  
 c.  $30^9 = (6 \cdot 5)^9 = (2 \cdot 3 \cdot 5)^9 = 2^9 \cdot 3^9 \cdot 5^9$ . Therefore,  $30^9$  has  $(9 + 1)(9 + 1)(9 + 1) = 1,000$  positive divisors.

### 4.4.3 Determining if a number is prime

**Theorem 4.67** If  $d$  is a divisor of  $n$ , then  $\frac{n}{d}$  is also a divisor of  $n$ .

**Example 4.68** 12 is divisible by 2. By the above theorem, we conclude that 12 is also divisible by 6 (because  $\frac{12}{2} = 6$ ). Also, 12 is divisible by 3. By the above theorem, we conclude that 12 is also divisible by 4 (because  $\frac{12}{3} = 4$ ).

**Theorem 4.69** If  $n$  is composite, then  $n$  has a prime factor  $p$  such that  $p^2 \leq n$ .

**Example 4.70** 12 is a composite number. By the above theorem 12 must have a prime factor  $p$  such that  $p^2 \leq n$ . For the number 12, there are two possible  $p$ : 2 and 3. Indeed,  $2^2 = 4$  is less than or equal to 12, and  $3^2 = 9$  is less than or equal to 12.

**Theorem 4.71** If  $n$  is an integer greater than 1 and not divisible by any prime  $p$ , such that  $p^2 \leq n$ , then  $n$  is prime.

#### Examples 4.72

- We can use the above theorem to prove that 23 is a prime number. The possible primes  $p$  such that  $p^2 \leq 23$  are 2 and 3. Since 23 is not divisible by neither 2 nor 3, we conclude that 23 is a prime.
- We can use the above theorem to prove that 61 is a prime number. The possible primes  $p$  such that  $p^2 \leq 61$  are 2, 3, 5 and 7. Since 61 is not divisible by 2, 3, 5 and by 7, we conclude that 61 is a prime.

## 4.5 Greatest common divisor and least common multiple

### 4.5.1 Greatest common divisor

**Definition 4.73 — The greatest common divisor.** The greatest common divisor of two natural numbers  $a$  and  $b$ , written  $\text{GCD}(a, b)$ , is the greatest natural number that divides both  $a$  and  $b$ .

**Example 4.74** The divisors of 12 are: 1, 2, 3, 4, 6, 12. The divisors of 18 are: 1, 2, 3, 6, 9, 18. The common divisors of 12 and 18 are 1, 2, 3 and 6. Hence, the greater common divisor of 12 and 18 is  $\text{GCD}(12, 18) = 6$ .

We can also find the greatest common divisor of 12 and 18 using the prime factorization.  $12 = 2^2 \cdot 3$  and  $18 = 2 \cdot 3^2$ . We multiply then the common prime factors with the lowest exponent:  $\text{GCD}(12, 18) = 2 \cdot 3 = 6$ .

**Exercise 4.75** Using the prime factorization method, find the greatest common divisor of 54 and 126.

**Solution.**  $54 = 2 \cdot 27 = 2 \cdot 3^3$ , and  $126 = 2 \cdot 63 = 2 \cdot 3^2 \cdot 7$ . Therefore,  $(54, 126) = 2 \cdot 3^2 = 2 \cdot 9 = 18$ .

**Exercise 4.76** Find each of the following:

- $\text{GCD}(630, 350)$ ;
- $\text{GCD}(x, y)$  if  $x = 2^5 \cdot 3^4 \cdot 7^2 \cdot 13$  and  $y = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17$ ;
- $\text{GCD}(x, y)$  if  $x = 2^5 \cdot 7^2$  and  $y = 5^2 \cdot 11^4$ .

**Solution.**

a.  $630 = 70 \cdot 9 = 7 \cdot 10 \cdot 9 = 2 \cdot 3^2 \cdot 5 \cdot 7$  and  $350 = 70 \cdot 5 = 7 \cdot 10 \cdot 5 = 7 \cdot 2 \cdot 5 \cdot 5 = 2 \cdot 5^2 \cdot 7$ . Therefore,  $\text{GCD}(630, 350) = 2 \cdot 5 \cdot 7 = 70$ ;

b. If  $x = 2^5 \cdot 3^4 \cdot 7^2 \cdot 13$  and  $y = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17$ , then  $\text{GCD}(x, y) = 2 \cdot 3^2 = 2 \cdot 9 = 18$ ;

c. If  $x = 2^5 \cdot 7^2$  and  $y = 5^2 \cdot 11^4$ , then  $\text{GCD}(x, y) = 1$  since  $x$  and  $y$  have no common prime factors.

### 4.5.2 Least common multiple

**Definition 4.77** Suppose that  $a$  and  $b$  are natural numbers. Then the least common multiple of  $a$  and  $b$ , written  $\text{LCM}(a, b)$ , is the least natural number that is simultaneously a multiple of  $a$  and a multiple of  $b$ .

**Example 4.78** The set of multiples of 12 is  $M_{12} = \{12, 24, 36, 48, 60, 72, \dots\}$  and the set of multiples of 18 is  $M_{18} = \{18, 36, 54, 72, \dots\}$ . Therefore, the set of common multiples of 12 and 18 is  $M_{12} \cap M_{18} = \{36, 72, 108, \dots\}$ . Consequently, the least common multiple of 12 and 18 is  $\text{LCM}(12, 18) = 36$ .

We can also find the least common multiple of 12 and 18 using the prime factorization.  $12 = 2^2 \cdot 3$  and  $18 = 2 \cdot 3^2$ . We multiply then the common prime factors with the highest exponent and the non-common prime factors:  $\text{LCM}(12, 18) = 2^2 \cdot 3^2 = 4 \cdot 9 = 36$ .

**Exercise 4.79** Using the prime factorization method, find the greatest least common multiple of 54 and 126.

**Solution.**  $54 = 2 \cdot 27 = 2 \cdot 3^3$ , and  $126 = 2 \cdot 63 = 2 \cdot 3^2 \cdot 7$ . Therefore,  $\text{LCM}(54, 126) = 2 \cdot 3^3 \cdot 7 = 2 \cdot 27 \cdot 7 = 378$ .

**Theorem 4.80** For any two natural numbers  $a$  and  $b$ ,  $\text{GCD}(a, b) \cdot \text{LCM}(a, b) = ab$ .

The above theorem can be used to compute the least common multiple of two given numbers knowing their greatest common divisor, and conversely.

#### Examples 4.81

- Since  $\text{GCD}(12, 18) = 6$ , then  $\text{LCM}(12, 18) = (12 \cdot 18) \div \text{GCD}(12, 18) = 216 \div 6 = 36$ .
- Since  $\text{LCM}(54, 126) = 378$ , then  $\text{GCD}(54, 126) = (54 \cdot 126) \div \text{LCM}(54, 126) = 6804 \div 378 = 18$ .



# Chapter 5

## Rational numbers as fractions

We use natural numbers to count how many objects we have: I have 3 apples, you have 5 dollars, etc. Integers can be used to express the notion of gaining (positive numbers) and losing objects (negative numbers): in the checking account, we can see +\$12 and -\$50. Fractions are used to express not only how many objects we have gained or lost, but also gaining or losing a portion of an object: I ate a half of a pizza, I lost three fourth of a dollars, etc.

### 5.1 The set of rational numbers

The set of rational numbers is the set  $\mathbb{Q} = \{ \frac{a}{b} \mid a \text{ and } b \text{ are integers, and } b \neq 0 \}$ .

**Examples 5.1**  $\frac{1}{2}, \frac{3}{4}, -\frac{7}{4}, -\frac{1}{5}, \dots$  are rational numbers.

**Remark 5.2 – Proper and improper fractions.** A fraction  $\frac{a}{b}$ , where  $0 \leq a < b$ , is a **proper fraction**. Otherwise, the fraction is an **improper fraction**.

**Examples 5.3** The fractions  $\frac{2}{3}, \frac{3}{5}, \frac{17}{19}$  and  $\frac{21}{189}$  are proper. The fractions  $\frac{4}{3}, \frac{7}{5}, \frac{21}{19}$  and  $\frac{321}{189}$  are improper.

#### 5.1.1 Equivalent or equal fractions

**Theorem 5.4 – Fundamental law of fractions.** Let  $\frac{a}{b}$  be any fraction and  $n$  a nonzero integer. Then  $\frac{a}{b} = \frac{an}{bn}$ .

**Examples 5.5**

$$\bullet \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \dots \quad \bullet \frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \dots \quad \bullet \frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \frac{12}{20} = \dots$$

**Exercise 5.6** Find the a value for  $x$  such that  $\frac{x}{36} = \frac{2}{3}$ .

**Solution.**  $36 = 3 \cdot 12$ . Using the fundamental law of fractions,  $\frac{2}{3} = \frac{2 \cdot 12}{3 \cdot 12} = \frac{24}{36}$ . Hence,  $\frac{24}{36} = \frac{x}{36}$ , and  $x = 24$ .

#### 5.1.2 Simplifying fractions

**Definition 5.7 – Simplest form.** A rational number  $\frac{a}{b}$  is in simplest form if  $b > 0$  and  $\text{GCD}(a, b) = 1$ ; that is, if  $a$  and  $b$  have no common factor greater than 1, and  $b > 0$ .

**Examples 5.8**

- $\frac{3}{4}$  is in simplest form because  $\text{GCD}(3, 4) = 1$ .
- $\frac{12}{18}$  is not in simplest form because  $\text{GCD}(12, 18) \neq 1$ . Its simplest form is  $\frac{2}{3}$ . This simplest form is obtained by dividing the numerator 12 and the denominator 18 by the greatest common divisor of 12 and 18. Indeed,  $\text{GCD}(12, 18) = 6$ , and hence  $12 \div 6 = 2$  and  $18 \div 6 = 3$  and  $\frac{12}{18} = \frac{2}{3}$ .

### 5.1.3 Equality of fractions

**Theorem 5.9** Two fractions  $\frac{a}{b}$  and  $\frac{c}{d}$  are equal if, and only if,  $ad = bc$ .

#### Examples 5.10

- $\frac{3}{5} = \frac{21}{35}$  because  $3 \cdot 35 = 105$  is equal to  $5 \cdot 21 = 105$ .
- $\frac{4}{7} = \frac{16}{28}$  because  $4 \cdot 28 = 112$  is equal to  $7 \cdot 16 = 112$ .
- $\frac{3}{5} \neq \frac{7}{13}$  because  $3 \cdot 13 = 39$  is not equal to  $5 \cdot 7 = 35$ .

### 5.1.4 Ordering rational numbers

**Theorem 5.11** If  $a, b$ , and  $c$  are integers and  $b > 0$ , then  $\frac{a}{b} > \frac{c}{b}$  if, and only if,  $a > c$ .

#### Examples 5.12

- $\frac{1}{7} < \frac{3}{7}$ .
- $\frac{4}{3} > \frac{2}{3}$ .
- $\frac{3}{10} < \frac{6}{10}$ .

**Theorem 5.13** If  $a, b, c$ , and  $d$  are integers with  $b > 0$  and  $d > 0$ , then  $\frac{a}{b} > \frac{c}{d}$  if, and only if,  $ad > bc$ .

#### Examples 5.14

- $\frac{4}{7} < \frac{3}{5}$  because  $4 \cdot 5 = 20$  is less than  $7 \cdot 3 = 21$ .
- $\frac{2}{3} < \frac{3}{4}$  because  $2 \cdot 4 = 8$  is less than  $3 \cdot 3 = 9$ .
- $\frac{3}{5} > \frac{1}{2}$  because  $3 \cdot 2 = 6$  is greater than  $5 \cdot 1 = 5$ .

### 5.1.5 Denseness of rational numbers

**Theorem 5.15 – Denseness property for rational numbers.** Given two different rational numbers  $\frac{a}{b}$  and  $\frac{c}{d}$  there is another rational number between these two numbers.

**Exercise 5.16** Find two fractions between  $\frac{5}{24}$  and  $\frac{1}{2}$ .

**Solution.**  $\frac{1}{2} = \frac{1 \cdot 12}{2 \cdot 12} = \frac{12}{24}$ . Then  $\frac{6}{24}$  and  $\frac{7}{24}$  are two fractions between  $\frac{5}{24}$  and  $\frac{1}{2}$ . Other fractions between  $\frac{5}{24}$  and  $\frac{1}{2}$  are:  $\frac{8}{24}, \frac{9}{24}, \frac{10}{24}$  and  $\frac{11}{24}$ .

**Theorem 5.17** Let  $\frac{a}{b}$  and  $\frac{c}{d}$  be any rational numbers with positive denominators, where  $\frac{a}{b} < \frac{c}{d}$ . Then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \quad (5.1)$$

### Examples 5.18

$$\bullet \frac{1}{3} < \frac{2}{5} < \frac{1}{2}.$$

$$\bullet \frac{3}{7} < \frac{6}{11} < \frac{3}{4}.$$

$$\bullet \frac{2}{3} < \frac{9}{7} < \frac{7}{4}.$$

## 5.2 Addition, subtraction, and estimation with rational numbers

### 5.2.1 Addition of rational numbers

**Definition 5.19 – Addition of rational numbers with like denominators.** If  $\frac{a}{b}$  and  $\frac{c}{b}$  are rational numbers, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}. \quad (5.2)$$

### Examples 5.20

$$\bullet \frac{2}{8} + \frac{3}{8} = \frac{2+3}{8} = \frac{5}{8}.$$

$$\bullet \frac{3}{11} + \frac{7}{11} = \frac{3+7}{11} = \frac{10}{11}.$$

$$\bullet \frac{2}{3} + \frac{4}{3} = \frac{2+4}{3} = \frac{6}{3} = 2.$$

**Theorem 5.21** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are two rational numbers, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}. \quad (5.3)$$

**Exercise 5.22** Find each of the following sums:

a.  $\frac{3}{7} + \frac{5}{12}.$

c.  $\frac{7}{13} + \frac{-1}{4}.$

e.  $\frac{5}{x} + \frac{7}{y}.$

b.  $\frac{3}{-4} + \frac{1}{5}.$

d.  $\left(\frac{2}{3} + \frac{1}{5}\right) + \frac{3}{8}.$

f.  $\frac{3}{x^2y} + \frac{5}{xy^2}.$

**Solution.**

a.  $\frac{3}{7} + \frac{5}{12} = \frac{3 \cdot 12 + 7 \cdot 5}{7 \cdot 12} = \frac{36 + 35}{84} = \frac{71}{84}.$

b.  $\frac{3}{-4} + \frac{1}{5} = \frac{3 \cdot 5 + (-4) \cdot 1}{(-4) \cdot 5} = \frac{15 + (-4)}{-20} = \frac{11}{-20} = -\frac{11}{20}.$

c.  $\frac{7}{13} + \frac{-1}{4} = \frac{7 \cdot 4 + 13 \cdot (-1)}{13 \cdot 4} = \frac{28 + (-13)}{52} = \frac{15}{52}.$

d.  $\left(\frac{2}{3} + \frac{1}{5}\right) + \frac{3}{8} = \frac{2 \cdot 5 + 3 \cdot 1}{3 \cdot 5} + \frac{3}{8} = \frac{10 + 3}{15} + \frac{3}{8} = \frac{13}{15} + \frac{3}{8} = \frac{13 \cdot 8 + 15 \cdot 3}{15 \cdot 8} = \frac{104 + 45}{120} = \frac{149}{120}.$

e.  $\frac{5}{x} + \frac{7}{y} = \frac{5y + 7x}{xy}.$

$$f. \frac{3}{x^2y} + \frac{5}{xy^2} = \frac{3xy^2 + 5x^2y}{x^2y \cdot xy^2} = \frac{3xy^2 + 5x^2y}{x^3y^3}.$$

Note that  $3xy^2 + 5x^2y$  factors as  $xy(3y + 5x)$ . Consequently  $\frac{3xy^2 + 5x^2y}{x^3y^3} = \frac{xy(3y + 5x)}{x^3y^3} = \frac{3y + 5x}{x^2y^2}$ . This result could have been obtained directly if we performed differently the addition. Indeed,  $\frac{3}{x^2y} = \frac{3y}{x^2y^2}$  and  $\frac{5}{xy^2} = \frac{5x}{x^2y^2}$ . Consequently,  $\frac{3}{x^2y} + \frac{5}{xy^2} = \frac{3y}{x^2y^2} + \frac{5x}{x^2y^2} = \frac{3y + 5x}{x^2y^2}$ .

### 5.2.2 Mixed numbers

In everyday life, we often use **mixed numbers**; that is numbers that are made up of an integer and a proper fraction.

#### Examples 5.23

- The mixed number  $1\frac{1}{4}$  means  $1 + \frac{1}{4}$ . Since  $1 = \frac{4}{4}$ , the mixed number  $1\frac{1}{4} = \frac{4}{4} + \frac{1}{4} = \frac{5}{4}$ .

Consequently,  $1\frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}$ .

- The mixed number  $2\frac{3}{4}$  means  $2 + \frac{3}{4}$ . Since  $2 = \frac{8}{4}$ , the mixed number  $2\frac{3}{4} = \frac{8}{4} + \frac{3}{4} = \frac{11}{4}$ .

Consequently,  $2\frac{3}{4} = 2 + \frac{3}{4} = \frac{11}{4}$ .

**Remark 5.24** It is sometimes inferred that  $2\frac{3}{4}$  means 2 times  $\frac{3}{4}$ , since  $xy$  means  $x \cdot y$ , but this is not correct.<sup>1</sup>

**Remark 5.25**  $-2\frac{3}{4}$  means  $-(2 + \frac{3}{4})$  or equivalently  $-2 - \frac{3}{4}$ , but NOT  $-2 + \frac{3}{4}$ .

**Exercise 5.26** Change each of the following mixed numbers to the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers:

a.  $5\frac{1}{3}$ .

b.  $3\frac{1}{4}$ .

c.  $-3\frac{2}{5}$ .

**Solution.**

a.  $5\frac{1}{3} = 5 + \frac{1}{3} = \frac{15}{3} + \frac{1}{3} = \frac{16}{3}$ .

b.  $3\frac{1}{4} = 3 + \frac{1}{4} = \frac{12}{4} + \frac{1}{4} = \frac{13}{4}$ .

c.  $-3\frac{2}{5} = -(3 + \frac{2}{5}) = -(\frac{15}{5} + \frac{2}{5}) = -(\frac{17}{5}) = \frac{-17}{5}$ .

**Exercise 5.27** Change  $\frac{37}{5}$  to a mixed number.

**Solution.** 37 divided by 5 is 7 and the remainder is 2. Hence,  $37 = 5 \cdot 7 + 2$ . Consequently,  $\frac{37}{5} = \frac{5 \cdot 7 + 2}{5} = \frac{5 \cdot 7}{5} + \frac{2}{5} = 7 + \frac{2}{5} = 7\frac{2}{5}$ .

<sup>1</sup>In France, and probably in all the countries using the metric system (meter, gramme, litre, ... instead of feet, inch, gallon, ...), mixed numbers are not used, and  $2\frac{3}{4}$  means 2 times  $\frac{3}{4}$ .

### 5.2.3 Properties of addition of rational numbers

**Theorem 5.28 – Additive inverse property of rational numbers.** For any rational number  $\frac{a}{b}$ , there exists a unique rational number  $-\frac{a}{b}$ , the additive inverse of  $\frac{a}{b}$ , such that

$$\frac{a}{b} + \left(-\frac{a}{b}\right) = 0 = \left(-\frac{a}{b}\right) + \frac{a}{b}. \quad (5.4)$$

**Exercise 5.29** Find the additive inverse of each of the following:

a.  $\frac{3}{7}$ .

b.  $\frac{-2}{5}$ .

c.  $2\frac{1}{3}$ .

**Solution.**

a.  $-\frac{3}{7}$  or  $\frac{-3}{7}$ .

b.  $-\left(\frac{-2}{5}\right) = \frac{-(-2)}{5} = \frac{2}{5}$ .

c.  $-2\frac{1}{3}$  or  $\frac{-7}{3}$ .

**Theorem 5.30 – Addition property of equality.** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers such that  $\frac{a}{b} = \frac{c}{d}$ , and if  $\frac{e}{f}$  is any rational number, then

$$\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}. \quad (5.5)$$

### 5.2.4 Subtraction of rational numbers

**Definition 5.31 – Subtraction of rational numbers in terms of addition.** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers, then  $\frac{a}{b} - \frac{c}{d}$  is the unique rational number  $\frac{e}{f}$  such that  $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$ .

**Definition 5.32 – Subtraction of rational numbers with like denominators.** If  $\frac{a}{b}$  and  $\frac{c}{b}$  are rational numbers, then

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}. \quad (5.6)$$

#### Examples 5.33

$$\bullet \frac{2}{8} - \frac{3}{8} = \frac{2-3}{8} = \frac{-1}{8}.$$

$$\bullet \frac{3}{11} - \frac{7}{11} = \frac{3-7}{11} = \frac{-4}{11}.$$

$$\bullet \frac{4}{3} - \frac{2}{3} = \frac{4-2}{3} = \frac{2}{3}.$$

**Theorem 5.34** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers, then

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}. \quad (5.7)$$

**Exercise 5.35** Find each of the following differences:

a.  $\frac{3}{7} - \frac{5}{12}$ .

c.  $\frac{7}{13} - \frac{-1}{4}$ .

e.  $\frac{5}{x} - \frac{7}{y}$ .

b.  $\frac{3}{-4} - \frac{1}{5}$ .

d.  $\left(\frac{2}{3} - \frac{1}{5}\right) - \frac{3}{8}$ .

f.  $\frac{3}{x^2y} - \frac{5}{xy^2}$ .

**Solution.**

$$\text{a. } \frac{3}{7} - \frac{5}{12} = \frac{3 \cdot 12 - 7 \cdot 5}{7 \cdot 12} = \frac{36 - 35}{84} = \frac{1}{84}.$$

$$\text{b. } \frac{3}{-4} - \frac{1}{5} = \frac{3 \cdot 5 - (-4) \cdot 1}{(-4) \cdot 5} = \frac{15 - (-4)}{-20} = \frac{19}{-20} = -\frac{19}{20}.$$

$$\text{c. } \frac{7}{13} - \frac{-1}{4} = \frac{7 \cdot 4 - 13 \cdot (-1)}{13 \cdot 4} = \frac{28 - (-13)}{52} = \frac{41}{52}.$$

$$\text{d. } \left(\frac{2}{3} - \frac{1}{5}\right) - \frac{3}{8} = \frac{2 \cdot 5 - 3 \cdot 1}{3 \cdot 5} - \frac{3}{8} = \frac{10 - 3}{15} - \frac{3}{8} = \frac{7}{15} - \frac{3}{8} = \frac{7 \cdot 8 - 15 \cdot 3}{15 \cdot 8} = \frac{56 - 45}{120} = \frac{11}{120}.$$

$$\text{e. } \frac{5}{x} - \frac{7}{y} = \frac{5y - 7x}{xy}.$$

$$\text{f. } \frac{3}{x^2y} - \frac{5}{xy^2} = \frac{3xy^2 - 5x^2y}{x^2y \cdot xy^2} = \frac{3xy^2 - 5x^2y}{x^3y^3} = \frac{3y - 5x}{x^2y^2}.$$

### 5.2.5 Definition of "greater than" and "less than" in terms of subtraction

**Definition 5.36 — Less than and greater than for rational numbers.** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers, then  $\frac{a}{b} < \frac{c}{d}$  if  $\frac{c}{d} - \frac{a}{b} > 0$ , and  $\frac{c}{d} > \frac{a}{b}$  if, and only if,  $\frac{a}{b} < \frac{c}{d}$ .

#### Exercise 5.37

1. Verify each of the following:

$$\text{a. } \frac{2}{3} + \frac{3}{2} > 2$$

$$\text{b. } \frac{3}{5} + \frac{5}{3} > 2$$

$$\text{c. } \frac{7}{2} + \frac{2}{7} > 2$$

2. Based on these examples, conjecture a general statement and justify it.

#### Solution.

$$1. \text{ a. } \frac{2}{3} + \frac{3}{2} - 2 = \frac{4}{6} + \frac{9}{6} - \frac{12}{6} = \frac{4 + 9 - 12}{6} = \frac{1}{6}, \text{ and } \frac{1}{6} > 0. \text{ Consequently, } \frac{2}{3} + \frac{3}{2} > 2.$$

$$\text{b. } \frac{3}{5} + \frac{5}{3} - 2 = \frac{9}{15} + \frac{25}{15} - \frac{30}{15} = \frac{9 + 25 - 30}{15} = \frac{4}{15}, \text{ and } \frac{4}{15} > 0. \text{ Consequently, } \frac{3}{5} + \frac{5}{3} > 2.$$

$$\text{c. } \frac{7}{2} + \frac{2}{7} - 2 = \frac{49}{14} + \frac{4}{14} - \frac{28}{14} = \frac{49 + 4 - 28}{14} = \frac{25}{14}, \text{ and } \frac{25}{14} > 0. \text{ Consequently, } \frac{7}{2} + \frac{2}{7} > 2.$$

2. Based on these examples, we can conjecture that for all positive integers  $a$  and  $b$ , such that  $a \neq b$ ,  $\frac{a}{b} + \frac{b}{a} > 2$ . This conjecture can be justified as follows:

$\frac{a}{b} + \frac{b}{a} - 2 = \frac{a^2}{ab} + \frac{b^2}{ab} - \frac{2ab}{ab} = \frac{a^2 + b^2 - 2ab}{ab}$ . We can use a result of the previous chapter, Exercise 4.20,  $(a - b)^2 = a^2 + b^2 - 2ab$ . Therefore,  $\frac{a}{b} + \frac{b}{a} - 2 = \frac{(a - b)^2}{ab}$ . The fraction  $\frac{(a - b)^2}{ab}$  is positive because  $(a - b)^2$  is positive and also  $ab$  is positive (both  $a$  and  $b$  are positive integers, and hence their product is a positive integer). Consequently,  $\frac{a}{b} + \frac{b}{a} - 2 > 0$  and hence  $\frac{a}{b} + \frac{b}{a} > 2$ .

### 5.2.6 Estimation with rational numbers

Sometimes to obtain an estimate it is desirable to round fractions to a *convenient* fraction, such as  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{3}, \frac{3}{4}, \dots$  or 1. If a student has 39 out of 60 questions, the student answered  $\frac{39}{60}$  of the questions correctly, which is approximately  $\frac{40}{60}$ , or  $\frac{2}{3}$ . The fraction  $\frac{2}{3}$  is greater than  $\frac{39}{60}$ . On a number line, the greater fraction is to the

right of the lesser. The estimate  $\frac{2}{3}$  for  $\frac{39}{60}$  is a high estimate. In a similar way, we can estimate  $\frac{41}{60}$  by  $\frac{40}{60}$ , or  $\frac{2}{3}$ . In this case, the estimate of  $\frac{2}{3}$  is a low estimate.

**Exercise 5.38** Estimate  $2\frac{2}{7} + 4\frac{4}{10} + 5\frac{8}{9} + \frac{6}{11}$ .

**Solution.** We can estimate this number by using the front-end estimation and then adjusting by using  $0, \frac{1}{2}$ , and 1. The front-end estimate is  $(2 + 4 + 5)$ , or 11. The adjustment is  $0 + \frac{1}{2} + 1 + \frac{1}{2}$ , or 2. An adjusted estimate would be  $11 + 2$  or 13.

**Exercise 5.39** Estimate each of the following:

a.  $\frac{9}{5} + \frac{2}{3}$ .

b.  $4\frac{1}{2} + 3\frac{1}{6}$ .

c.  $2\frac{9}{10} + 3\frac{5}{6} + \frac{9}{21}$ .

**Solution.**

a.  $\frac{9}{5}$  is slightly less than 2 and  $\frac{2}{3}$  is more than  $\frac{1}{2}$  and less than 1. Consequently, a high estimate of  $\frac{9}{5} + \frac{2}{3}$  can be 3 and a low estimate of  $\frac{9}{5} + \frac{2}{3}$  can be  $2\frac{1}{2}$ .

b. An estimate of  $4\frac{1}{2} + 3\frac{1}{6}$  is  $7\frac{1}{2}$ .

c.  $2\frac{9}{10} + 3\frac{5}{6} + \frac{9}{21}$ . The front-end estimate is  $(2 + 3)$ , or 5. The adjustment is  $1 + 1 + \frac{1}{2}$ , or  $2\frac{1}{2}$ . An adjusted estimate would be  $5 + 2\frac{1}{2}$  or  $7\frac{1}{2}$ .

## 5.3 Multiplication and division of rational numbers

### 5.3.1 Multiplication of rational numbers

To motivate the definition of multiplication of rational numbers, we use the interpretation of multiplication as repeated addition. We can interpret  $5 \cdot \left(\frac{2}{3}\right)$  as  $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{2+2+2+2+2}{3} = \frac{10}{3} = 3\frac{1}{3}$ . We next consider  $\left(\frac{2}{3}\right) \cdot 5$  as two thirds of 5. We divide 5 by 3, which gives  $\frac{5}{3}$  and we double this number. This discussion leads to the following definition:

**Definition 5.40 – Multiplication of rational numbers.** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}. \quad (5.8)$$

**Exercise 5.41** Perform the following:

a.  $\frac{12}{5} \cdot \frac{2}{3}$ .

b.  $\frac{1}{2} \cdot \frac{5}{6}$ .

c.  $\frac{9}{10} \cdot \frac{5}{6}$ .

**Solution.**

a.  $\frac{12}{5} \cdot \frac{2}{3} = \frac{12 \cdot 2}{5 \cdot 3} = \frac{24}{15} = \frac{8}{5}$ .

b.  $\frac{1}{2} \cdot \frac{5}{6} = \frac{1 \cdot 5}{2 \cdot 6} = \frac{5}{12}$ .

c.  $\frac{9}{10} \cdot \frac{5}{6} = \frac{9 \cdot 5}{10 \cdot 6} = \frac{45}{60} = \frac{3}{4}$ .

### 5.3.2 Properties of multiplication of rational numbers

#### Theorem 5.42 – Multiplicative identity and multiplicative inverse of rational inverse.

1. The number 1 is the unique number such that for every rational number  $\frac{a}{b}$ ,

$$1 \cdot \left(\frac{a}{b}\right) = \frac{a}{b} = \left(\frac{a}{b}\right) \cdot 1. \quad (5.9)$$

2. For any nonzero rational number  $\frac{a}{b}$ ,  $\frac{b}{a}$  is the unique rational number such that  $\frac{a}{b} \cdot \frac{b}{a} = 1 = \frac{b}{a} \cdot \frac{a}{b}$ .

**Exercise 5.43** Find the multiplicative inverse, if possible, for each of the following rational numbers:

a.  $\frac{2}{3}$ ;

c. 4;

e. 1;

g.  $3\frac{1}{2}$ ;

b.  $\frac{-3}{4}$ ;

d. -2;

f. 0;

h.  $-2\frac{1}{3}$ .

**Solution.**

a.  $\frac{3}{2}$ ;

b.  $\frac{4}{-3}$  or  $\frac{-4}{3}$ ;

c.  $\frac{1}{4}$ ;

d.  $\frac{-1}{2}$ ;

e. 1;

f. There is no multiplicative inverse of 0, because even though  $0 = \frac{0}{1}$ ,  $\frac{1}{0}$  is undefined;

g.  $\frac{2}{7}$  because  $3\frac{1}{2} = 3 + \frac{1}{2} = \frac{6}{2} + \frac{1}{2} = \frac{7}{2}$ ;

h.  $\frac{3}{-7}$  or  $\frac{-3}{7}$  because  $-2\frac{1}{3} = -2 - \frac{1}{3} = \frac{-6}{3} - \frac{1}{3} = \frac{-6-1}{3} = \frac{-7}{3}$ .

#### Theorem 5.44

1. **Distributive property of multiplication over addition for rational numbers** If  $\frac{a}{b}$ ,  $\frac{c}{d}$ , and  $\frac{e}{f}$  are rational numbers, then

$$\frac{a}{b} \left( \frac{c}{d} + \frac{e}{f} \right) = \left( \frac{a}{b} \cdot \frac{c}{d} \right) + \left( \frac{a}{b} \cdot \frac{e}{f} \right) \quad (5.10)$$

2. **Multiplication property of equality for rational numbers** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers such that  $\frac{a}{b} = \frac{c}{d}$ , and  $\frac{e}{f}$  is any rational number, then

$$\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}. \quad (5.11)$$

3. **Multiplication property of inequality for rational numbers**

(i) If  $\frac{a}{b} > \frac{c}{d}$  and  $\frac{e}{f} > 0$ , then  $\frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f}$ .

(ii) If  $\frac{a}{b} > \frac{c}{d}$  and  $\frac{e}{f} < 0$ , then  $\frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f}$ .

#### 4. Multiplication property of zero for rational numbers

If  $\frac{a}{b}$  is any rational number, then  $\frac{a}{b} \cdot 0 = 0 = 0 \cdot \frac{a}{b}$ .

**Exercise 5.45** A stroller is on sale at  $\frac{3}{5}$  of its original price. If the sale price is \$123, what was the original price?

**Solution.** Let  $x$  be the original price. Then  $\frac{3}{5}$  of the original price is  $\frac{3}{5}x$ . Since the sale price is \$123, we have  $\frac{3}{5}x = 123$ . Solving for  $x$  gives

$$\begin{array}{l} \frac{3}{5}x = 123 \\ \frac{5}{3} \cdot \frac{3}{5}x = \frac{5}{3} \cdot 123 \\ x = \frac{5 \cdot 123}{3} = \frac{615}{3} = 205 \end{array} \quad \begin{array}{l} \text{Multiply both sides of the equation by } \frac{5}{3} \\ \text{Simplify} \end{array}$$

The original price of the stroller is \$205.

**Exercise 5.46 – Multiplication with mixed numbers.** Multiply  $2\frac{1}{3}$  and  $3\frac{1}{5}$ .

**Solution.** There are two ways to perform this addition:

- One way is to change the mixed numbers to improper fractions and use the definition of the multiplication:

$$2\frac{1}{3} \cdot 3\frac{1}{5} = \left(2 + \frac{1}{3}\right) \left(3 + \frac{1}{5}\right) = \left(\frac{6}{3} + \frac{1}{3}\right) \left(\frac{15}{5} + \frac{1}{5}\right) = \frac{7}{3} \cdot \frac{16}{5} = \frac{7 \cdot 16}{3 \cdot 5} = \frac{112}{15}.$$

- Another way is to use the distributive property of multiplication over addition:

$$\begin{aligned} 2\frac{1}{3} \cdot 3\frac{1}{5} &= \left(2 + \frac{1}{3}\right) \left(3 + \frac{1}{5}\right) = \left(2 + \frac{1}{3}\right) \cdot 3 + \left(2 + \frac{1}{3}\right) \cdot \frac{1}{5} = 2 \cdot 3 + \frac{1}{3} \cdot 3 + 2 \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{5} = 6 + \frac{3}{3} + \frac{2}{5} + \frac{1}{15} \\ &= 6 + 1 + \frac{2}{5} + \frac{1}{15} = 7 + \frac{2}{5} + \frac{1}{15} = \frac{7 \cdot 15}{15} + \frac{2 \cdot 3}{5 \cdot 3} + \frac{1}{15} = \frac{105}{15} + \frac{6}{5 \cdot 3} + \frac{1}{15} = \frac{105 + 6 + 1}{15} = \frac{112}{15}. \end{aligned}$$

### 5.3.3 Division of rational numbers

**Definition 5.47 – Division of rational numbers.** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers, then  $\frac{a}{b} \div \frac{c}{d} = \frac{e}{f}$  if, and only if,  $\frac{e}{f}$  is the unique rational number such that  $\frac{c}{d} \cdot \frac{e}{f} = \frac{a}{b}$ .

**Example 5.48** To compute  $\frac{3}{4} \div \frac{5}{9}$ , we need to solve the following equation:  $\frac{3}{4} \div \frac{5}{9} = x$ . By the above definition,  $\frac{3}{4} = \frac{5}{9}x$ . To solve the latter equation, we multiply both sides of the equation by  $\frac{9}{5}$  (the multiplicative inverse of  $\frac{5}{9}$ ) and hence,  $\frac{9}{5} \cdot \frac{3}{4} = \frac{9}{5} \cdot \frac{5}{9}x$  and hence  $\frac{9 \cdot 3}{5 \cdot 4} = \frac{9 \cdot 5}{5 \cdot 9}x$ , or  $\frac{27}{20} = \frac{45}{45}x$  or  $\frac{27}{20} = x$ .

The above example show that dividing two rational number is equivalent to multiply the first fraction by the multiplicative inverse of the second fraction, and this leads to the following theorem:

**Theorem 5.49 – Algorithm for division of fractions.** If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any rational numbers and  $\frac{c}{d} \neq 0$ , then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}. \quad (5.12)$$

#### Examples 5.50

- $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{2 \cdot 7}{3 \cdot 5} = \frac{14}{15}$ .

- $\frac{3}{4} \div \frac{-3}{11} = \frac{3}{4} \cdot \frac{-11}{3} = \frac{3 \cdot -11}{4 \cdot 3} = \frac{-33}{12} = \frac{-11}{4}$ .
- $\frac{2}{7} \div 3 = \frac{2}{7} \cdot \frac{1}{3} = \frac{2 \cdot 1}{7 \cdot 3} = \frac{2}{21}$ .

**Exercise 5.51** Solve for  $x$  each of the following:

a.  $\frac{2}{3}x = \frac{11}{6}$ ;

b.  $\frac{5}{6} - \frac{2}{3}x = \frac{3}{4}$ ;

c.  $\frac{2x}{3} + \frac{1}{4} = \frac{x}{6} - \frac{1}{2}$ .

**Solution.**

a.

$$\begin{array}{l} \frac{2}{3}x = \frac{11}{6} \\ \frac{3}{2} \cdot \frac{2}{3}x = \frac{3}{2} \cdot \frac{11}{6} \quad \leftarrow \text{Multiply both sides of the equation by } \frac{3}{2} \\ x = \frac{3 \cdot 11}{2 \cdot 6} = \frac{33}{12} = \frac{11}{4} \quad \leftarrow \text{Simplify} \end{array}$$

b.

$$\begin{array}{l} \frac{5}{6} - \frac{2}{3}x = \frac{3}{4} \\ \frac{5}{6} - \frac{2}{3}x + \frac{2}{3}x = \frac{3}{4} + \frac{2}{3}x \quad \leftarrow \text{Add } \frac{2}{3}x \text{ to both sides of the equation} \\ \frac{5}{6} = \frac{3}{4} + \frac{2}{3}x \quad \leftarrow \text{Simplify} \\ \frac{5}{6} - \frac{3}{4} = \frac{3}{4} + \frac{2}{3}x - \frac{3}{4} \quad \leftarrow \text{Add } -\frac{3}{4} \text{ to both sides of the equation} \\ \frac{10}{12} - \frac{9}{12} = \frac{2}{3}x \\ \frac{1}{12} = \frac{2}{3}x \\ \frac{3}{2} \cdot \frac{1}{12} = \frac{3}{2} \cdot \frac{2}{3}x \quad \leftarrow \text{Multiply both sides of the equation by } \frac{3}{2} \\ \frac{1}{8} = \frac{3}{4} = x \quad \leftarrow \text{Simplify} \end{array}$$

c.

$$\begin{array}{l} \frac{2x}{3} + \frac{1}{4} = \frac{x}{6} - \frac{1}{2} \\ \frac{2x}{3} - \frac{x}{6} = \frac{-1}{4} - \frac{1}{2} \quad \leftarrow \text{Add } -\frac{x}{6} - \frac{1}{4} \text{ to both sides of the equation} \\ \frac{4x}{6} - \frac{x}{6} = \frac{-1}{4} - \frac{2}{4} \\ \frac{3x}{6} = \frac{-1-2}{4} \\ \frac{x}{2} = \frac{-3}{4} \\ 2 \cdot \frac{x}{2} = 2 \cdot \frac{-3}{4} \quad \leftarrow \text{Multiply both sides of the equation by 2} \\ x = \frac{-6}{4} = \frac{-3}{2} \quad \leftarrow \text{Simplify} \end{array}$$

### 5.3.4 Estimation and mental math with rational numbers

**Exercise 5.52** Use mental math to find

a.  $(24 \cdot 15) \cdot \frac{1}{4}$ ;

b.  $\left(3\frac{1}{2}\right) \cdot 6$ ;

c.  $\frac{3}{5} \cdot 20$ .

**Solution.**

a.  $(24 \cdot 15) \cdot \frac{1}{4} = 15 \cdot \left(24 \cdot \frac{1}{6}\right) = 15 \cdot 4 = 60$ ;

b.  $\left(3\frac{1}{2}\right) \cdot 6 = \left(3 + \frac{1}{2}\right) \cdot 6 = 3 \cdot 6 + \frac{1}{2} \cdot 6 = 18 + 3 = 21$ ;

c.  $\frac{3}{5} \cdot 20 = 3 \cdot \left(\frac{1}{5} \cdot 20\right) = 3 \cdot 4 = 12$ .

### 5.3.5 Extending the notion of exponents

**Definition 5.53 – a to the m<sup>th</sup> power.**  $a^m = \underbrace{a \times a \times \cdots \times a}_{m \text{ factors}}$ , where  $a$  is any rational number and  $m$  is any natural number.

**Theorem 5.54** For any nonzero rational number  $a$  and any integers  $m$  and  $n$ ,  $a^m \cdot a^n = a^{m+n}$ .

**Examples 5.55**

•  $\left(\frac{2}{3}\right)^3 \cdot \left(\frac{2}{3}\right)^8 = \left(\frac{2}{3}\right)^{3+8} = \left(\frac{2}{3}\right)^{11}$ ;

•  $\left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right)^9 = \left(\frac{3}{4}\right)^{1+9} = \left(\frac{3}{4}\right)^{10}$ .

**Theorem 5.56** For any nonzero rational number  $a$  and any integers  $m$  and  $n$ ,  $\frac{a^m}{a^n} = a^{m-n}$ .

**Examples 5.57**

•  $\frac{\left(\frac{2}{3}\right)^8}{\left(\frac{2}{3}\right)^3} = \left(\frac{2}{3}\right)^{8-3} = \left(\frac{2}{3}\right)^5$ ;

•  $\frac{\left(\frac{3}{4}\right)^9}{\left(\frac{3}{4}\right)} = \left(\frac{3}{4}\right)^{9-1} = \left(\frac{3}{4}\right)^8$ .

**Theorem 5.58** For any rational number nonzero  $a$  and any integers  $m$  and  $n$ ,  $(a^m)^n = a^{mn}$ .

**Examples 5.59**

•  $\left(\left(\frac{2}{3}\right)^3\right)^8 = \left(\frac{2}{3}\right)^{3 \cdot 8} = \left(\frac{2}{3}\right)^{24}$ ;

•  $\left(\left(\frac{3}{4}\right)^2\right)^9 = \left(\frac{3}{4}\right)^{2 \cdot 9} = \left(\frac{3}{4}\right)^{18}$ .

**Theorem 5.60** For any nonzero rational number  $\frac{a}{b}$  and any integer  $m$ ,  $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ .

**Examples 5.61**

•  $\left(\frac{2}{3}\right)^{11} = \frac{2^{11}}{3^{11}}$ ;

•  $\left(\frac{3}{4}\right)^{10} = \frac{3^{10}}{4^{10}}$ .

**Theorem 5.62** For any nonzero rational number  $\frac{a}{b}$  and any integer  $m$ ,  $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ .

**Examples 5.63**

- $\left(\frac{2}{3}\right)^{-11} = \left(\frac{3}{2}\right)^{11} = \frac{3^{11}}{2^{11}};$
- $2^{-5} = \left(\frac{1}{2}\right)^5 = \frac{1^5}{2^5} = \frac{1}{2^5};$
- $\left(\frac{3}{4}\right)^{-10} = \left(\frac{4}{3}\right)^{10} = \frac{4^{10}}{3^{10}};$
- $\left(\frac{1}{3}\right)^{-5} = 3^5.$

**Definition 5.64 – a to the an integer power.**

1.  $a^m = \underbrace{a \times a \times \cdots \times a}_{m \text{ factors}}$ , where  $m$  is a positive integer and  $a$  any rational number.
2.  $a^0 = 1.$
3.  $a^{-m} = \frac{1}{a^m}.$

**Theorem 5.65 – Properties of exponents.**

$$a^m \cdot a^n = a^{m+n} \quad (5.13)$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (5.14)$$

$$(a^m)^n = a^{mn} \quad (5.15)$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (5.16)$$

$$(ab)^m = a^m b^m \quad (5.17)$$

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m \quad (5.18)$$

**Examples 5.66**

- $\frac{2^7}{2^4} = 2^{7-4} = 2^3;$
- $\left(\frac{2}{3} \cdot \frac{5}{7}\right)^{10} = \left(\frac{2}{3}\right)^{10} \cdot \left(\frac{5}{7}\right)^{10}.$

# Chapter 6

## Decimal and real numbers

### 6.1 Introduction to decimals

We encounter decimals when dealing with money. For example, looking at my electricity bill, I see \$87.63. The dot in 87.63 is the **decimal point**. The number \$87.63 means \$87 and  $\frac{63}{100}$  of a dollars, i.e. you divide a dollar into 100 pieces, here a piece is a cent, and we take 63 pieces. Consequently,  $87.63 = 87\frac{63}{100}$ . Note also that 63 cents is 6 dimes and 3 cents. Since 10 dimes is a dollars, then 6 dimes is  $\frac{6}{10}$  of a dollar. Consequently, 87.63 is also equal to  $87 + \frac{6}{10} + \frac{3}{100}$ .

#### Examples 6.1

$$\bullet \frac{3}{10} = \frac{3}{10^1} = 0.3$$

$$\bullet \frac{7}{10} = \frac{7}{10^1} = 0.7$$

$$\bullet \frac{3}{100} = \frac{3}{10^2} = 0.03$$

$$\bullet \frac{7}{100} = \frac{7}{10^2} = 0.07$$

$$\bullet \frac{3}{1,000} = \frac{3}{10^3} = 0.003$$

$$\bullet \frac{7}{1,000} = \frac{7}{10^3} = 0.007$$

$$\bullet \frac{3}{10,000} = \frac{3}{10^4} = 0.0003$$

$$\bullet \frac{7}{10,000} = \frac{7}{10^4} = 0.0007$$

#### Examples 6.2

$$\bullet 587.9367 = 5 \cdot 10^2 + 8 \cdot 10^1 + 7 \cdot 10^0 + 9 \cdot \frac{1}{10} + 3 \cdot \frac{1}{10^2} + 6 \cdot \frac{1}{10^3} + 7 \cdot \frac{1}{10^4} = 587\frac{9367}{10,000}.$$

We can also write  $587.9367 = 5 \cdot 10^2 + 8 \cdot 10^1 + 7 \cdot 10^0 + 9 \cdot 10^{-1} + 3 \cdot 10^{-2} + 6 \cdot 10^{-3} + 7 \cdot 10^{-4}$ ;

$$\bullet 1234.56789 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 + 5 \cdot \frac{1}{10} + 6 \cdot \frac{1}{10^2} + 7 \cdot \frac{1}{10^3} + 8 \cdot \frac{1}{10^4} + 9 \cdot \frac{1}{10^5} = 1234\frac{56789}{100,000}.$$

We can also write  $1234.56789 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2} + 7 \cdot 10^{-3} + 8 \cdot 10^{-4} + 9 \cdot 10^{-5}$ ;

**Exercise 6.3** Convert each of the following to decimals:

a.  $\frac{17}{10}$ ;

b.  $\frac{34}{100}$ ;

c.  $\frac{178}{1,000}$ ;

d.  $\frac{203}{10,000}$ .

**Solution.**

$$\text{a. } \frac{17}{10} = 1.7; \quad \text{b. } \frac{34}{100} = 0.34; \quad \text{c. } \frac{178}{1,000} = 0.178; \quad \text{d. } \frac{203}{10,000} = 0.0203.$$

**Exercise 6.4** Express each of the following as decimals:

$$\text{a. } \frac{4}{5}; \quad \text{b. } \frac{9}{250}; \quad \text{c. } \frac{3}{125}; \quad \text{d. } \frac{1}{2^6 \cdot 5^4}; \quad \text{e. } \frac{3}{5^6}; \quad \text{f. } \frac{34}{2^3}.$$

**Solution.**

$$\text{a. } \frac{4}{5} = \frac{4 \cdot 2}{5 \cdot 2} = \frac{8}{10} = 0.8;$$

$$\text{b. } \frac{9}{250} = \frac{9 \cdot 4}{250 \cdot 4} = \frac{36}{1,000} = \frac{36}{10^3} = 0.036;$$

$$\text{c. } \frac{3}{125} = \frac{3 \cdot 8}{125 \cdot 8} = \frac{24}{1,000} = \frac{24}{10^3} = 0.024;$$

$$\text{d. } \frac{1}{2^6 \cdot 5^4} = \frac{1 \cdot 5^2}{2^6 \cdot 5^4 \cdot 5^2} = \frac{25}{2^6 \cdot 5^6} = \frac{25}{(2 \cdot 5)^6} = \frac{25}{10^6} = \frac{25}{1,000,000} = \frac{25}{10^6} = 0.000025;$$

$$\text{e. } \frac{3}{5^6} = \frac{3 \cdot 2^6}{5^6 \cdot 2^6} = \frac{3 \cdot 64}{(5 \cdot 2)^6} = \frac{192}{1,000,000} = \frac{192}{10^6} = 0.000192;$$

$$\text{f. } \frac{1}{2^3} = \frac{1 \cdot 5^3}{2^3 \cdot 5^3} = \frac{125}{(2 \cdot 5)^3} = \frac{125}{1,000} = \frac{125}{10^3} = 0.125.$$

**Theorem 6.5** A rational number  $\frac{a}{b}$  in simplest form can be written as a terminating decimal if, and only if, the prime factorization of the denominator contains no primes other than 2 or 5.

**Exercise 6.6** Which of the following fractions can be written as terminating decimals?

$$\text{a. } \frac{4}{3}; \quad \text{b. } \frac{18}{45}; \quad \text{c. } \frac{17}{45}; \quad \text{d. } \frac{37.98}{1,000}$$

**Solution.**

a. The fraction  $\frac{4}{3}$  is in simplest form, and its denominator is 3. Consequently,  $\frac{4}{3}$  can not be written as a terminating decimal;

b. The fraction  $\frac{18}{45}$  is not in simplest form.  $\frac{18}{45} = \frac{2 \cdot 9}{5 \cdot 9} = \frac{2}{5}$ , and since the denominator is 5, the fraction  $\frac{18}{45}$  can be written as a terminating decimal;

c. The fraction  $\frac{17}{45}$  is in simplest form, and since  $45 = 3^2 \cdot 5$ , the fraction  $\frac{17}{45}$  can not be written as a terminating decimal;

$$\text{d. } \frac{37.98}{1,000} = \frac{37.98 \cdot 100}{1,000 \cdot 100} = \frac{3798}{100,000} = \frac{3798}{10^5} = 0.03798.$$

### 6.1.1 Ordering terminating decimals

**Exercise 6.7** In each of the following, order the decimals from the least to the greatest:

$$\text{a. } 13.4919, 13.492, 13.49183, 13.49199; \quad \text{b. } -1.453, -1.45, -1.4053, -1.493.$$

**Solution.**

a.  $13.49183 < 13.4919 < 13.49199 < 13.492$ ;

b.  $-1.493 < -1.453 < -1.45 < -1.4053$ .

## 6.2 Operations on decimals

### 6.2.1 Addition decimals

**Example 6.8** We can compute  $89.47 + 32.16$  in two ways:

$$\bullet \quad 89.47 + 32.16 = \frac{8947}{100} + \frac{3216}{100} = \frac{8947 + 3216}{100} = \frac{12163}{100} = \frac{12163}{10^2} = 121.63;$$

$$\bullet \quad \begin{array}{r} 89.47 \\ + 32.16 \\ \hline = 121.63 \end{array}$$

#### Exercise 6.9

a.  $18.0947 + 172.98911$ ;

b.  $7.2866 + 5.676$ ;

c.  $9.9746 + 7.63$ ;

d.  $76.2598 + 6.986$

**Solution.**

$$\text{a. } \begin{array}{r} 18.09470 \\ + 172.98911 \\ \hline = 191.08381 \end{array};$$

$$\text{b. } \begin{array}{r} 7.2866 \\ + 5.6760 \\ \hline = 12.9626 \end{array};$$

$$\text{c. } \begin{array}{r} 9.9746 \\ + 7.6300 \\ \hline = 17.6046 \end{array};$$

$$\text{d. } \begin{array}{r} 76.2598 \\ + 6.9860 \\ \hline = 83.2458 \end{array}.$$

### 6.2.2 Subtraction decimals

**Example 6.10** We can compute  $89.47 - 32.16$  in two ways:

$$\bullet \quad 89.47 - 32.16 = \frac{8947}{100} - \frac{3216}{100} = \frac{8947 - 3216}{100} = \frac{5731}{100} = 57.31;$$

$$\bullet \quad \begin{array}{r} 89.47 \\ - 32.16 \\ \hline = 57.31 \end{array}$$

#### Exercise 6.11

a.  $18.0947 - 172.98911$ ;

b.  $7.2866 - 5.676$ ;

c.  $9.9746 - 7.63$ ;

d.  $76.2598 - 6.986$ .

**Solution.**

$$\text{a. } 18.0947 - 172.98911 = -(172.98911 - 18.0947). \text{ Since } \begin{array}{r} 172.98911 \\ - 18.09470 \\ \hline = 154.91441 \end{array}, 18.0947 - 172.98911 = -154.91441;$$

$$\text{b. } \begin{array}{r} 7.2866 \\ - 5.6760 \\ \hline = 1.6106 \end{array};$$

$$\text{c. } \begin{array}{r} 9.9746 \\ - 7.6300 \\ \hline = 2.3446 \end{array};$$

$$\text{d. } \begin{array}{r} 76.2598 \\ - 6.9860 \\ \hline = 69.2738 \end{array}.$$

### 6.2.3 Multiplying decimals

**Example 6.12** We can compute  $9.4 \times 2.16$  in two ways:

$$\bullet \quad 9.4 \times 2.16 = \frac{94}{10} \cdot \frac{216}{100} = \frac{94 \times 216}{10 \times 100} = \frac{20,304}{1,000} = 20.304;$$

$$\bullet \quad \begin{array}{r} 2.16 \\ \times 9.4 \\ \hline 864 \\ + 19440 \\ \hline = 20.304 \end{array}$$

#### Exercise 6.13

a.  $9.22 \times 7.9$ ;

b.  $0.893 \times 0.4$ ;

c.  $5.569 \times 60.4$ ;

d.  $3.264 \times 3.36$ .

**Solution.**

$$\text{a. } \begin{array}{r} 9.22 \\ \times 7.9 \\ \hline 8298 \\ + 64540 \\ \hline = 72.838 \end{array};$$

$$\text{b. } \begin{array}{r} 0.893 \\ \times 0.4 \\ \hline 03572 \\ + 00000 \\ \hline = 0.3572 \end{array};$$

$$\text{c. } \begin{array}{r} 5.569 \\ \times 60.4 \\ \hline 22276 \\ + 00000 \\ + 3341400 \\ \hline = 336.3676 \end{array};$$

$$\text{d. } \begin{array}{r} 3.264 \\ \times 3.36 \\ \hline 19584 \\ + 97920 \\ + 979200 \\ \hline = 10.96704 \end{array}.$$

#### Exercise 6.14 — Multiplying by $10^n$ .

a.  $8.811 \times 10$ ;

b.  $8.811 \times 100$ ;

c.  $8.811 \times 1,000$ ;

d.  $8.811 \times 10,000$ ;

d.  $8.811 \times 100,000$ .

**Solution.**

a.  $8.811 \times 10 = 8.811 \times 10^1 = 88.11$ ;

b.  $8.811 \times 100 = 8.811 \times 10^2 = 881.1$ ;

c.  $8.811 \times 1,000 = 8.811 \times 10^3 = 8811$ ;

d.  $8.811 \times 10,000 = 8.811 \times 10^4 = 88110$ ;

d.  $8.811 \times 100,000 = 8.811 \times 10^5 = 881100$ .

### 6.2.4 Dividing decimals

#### Exercise 6.15 — Dividing by $10^n$ .

a.  $881.1 \div 10$ ;

b.  $881.1 \div 100$ ;

c.  $881.1 \div 1,000$ ;

d.  $881.1 \div 10,000$ ;

d.  $881.1 \div 100,000$ .

**Solution.**

a.  $881.1 \div 10 = 881.1 \div 10^1 = 88.11$ ;

b.  $881.1 \div 100 = 881.1 \div 10^2 = 8.811$ ;

c.  $881.1 \div 1,000 = 881.1 \div 10^3 = 0.8811$ ;

d.  $881.1 \div 10,000 = 881.1 \div 10^4 = 0.08811$ ;

d.  $881.1 \div 100,000 = 881.1 \div 10^5 = 0.008811$ .

**Example 6.16** We can compute  $28.714 \div 9.8$  as follows:

$$\bullet 28.714 \div 9.8 = \frac{28714}{1,000} \div \frac{98}{10} = \frac{28714}{1,000} \cdot \frac{10}{98} = \frac{28714 \cdot 10}{1,000 \cdot 98} = \frac{28714}{100 \cdot 98} = \frac{28714}{9800} = 2.93;$$

**Remark 6.17**  $28.714 \div 9.8 = 287.14 \div 98 = 2871.4 \div 980 = 28714 \div 9800$ .

**Exercise 6.18**

- a.  $565.23 \div 0.83$ ;                      b.  $4202.8 \div 553$ ;                      c.  $19.436 \div 48.59$ ;                      d.  $57.754 \div 6.7$ .

**Solution.**

- a.  $565.23 \div 0.83 = 681$ ;                      b.  $4202.8 \div 553 = 7.6$ ;                      c.  $19.436 \div 48.59 = 0.4$ ;                      d.  $57.754 \div 6.7 = 8.62$ .

**Exercise 6.19** Compute  $47.06874 \div 12.7$ .

**Solution.**  $47.06874 \div 12.7 = 4706874 \div 1270000 = 3.7062$ .

## 6.2.5 Rounding decimals

**Exercise 6.20** Round each of the following numbers:

- a. 56.7438 to the nearest thousandth;                      e. 567,438 to the nearest ten;  
 b. 56.7438 to the nearest hundredth;                      f. 567,438 to the nearest hundred;  
 c. 56.7438 to the nearest tenth;                      g. 567,438 to the nearest thousand;  
 d. 56.7438 to the nearest unit;

**Solution.**

- a.  $56.7438 \approx 56.744$ ;                      c.  $56.7438 \approx 56.7$ ;                      e.  $567,438 \approx 567,440$ ;                      g.  $567,438 \approx 567,000$ .  
 b.  $56.7438 \approx 56.74$ ;                      d.  $56.7438 \approx 57$ ;                      f.  $567,438 \approx 567,400$ ;

## 6.3 Nonterminating decimals

### 6.3.1 Repeating decimals

$\frac{7}{4} = \frac{7 \cdot 25}{4 \cdot 25} = \frac{175}{100} = 1.75$ . However, we showed that  $\frac{3}{11}$  was not a terminating decimal because  $\frac{3}{11}$  is in simplest form and the denominator 11 does not contain any primes other than 2 and 5.

**Examples 6.21**

$$\bullet \frac{3}{11} = 0.272727 \dots = 0.\overline{27}; \quad \bullet \frac{1}{3} = 0.333333 \dots = 0.\overline{3}; \quad \bullet \frac{22}{7} = 3.142857142857 \dots = \underline{3.142857}.$$

### 6.3.2 A surprising result

$1 = \frac{3}{3} = 3 \cdot \frac{1}{3} = 3 \cdot 0.\overline{3} = 0.\overline{9}$ . Consequently,  $1 = 0.\overline{9}$ . Note that  $0.9 \approx 1$  and  $0.9 \neq 1$ ,  $0.99 \approx 1$  and  $0.99 \neq 1$ ,  $0.999 \approx 1$  and  $0.999 \neq 1$ , etc. However, the repeating decimal  $0.\overline{9} = 0.999999 \dots$  is exactly equal to 1.

### 6.3.3 Ordering repeating decimals

**Exercise 6.22** Compare  $2.\overline{13}$  and  $2.\overline{132}$ .

**Solution.**  $2.\overline{13} = 2.1313131313\dots$ , and  $2.\overline{132} = 2.132132312\dots$ . Consequently,  $2.\overline{13} < 2.\overline{132}$ .

## 6.4 Real numbers

Every rational number can be expressed either as a terminating decimal or as a repeating decimal. The Greeks discovered numbers that are not rational. Such numbers must have a decimal representation that neither terminates nor repeats. To find such decimals, we focus on the characteristics they must have:

1. There must be an infinite number of nonzero digits to the rights of the decimal points.
2. There cannot be a repeating blocks of digits.

Famous and important irrational numbers are:  $\sqrt{2}$  and  $\pi$

### 6.4.1 Square roots

**Definition 6.23 – Principal square root.** If  $a$  is any nonnegative number, the **principal square root** of  $a$  (denoted  $\sqrt{a}$ ) is the nonnegative number  $b$  such that  $b^2 = a$ .

#### Examples 6.24

- The square roots of 4 are 2 and  $-2$ , because  $2^2 = (-2)^2 = 4$ . The principal square root of 4, denoted  $\sqrt{4}$ , is 2. Therefore,  $\sqrt{4} = 2$ .
- The square roots of 9 are 3 and  $-3$ , because  $3^2 = (-3)^2 = 9$ . The principal square root of 9, denoted  $\sqrt{9}$ , is 3. Therefore,  $\sqrt{9} = 3$ .
- The square roots of 16 are 4 and  $-4$ , because  $4^2 = (-4)^2 = 16$ . The principal square root of 16, denoted  $\sqrt{16}$ , is 4. Therefore,  $\sqrt{16} = 4$ .
- The square roots of 25 are 5 and  $-5$ , because  $5^2 = (-5)^2 = 25$ . The principal square root of 25, denoted  $\sqrt{25}$ , is 5. Therefore,  $\sqrt{25} = 5$ .

### 6.4.2 The system of real number

The set of real numbers, denoted  $\mathbb{R}$ , is the union of the set of rational numbers and the set of irrational numbers.

#### Theorem 6.25 – Properties of real numbers.

1. **Closure properties** For real numbers  $a$  and  $b$ ,  $a + b$  and  $ab$  are unique real numbers.
2. **Commutative properties** For real numbers  $a$  and  $b$ ,  $a + b = b + a$  and  $ab = ba$ .
3. **Associative properties** For real numbers  $a, b$  and  $c$ ,  $a + (b + c) = (a + b) + c$  and  $a(bc) = (ab)c$ .
4. **Identity properties** The number 0 is the unique additive identity and 1 is the unique multiplicative identity such that, for any real number  $a$ ,  $0 + a = a = a + 0$  and  $1 \cdot a = a = a \cdot 1$ .
5. **Inverse properties**
  - (i) For every real number  $a$ ,  $-a$  is its unique additive inverse; that is,  $a + (-a) = 0 = (-a) + a$ .
  - (ii) For every nonzero real number  $a$ ,  $\frac{1}{a}$  is its unique multiplicative inverse; that is,  $a\left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right)a$ .
6. **Distributive property of multiplication over addition** For real numbers  $a, b$ , and  $c$ ,  $a(b+c) = ab+bc$ .
7. **Denseness property** For real numbers  $a$  and  $b$ , there exists a real number  $c$  such that  $a < c < b$ .

# Proportional reasoning, percents, and applications

Ratios, proportions are very important and are encountered in everyday life.

## 7.1 Ratios, proportions, and proportional reasoning

### 7.1.1 Ratios

A ratio of 1 : 6 for boy to girls means that the number of boys is one sixth of the number of girls. That is, there is 1 boy for every 6 girls. We can also say that the ration for girls to boys is 6 : 1, or that there are 6 times as many girls as boys. Consequently, ration can represent **part-to-whole**, for instance  $\frac{1}{6}$ , or **whole-to-part**  $\frac{6}{1}$ .

**Example 7.1** In this section, among the 59 registered students, 4 students are male students. There are then 4 male students and 55 female students. The ratio of boys to girls is then  $\frac{4}{55}$ .

**Exercise 7.2** On an american flag, what is the ratio of stars to stripes? What is the ratio of stripes to stars?

**Solution.** There are 13 stripes and 50 stars on an american flag. The ratio of stars to stripes is  $\frac{50}{13}$  and the ration of stripes to stars is  $\frac{13}{50}$ .

**Exercise 7.3** There are 5 adult drivers to each teenage driver in Aluossim. If there are 12,345 adult drivers in Aluossim, how many teenage drivers are there?

**Solution.** Let T be the number of teenage drivers in Aluossim. There are 5 adult drivers to each teenage driver. therefore, the ratio of adults drivers to teenage drivers is 5 : 1, or equivalently, the ratio of teenage drivers to adult drivers is 1 : 5 (there is 1 teenage driver for every 5 adult drivers). Consequently, if there are 12,345 adult drivers, then there are  $\frac{1}{5}$  of 12,345 teenage drivers.  $\frac{1}{5}$  of 12,345 is  $\frac{1}{5} \cdot 12,345 = \frac{12,345}{5} = 2469$ . Finally, there are 2469 teenage drivers in Aluossim.

### 7.1.2 Proportions

**Definition 7.4** A **proportion** is a statement that two given ratios are equal.

**Theorem 7.5** If  $a, b, c,$  and  $d$  are all real numbers and  $b \neq 0$  and  $d \neq 0$ , then

$$\frac{a}{b} = \frac{c}{d} \text{ is a proportion if, and only if, } ad = bc. \quad (7.1)$$

**Exercise 7.6** Solve for  $x$  in each of the following proportions:

- a.  $\frac{5}{x} = \frac{30}{42}$ .                      b.  $\frac{7}{8} = \frac{3x}{48}$ .                      c.  $\frac{x}{8} = \frac{-12}{32}$ .                      d.  $3\frac{1}{2}$  is to 8 as  $x$  is to 24.

**Solution.**

- a.  $\frac{5}{x} = \frac{30}{42}$ , and hence  $x = \frac{5 \cdot 42}{30} = \frac{210}{30} = 7$ . These computations can be easier. Indeed, the fraction  $\frac{30}{42}$  is not in simplest form. Its simplest form is  $\frac{5}{7}$ . Therefore, the equation  $\frac{5}{x} = \frac{30}{42}$  becomes  $\frac{5}{x} = \frac{5}{7}$ . The solution is then  $x = 7$ .
- b.  $\frac{7}{8} = \frac{3x}{48}$ , and hence  $3x = \frac{7 \cdot 48}{8} = \frac{336}{8} = 42$ . Consequently,  $x = \frac{42}{3} = 14$ . Again, these computations can be easier. Indeed, the fraction  $\frac{3x}{48}$  is not in simplest form. Its simplest form is  $\frac{x}{16}$ . Therefore, the equation  $\frac{7}{8} = \frac{3x}{48}$  becomes  $\frac{7}{8} = \frac{x}{16}$ . The solution is then  $x = \frac{7 \times 16}{8} = \frac{7 \times 2 \times 8}{8} = 7 \times 2 = 14$ .
- c.  $\frac{x}{8} = \frac{-12}{32}$ , and hence  $x = \frac{8 \cdot -12}{32} = -3$ . Again, these computations can be easier. Indeed, the fraction  $\frac{-12}{32}$  is not in simplest form. Its simplest form is  $\frac{-3}{8}$ . Therefore, the equation  $\frac{x}{8} = \frac{-12}{32}$  becomes  $\frac{x}{8} = \frac{-3}{8}$ . The solution is then  $x = -3$ .
- d.  $3\frac{1}{2}$  is to 8 as  $x$  is to 24 means that  $\frac{3\frac{1}{2}}{8} = \frac{x}{24}$ . Before finding  $x$ , lets simplify  $\frac{3\frac{1}{2}}{8} = \frac{\frac{7}{2}}{8} = \frac{7}{2} \cdot \frac{1}{8} = \frac{7}{16}$ . We have then  $\frac{7}{16} = \frac{x}{24}$ . Therefore,  $x = \frac{7 \cdot 24}{16} = \frac{7 \cdot 3 \cdot 8}{2 \cdot 8} = \frac{21}{2} = 10.5$ .

**Exercise 7.7** Jim found out that after working for 9 months he had earned 6 days of vacation time. How many day per year does he earn at this rate?

**Solution.** The ratio of vacation time to worked days is  $\frac{6}{9}$  (days per months). Lets denote by  $x$  vacation time per year (12 months). If the ratio is the same, then  $\frac{6}{9} = \frac{x}{12}$ . Therefore,  $x = \frac{6 \times 12}{9} = \frac{72}{9} = 8$ . Finally, the number of days he earns per year is 8.

**Definition 7.8** If the variable  $x$  and  $y$  are related by the equality  $y = kx$ , ( $k = \frac{y}{x}$ ), then  **$y$  is said to be proportional to  $x$**  and  $k$  is the **constant of proportionality** between  $y$  and  $x$ .

**Example 7.9** Lets consider these data.

x	1	2	3	4	5	6	7
y	7	14	21	28	35	42	49

$y$  is proportional to  $x$  and the constant of proportionality between  $y$  and  $x$  is 7, because  $y = 7x$ .

**Theorem 7.10** For any rational numbers  $\frac{a}{b}$  and  $\frac{c}{d}$ , with  $a \neq 0$  and  $c \neq 0$ ,  $\frac{a}{b} = \frac{c}{d}$  if, and only if,  $\frac{b}{a} = \frac{d}{c}$ .

**Theorem 7.11** For any rational numbers  $\frac{a}{b}$  and  $\frac{c}{d}$ , with  $b, c, d \neq 0$ ,  $\frac{a}{b} = \frac{c}{d}$  if, and only if,  $\frac{a}{c} = \frac{b}{d}$ .

## 7.2 Percents

**Definition 7.12 – Percent.**  $n\% = \frac{n}{100}$ .

### Examples 7.13

- $2\% = \frac{2}{100} = \frac{1}{50}$ ;
- $25\% = \frac{25}{100} = \frac{1}{4}$ ;
- $75\% = \frac{75}{100} = \frac{3}{4}$ ;
- $17\% = \frac{17}{100}$ .

**Exercise 7.14** Write each of the following as a percent:

- |                  |                  |                     |                     |
|------------------|------------------|---------------------|---------------------|
| a. 1.            | e. 1.7.          | i. $\frac{3}{5}$ .  | l. $2\frac{1}{3}$ . |
| b. 0.            | f. $1.\bar{7}$   | j. $\frac{3}{4}$ .  | m. $3\frac{1}{5}$ . |
| c. 0.7.          | g. 0.00027.      | k. $1\frac{1}{4}$ . |                     |
| d. $0.\bar{3}$ . | h. $\frac{2}{3}$ |                     |                     |

**Solution.**

$$\text{a. } 1 = \frac{100}{100} = 100\%.$$

$$\text{b. } 0 = \frac{0}{100} = 0\%.$$

$$\text{c. } 0.7 = \frac{0.7 \cdot 100}{100} = \frac{70}{100} = 70\%.$$

$$\text{d. } 0.\bar{3} = \frac{0.\bar{3} \cdot 100}{100} = \frac{33.\bar{3}}{100} = 33.\bar{3}\%.$$

$$\text{e. } 1.7 = \frac{1.7 \cdot 100}{100} = \frac{170}{100} = 170\%.$$

$$\text{f. } 1.\bar{7} = \frac{1.\bar{7} \cdot 100}{100} = \frac{177.\bar{7}}{100} = 177.\bar{7}\%$$

$$\text{g. } 0.00027 = \frac{0.00027 \cdot 100}{100} = \frac{0.027}{100} = 0.027\%.$$

$$\text{h. } \frac{2}{3} = 0.\bar{6} = \frac{0.\bar{6} \cdot 100}{100} = \frac{66.\bar{6}}{100} = 66.\bar{6}\%.$$

$$\text{i. } \frac{3}{5} = 0.6 = \frac{0.6 \cdot 100}{100} = \frac{60}{100} = 60\%.$$

$$\text{j. } \frac{3}{4} = 0.75 = \frac{0.75 \cdot 100}{100} = \frac{75}{100} = 75\%.$$

$$\text{k. } 1\frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4} = 1.25 = \frac{1.25 \cdot 100}{100} = \frac{125}{100} = 125\%.$$

$$\text{l. } 2\frac{1}{3} = 2 + \frac{1}{3} = 2.\bar{3} = \frac{2.\bar{3} \cdot 100}{100} = \frac{233.\bar{3}}{100} = 233.\bar{3}\%.$$

$$\text{m. } 3\frac{1}{5} = 3.2 = \frac{3.2 \cdot 100}{100} = \frac{320}{100} = 320\%.$$

**7.2.1 Applications involving percent**

**Exercise 7.15** A used car originally cost \$1,800. One year later, it was worth \$1,400. What is the percentage of depreciation?

**Solution.** After one year, the depreciation is  $\$1800 - \$1400 = \$400$ . Therefore, the depreciation is \$400 of \$1,800, and the percentage of the depreciation is  $\frac{400}{1800} = \frac{4}{18} = 0.\bar{2} = 22.\bar{2}\%$ .

**Exercise 7.16** What is the sale price of a softball if the regular price is \$12 and there is a 25% discount?

**Solution.** If there is a 25% discount, then the sale price represents 75% of the original price. Therefore, 75% of \$12 is:  $75\% \cdot 12 = \frac{75 \cdot 12}{100} = 9$ . The sale price is then \$9.

**Exercise 7.17** If 30 is 120% of a number, is the number greater than or less than 30? Why?

**Solution.** If 30 is 120% of a number, then the number is less than 30. If we denote by  $x$  a number and since  $120\% = 1.2$ , then  $30 = 1.2x$ . Multiplying a number by a number greater than 1 gives a greater number. Note that we can find the number by solving this equation:  $30 = 1.2x$ . Consequently,  $x = 30 \div 1.2 = 300 \div 12 = 25$ . The number 25 is less than 30.

**Exercise 7.18** What is 40% of 50%.

**Solution.** 40% of 50% is  $\frac{40}{100} \cdot \frac{50}{100} = \frac{4}{10} \cdot \frac{5}{10} = \frac{20}{100} = 20\%$ .

**7.2.2 Mental math with percents**

Percent	25%	50%	75%	$33\frac{1}{3}\%$	$66\frac{2}{3}\%$	10%	1%
Equivalent fraction	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{10}$	$\frac{1}{100}$

# Probability

## 8.1 How probabilities are determined

Probabilities are ratios, expressed as fractions, decimals, or percents, determined by considering results or outcomes of experiments.

**Definition 8.1 – Experiment.** An **experiment**  $M$  is an activity whose results can be observed and recorded.

**Definition 8.2 – Outcome.** Each of the possible results of an experiment is an **outcome**.

**Definition 8.3 – Sample space.** A set of all possible outcomes for an experiment is a **sample space**.

### Remarks 8.4

- The sample space can be modeled by a tree diagram. Each outcome of the experiment is designated by a separate branch in the tree diagram.
- The outcomes in the sample space cannot overlap.

### Examples 8.5

- For the experiment “single coin toss”, there are two distinct possible outcomes: heads (H) and tails (T). Therefore, the sample space is  $S = \{H, T\}$ .

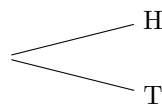


Figure 8.1: Tree diagram for a single coin toss

- For the experiment “rolling a standard die”, there are six distinct possible outcomes: 1, 2, 3, 4, 5 and 6. Therefore, the sample space is  $S = \{1, 2, 3, 4, 5, 6\}$ .

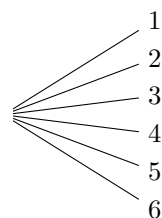


Figure 8.2: Tree diagram for a single die roll

**Definition 8.6 – Event.** Any subset of a sample space is an **event**.

**Example 8.7** The set of all odd-numbered rolls  $E_1 = \{1, 3, 5\}$  is a subset of all possible rolls of a die  $S = \{1, 2, 3, 4, 5, 6\}$ . Therefore,  $E_1$  is an event. The set of all even-numbered rolls  $E_2 = \{2, 4, 6\}$  is a subset of all possible rolls of a die  $S = \{1, 2, 3, 4, 5, 6\}$ . Therefore,  $E_2$  is also an event.

**Exercise 8.8** Suppose an experiment consists of drawing 1 slip of paper from a jar containing 7 slips of paper, each with a different day of the week written on it. Find each of the following:

1. The sample space  $S$  of the experiment.
2. The event  $A$  consisting of outcomes having a day beginning with T.
3. The event  $B$  consisting of outcomes having the name of a day that has exactly 6 letters.
4. The event  $C$  consisting of outcomes having a day that begins with F or S.
5. The event  $D$  consisting of outcomes having a day that begins with Z.

**Solution.**

1.  $S = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\}$ .
2.  $A = \{\text{Tuesday, Thursday}\}$ ;
3.  $B = \{\text{Monday, Friday, Sunday}\}$ ;
4.  $C = \{\text{Friday, Saturday, Sunday}\}$ ;
5.  $D = \emptyset$ .

### 8.1.1 Determining probabilities

Around 1900, the English statistician Karl Pearson tossed a coin 24,000 times and recorded 12,012 heads. During World War II, John Kerrich, a prisoner of war, tossed a coin 10,000 times. A subset of his results is in the following table:

Number of tosses	Number of heads	Relative frequency (rounded)
10	4	0.400
50	25	0.500
100	44	0.440
500	255	0.510
1,000	502	0.502
5,000	2,533	0.507
10,000	5,067	0.507

**Theorem 8.9 – Law of large numbers (Bernoulli's Theorem).** If an experiment is repeated a large number of times, the *experimental* or *empirical* probability of a particular outcomes approaches a fixed number as the number of repetitions increases.

When a fair coin is tossed many times, then the fraction  $\frac{\text{number of heads}}{\text{number of tosses}}$  is near  $\frac{1}{2}$ . Therefore, we say that the probability of heads occurring is  $\frac{1}{2}$  and we write  $P(\{H\}) = \frac{1}{2}$ , or we shorten the symbolism to  $P(H) = \frac{1}{2}$ .

**Definition 8.10 – Probability of an event with equally likely outcomes.** For an experiment with sample space  $S$  with equally likely outcomes, the **probability of an event  $A$**  is given by

$$P(A) = \frac{\text{Number of elements of } A}{\text{Number of elements of } S} = \frac{n(A)}{n(S)}. \quad (8.1)$$

**Example 8.11** A fair die is a die that is just as likely to land showing any of the numerals 1 through 6 on any toss. Its sample space is given by  $S = \{1, 2, 3, 4, 5, 6\}$ , and  $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$ . The probability of rolling an even number, that is, the probability of the event  $E = \{2, 4, 6\}$ , is  $\frac{3}{6} = \frac{1}{2}$ , because  $n(E) = 3$  and  $n(S) = 6$ .

### 8.1.2 Mutually exclusive events

**Definition 8.12 – Mutually exclusive events.** Events A and B are **mutually exclusive** if they have no elements in common; that is  $A \cap B = \emptyset$ .

**Theorem 8.13** If events A and B are mutually exclusive, then  $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$ .

**Example 8.14** Each outcome in the space  $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  is equally likely, with probability  $\frac{1}{10}$ . Let A be the event  $\{0, 2, 4, 8\}$  and B be the event  $\{3, 6, 9\}$ . Since  $A \cap B = \emptyset$ , the events A and B are mutually exclusive. We write then the probability of A or B as  $P(A \cup B)$ , and is determined as follows: Either by

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{7}{10},$$

or, since A and B are mutually exclusive, we have

$$P(A \cup B) = P(A) + P(B) = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}.$$

### 8.1.3 Complementary events

**Theorem 8.15** If A is an event and  $\bar{A}$  is its complement, then

$$P(A) + P(\bar{A}) = 1 \tag{8.2}$$

$$P(A) = 1 - P(\bar{A}) \tag{8.3}$$

$$P(\bar{A}) = 1 - P(A) \tag{8.4}$$

**Example 8.16** Each outcome in the space  $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  is equally likely, with probability  $\frac{1}{10}$ . Let A be the event  $\{0, 3, 6, 9\}$ . The probability of A is then  $P(A) = \frac{n(A)}{n(S)} = \frac{4}{10}$ . The probability of  $\bar{A}$  (the probability that the event A doesn't occur) is  $P(\bar{A}) = 1 - \frac{4}{10} = \frac{6}{10}$ .

### 8.1.4 Non-mutually exclusive events

**Example 8.17** Each outcome in the space  $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  is equally likely, with probability  $\frac{1}{10}$ . Let E be the event  $\{0, 2, 4, 6, 8\}$  and F is the event  $\{0, 3, 6, 9\}$ . Since  $E \cap F = \{0, 6\} \neq \emptyset$ , the events E and F are not mutually exclusive. On one hand,  $P(E) = \frac{n(E)}{n(S)} = \frac{5}{10} = \frac{1}{2}$ , and  $P(F) = \frac{n(F)}{n(S)} = \frac{4}{10}$ . On the other hand,  $E \cup F = \{0, 2, 3, 4, 6, 8, 9\}$ , and hence,  $P(E \cup F) = \frac{n(E \cup F)}{n(S)} = \frac{7}{10}$ . Therefore,  $P(E \cup F) = \frac{7}{10}$  does not equal to  $P(E) + P(F) = \frac{5}{10} + \frac{4}{10} = \frac{9}{10}$ . However, if we subtract  $P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{2}{10}$  from  $P(E) + P(F)$ , i.e.,  $P(E) + P(F) - P(E \cap F) = \frac{9}{10} - \frac{2}{10} = \frac{7}{10}$ .

**Theorem 8.18 – Summary of probability properties.**

1.  $P(\emptyset) = 0$  (impossible event).
2.  $P(S) = 1$ , where  $S$  is the sample space (certain event).
3. For any event  $A$ ,  $0 \leq P(A) \leq 1$ .
4. If  $A$  and  $B$  are events and  $A \cap B = \emptyset$ , then  $P(A \cup B) = P(A) + P(B)$ .
5. If  $A$  and  $B$  are any events, then  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .
6. If  $A$  is an event, then  $P(\bar{A}) = 1 - P(A)$ .

**Exercise 8.19** If there are 16 total bottles of diet soda, 8 total bottles of regular soda, and 4 total bottles of water, what is the probability of each of the following:

- a. Choosing a bottle of diet soda when a bottle is chosen at random;
- b. Choosing a bottle of regular soda when a bottle is chosen at random;
- c. Choosing a bottle of water when a bottle is chosen at random.
- d. Not choosing a bottle of water when a bottle is chosen at random.

**Solution.** The cardinal number of the sample space is  $n(S) = 16 + 8 + 4 = 28$  (there are 28 bottles in total). Let denote by  $D$  the event of choosing a bottle of diet soda,  $R$  the event of choosing a bottle of regular soda, and  $W$  the event of choosing a bottle of water.

a.  $P(D) = \frac{n(D)}{n(S)} = \frac{16}{28} = \frac{4}{7}$ ;

b.  $P(R) = \frac{n(R)}{n(S)} = \frac{8}{28} = \frac{2}{7}$ ;

c.  $P(W) = \frac{n(W)}{n(S)} = \frac{4}{28} = \frac{1}{7}$ ;

d.  $P(\bar{W}) = 1 - P(W) = 1 - \frac{1}{7} = \frac{6}{7}$ .

## 8.2 Multistage experiments with tree diagrams

In the previous section, we considered

experiments, this is, experiments that were over after one step. For example, drawing one ball from a box containing a red, white and green ball is a one-stage experiment. A tree diagram for this experiment is:

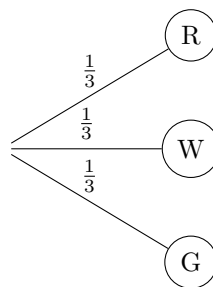
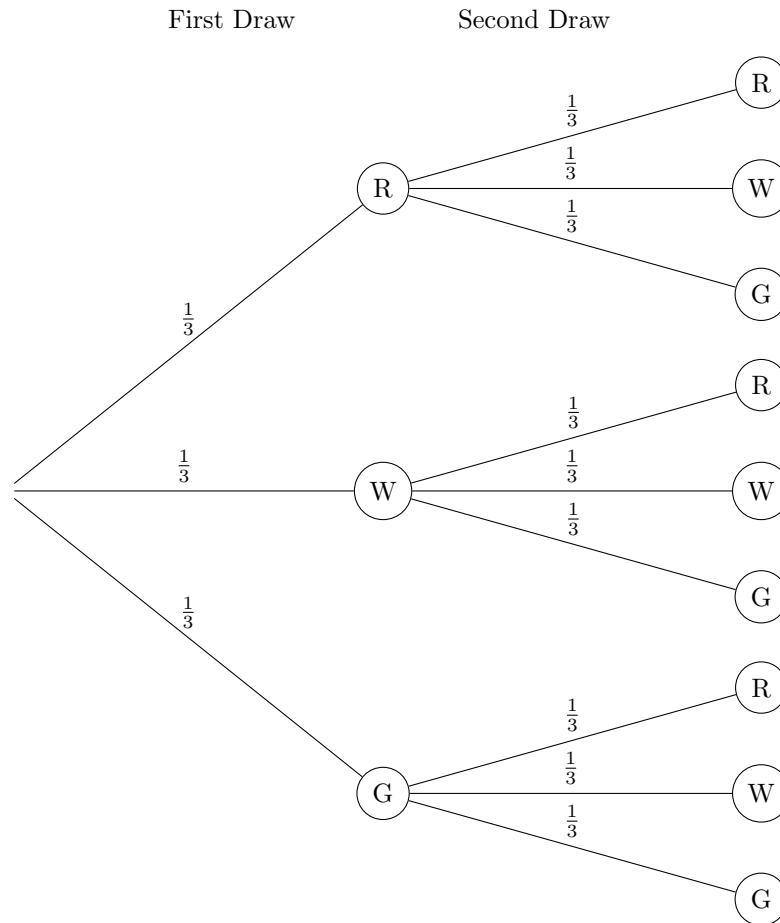


Figure 8.3: Tree diagram for one-stage drawing

Now consider a **two-stage experiment**: We have a box containing a red, white and green ball.

- **With replacement:** A ball is drawn from the above box and its color is recorded. Then the ball is *replaced*, and a second ball is drawn and its color is recorded. The tree diagram for this two-stage experiment is:



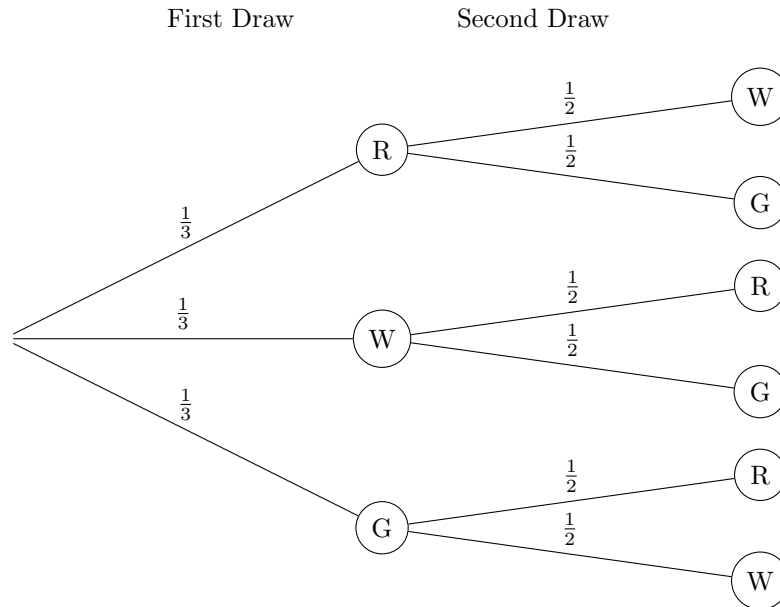
We can use the diagram tree to generate the sample space. The sample space for this experiment is  $\{RR, RW, RG, WR, WW, WG, GR, GW, GG\}$ . Each of the outcomes in the sample space is equally likely and there are nine total outcomes, so the probability for each outcome is  $\frac{1}{9}$ . The probability that:

- both balls are red is  $P(RR) = \frac{1}{9}$ .
- no ball is red is  $P(\{WW, WG, GW, GG\}) = \frac{4}{9}$ .
- At least one ball is red is  $P(\{RR, RW, RG, WR, GR\}) = \frac{5}{9}$ .
- both balls are the same color is  $P(\{RR, WW, GG\}) = \frac{3}{9} = \frac{1}{3}$ .
- balls with different colors is  $P(\{RW, RG, WR, WG, GR, GW\}) = \frac{6}{9} = \frac{2}{3}$ .

Note that the event  $\{RW, RG, WR, WG, GR, GW\}$  is the complement of the event  $\{RR, WW, GG\}$ .

Therefore,  $P(\{RW, RG, WR, WG, GR, GW\}) = 1 - P(\{RR, WW, GG\}) = 1 - \frac{1}{3} = \frac{2}{3}$ .

- **Without replacement:** A ball is drawn from the above box and its color is recorded, but the ball is *not replaced*, and a second ball is drawn and its color is recorded. The tree diagram for this two-stage experiment is:



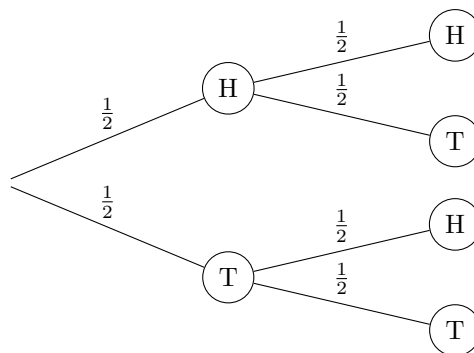
We can use the diagram tree to generate the sample space. The sample space for this experiment is  $\{RW, RG, WR, WG, GR, GW\}$ . Each of the outcomes in the sample space is equally likely and there are six total outcomes, so the probability for each outcome is  $\frac{1}{6}$ . The probability that:

- a. both balls are red is  $P(RR) = 0$  because it is an impossible outcome.
- b. no ball is red is  $P(\{WG, GW\}) = \frac{2}{6} = \frac{1}{3}$ .
- c. At least one ball is red is  $P(\{RW, RG, WR, GR\}) = \frac{4}{6} = \frac{2}{3}$ .
- d. both balls are the same color is  $P(\{RR, WW, GG\}) = 0$  because it is an impossible event.
- e. balls with different colors is  $P(\{RW, RG, WR, WG, GR, GW\}) = \frac{6}{6} = 1$ .

**Exercise 8.20** We toss a fair coin two times and record the results. Find each of the following:

- a. The sample space;
- b. The probability of tossing 1 head and 1 tail;
- c. The probability of tossing no tail;
- d. The probability of tossing the same face;
- e. The probability of tossing a head on the last toss.

**Solution.** The tree diagram for this two-stage experiment is:



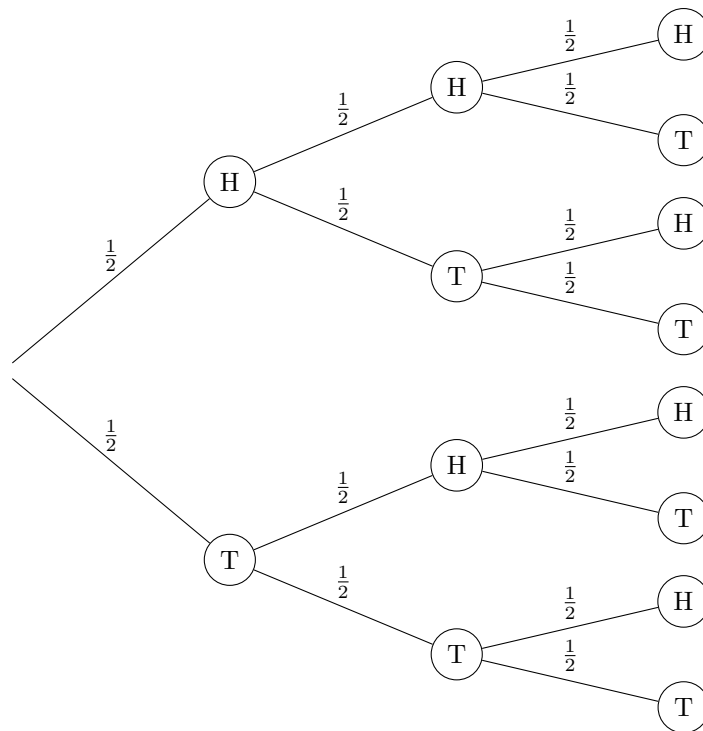
- a. The sample space is  $\{HH, HT, TH, TT\}$ ;

- b. The probability of tossing 1 head and 1 tail is  $P(\{HT, TH\}) = \frac{2}{4} = \frac{1}{2}$ ;
- c. The probability of tossing no tail is  $P(\{HH\}) = \frac{1}{4}$ ;
- d. The probability of tossing the same face is  $P(\{HH, TT\}) = \frac{2}{4} = \frac{1}{2}$ ;
- e. The probability of tossing a head on the last toss is  $P(\{HH, TH\}) = \frac{2}{4} = \frac{1}{2}$ .

**Exercise 8.21** We toss a fair coin three times and record the results. Find each of the following:

- a. The sample space;
- b. The probability of tossing 2 heads and 1 tail;
- c. The probability of tossing no tail;
- d. The probability of tossing the same face;
- e. The probability of tossing a head on the last toss.

**Solution.** The tree diagram for this three-stage experiment is:



- a. The sample space is  $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ ;
- b. The probability of tossing 2 heads and 1 tail is  $P(\{HHT, HTH, THH\}) = \frac{3}{8}$ ;
- c. The probability of tossing no tail is  $P(\{HHH\}) = \frac{1}{8}$ ;
- d. The probability of tossing the same face is  $P(\{HHH, TTT\}) = \frac{2}{8} = \frac{1}{4}$ ;
- e. The probability of tossing a head on the last toss is  $P(\{HHH, HTH, THH, TTH\}) = \frac{4}{8} = \frac{1}{2}$ .

### 8.2.1 More multistage experiments

Consider the experiment of drawing one ball from a box containing one black ball and two white balls. A tree diagram for this experiment is:

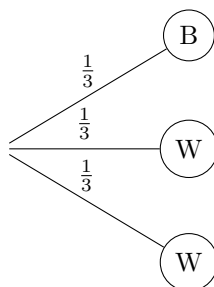


Figure 8.4: Tree diagram for one-stage drawing

Since each white ball have the same probability to be drawn, we can combine the branches to obtain the following tree diagram:

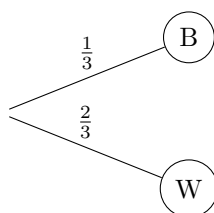
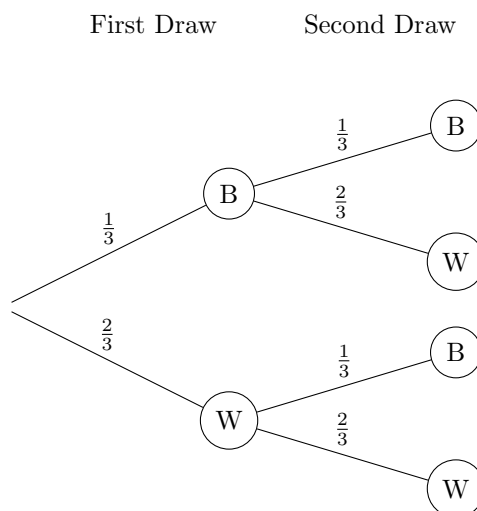


Figure 8.5: Tree diagram for one-stage drawing

Combining branches in this way is a common practice because it simplifies tree diagrams. Now consider a **two-stage experiment** with the box containing one black ball and two white balls.

- **With replacement:** A ball is drawn from the above box and its color is recorded. Then the ball is *replaced*, and a second ball is drawn and its color is recorded. The tree diagram for this two-stage experiment is:



From the tree diagram, the sample space is  $\{BB, BW, WB, WW\}$ . To assign the probability of the outcomes in this experiment, consider the path for the outcome BW (drawing a black ball first, then drawing a white ball). In the first draw, the probability of obtaining a black ball is  $\frac{1}{3}$ . Then the probability of obtaining

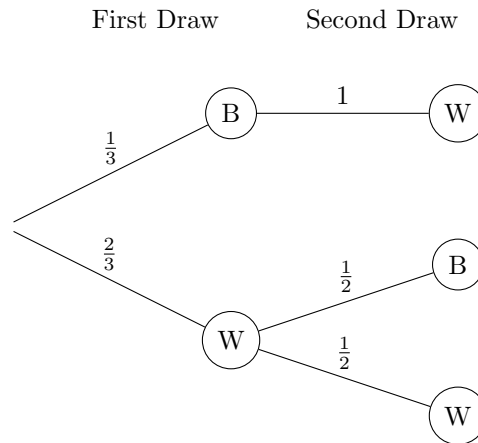
a white ball in the second draw is  $\frac{2}{3}$ . The probability of the outcome BW is:  $P(BW) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$ . The same reasoning leads to

1. The probability of drawing two black balls is:  $P(BB) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ .
2. The probability of drawing a black ball first, then a white ball is:  $P(BW) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$ .
3. The probability of drawing a white ball first, then a black ball is:  $P(WB) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ .
4. The probability of drawing two white balls is:  $P(WW) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$ .

Therefore, we conclude that:

5. The probability of drawing ball of the same color is:  $P(BB, WW) = P(BB) + P(WW) = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$ .
6. The probability of drawing ball of different colors is:  $P(BW, WB) = P(BW) + P(WB) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$ .

- **Without replacement:** A ball is drawn from the above box and its color is recorded, but the ball is *not replaced*, and a second ball is drawn and its color is recorded. The tree diagram for this two-stage experiment is:



From the tree diagram, the sample space is  $\{BW, WB, WW\}$ . To assign the probability of the outcomes in this experiment, consider the path for the outcome WB (drawing a white ball first, then drawing a black ball). In the first draw, the probability of obtaining a white ball is  $\frac{2}{3}$ . Then the probability of obtaining a black ball in the second draw is  $\frac{1}{2}$ . The probability of the outcome WB is:  $P(WB) = \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$ . The same reasoning leads to

1. The probability of drawing two black balls is:  $P(BB) = 0$ .
2. The probability of drawing a black ball first, then a white ball is:  $P(BW) = \frac{1}{3} \cdot 1 = \frac{1}{3}$ .
3. The probability of drawing a white ball first, then a black ball is:  $P(WB) = \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$ .
4. The probability of drawing two white balls is:  $P(WW) = \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$ .

Therefore, we conclude that:

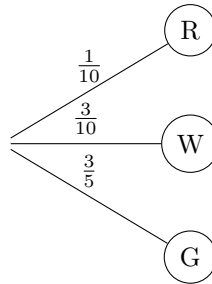
5. The probability of drawing balls of the same color is:  $P(WW) = \frac{1}{3}$ .
6. The probability of drawing balls of different colors is:  $P(BW, WB) = P(BW) + P(WB) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ .

**Theorem 8.22 – Multiplication rule for probabilities for tree diagrams.** For all multistage experiments, the probability of the outcomes along any path of a tree diagram is equal to the product of all the probabilities along the path.

**Examples 8.23** A box contains 1 red ball, 3 white balls and 6 green balls. The probability of drawing:

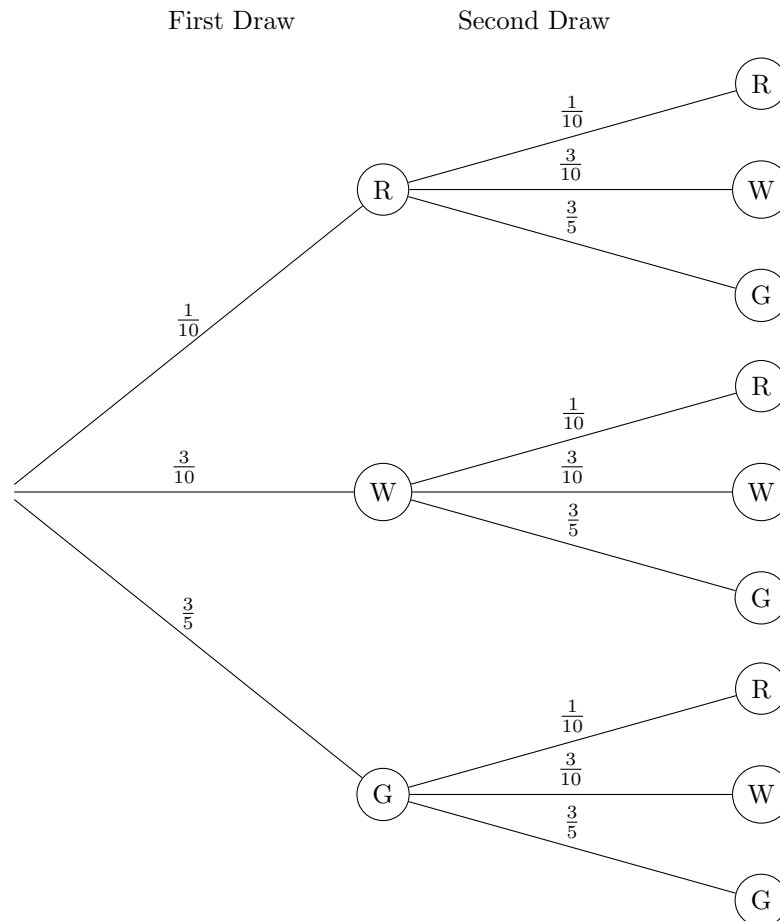
- a red ball is:  $P(R) = \frac{1}{10}$ .
- a white ball is:  $P(W) = \frac{3}{10}$ .
- a green ball is:  $P(G) = \frac{6}{10} = \frac{3}{5}$ .

The tree diagram for this one-stage experiment is



Now consider a **two-stage experiment**: We have a same box.

- **With replacement:** A ball is drawn from the above box and its color is recorded. Then the ball is *replaced*, and a second ball is drawn and its color is recorded. The tree diagram for this two-stage experiment is:



From the tree diagram, the sample space is  $\{RR, RW, RG, WR, WW, WG, GR, RW, RR\}$ .

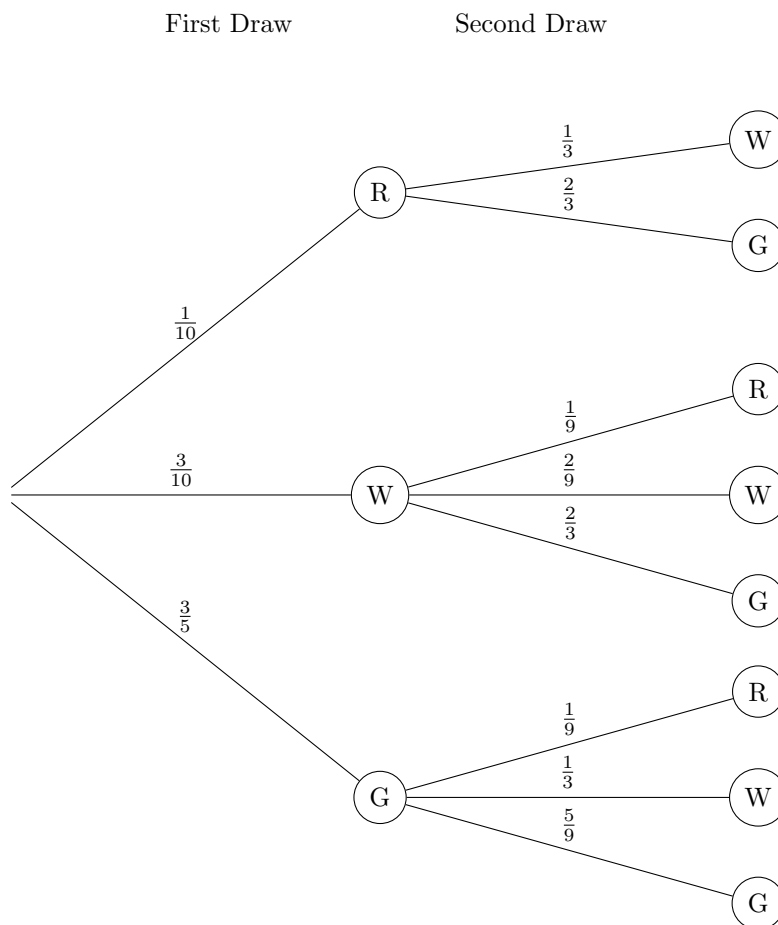
1. The probability of drawing two red balls is:  $P(RR) = \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{100}$ .
2. The probability of drawing first a red ball, then a white ball is:  $P(RW) = \frac{1}{10} \cdot \frac{3}{10} = \frac{3}{100}$ .
3. The probability of drawing first a red ball, then a green ball is:  $P(RG) = \frac{1}{10} \cdot \frac{3}{5} = \frac{3}{50}$ .
4. The probability of drawing first a white ball, then a red ball is:  $P(WR) = \frac{3}{10} \cdot \frac{1}{10} = \frac{3}{100}$ .
5. The probability of drawing two white balls is:  $P(WW) = \frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100}$ .
6. The probability of drawing first a white ball, then a green ball is:  $P(WG) = \frac{3}{10} \cdot \frac{3}{5} = \frac{9}{50}$ .
7. The probability of drawing first a green ball, then a red ball is:  $P(GR) = \frac{3}{5} \cdot \frac{1}{10} = \frac{3}{50}$ .
8. The probability of drawing first a green ball, then a white ball is:  $P(GW) = \frac{3}{5} \cdot \frac{3}{10} = \frac{9}{50}$ .
9. The probability of drawing two green balls is:  $P(GG) = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$ .

Therefore,

10. The probability of drawing two balls of the same color is:  $P(RR, WW, GG) = P(RR) + P(WW) + P(GG) = \frac{1}{100} + \frac{9}{100} + \frac{9}{25} = \frac{1}{100} + \frac{9}{100} + \frac{36}{100} = \frac{46}{100} = 46\%$ .
11. The probability of drawing two balls of different color can be determined by:
 
$$P(RW, RG, WR, WG, GR, GW) = P(RW) + P(RG) + P(WR) + P(WG) + P(GR) + P(GW).$$

Notice that the event “two balls of different colors” is the complement of the event “both ball of the same color”. Consequently,  $P(RW, RG, WR, WG, GR, GW) = 1 - P(RR, WW, GG) = 1 - 46\% = 54\%$ .
12. To draw a red and green ball, there are two possibilities: either drawing first a red ball then a green ball, or drawing first a green ball then a red ball. Consequently, the probability of drawing a red and green ball is  $P(RG, GR) = P(RG) + P(GR) = \frac{3}{50} + \frac{3}{50} = \frac{6}{50} = 12\%$ .

- **Without replacement:** A ball is drawn from the above box and its color is recorded, but the ball is *not replaced*, and a second ball is drawn and its color is recorded. The tree diagram for this two-stage experiment is:



From the tree diagram, the sample space is  $\{RW, RG, WR, WW, WG, GR, RW, RR\}$ .

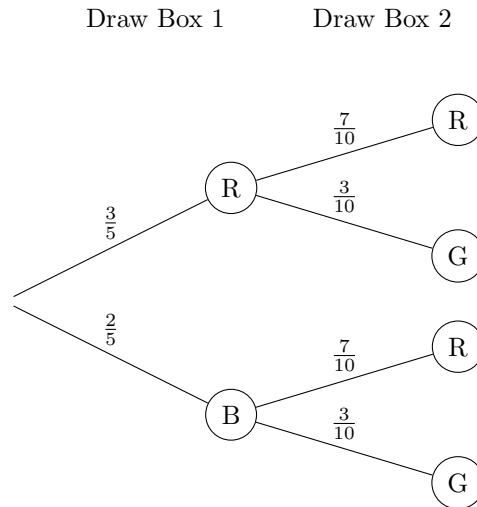
1. The probability of drawing two red balls is:  $P(RR) = 0$ .
2. The probability of drawing first a red ball, then a white ball is:  $P(RW) = \frac{1}{10} \cdot \frac{1}{3} = \frac{1}{30}$ .
3. The probability of drawing first a red ball, then a gree ball is:  $P(RG) = \frac{1}{10} \cdot \frac{2}{3} = \frac{2}{30} = \frac{1}{15}$ .
4. The probability of drawing first a white ball, then a red ball is:  $P(WR) = \frac{3}{10} \cdot \frac{1}{9} = \frac{3}{90} = \frac{1}{30}$ .
5. The probability of drawing two white balls is:  $P(WW) = \frac{3}{10} \cdot \frac{2}{9} = \frac{6}{90} = \frac{2}{30} = \frac{1}{15}$ .
6. The probability of drawing first a white ball, then a green ball is:  $P(WG) = \frac{3}{10} \cdot \frac{2}{3} = \frac{6}{30} = \frac{1}{5}$ .
7. The probability of drawing first a green ball, then a red ball is:  $P(GR) = \frac{3}{5} \cdot \frac{1}{9} = \frac{3}{45} = \frac{1}{15}$ .
8. The probability of drawing first a green ball, then a white ball is:  $P(GW) = \frac{3}{5} \cdot \frac{1}{3} = \frac{3}{15} = \frac{1}{5}$ .
9. The probability of drawing two green balls is:  $P(GG) = \frac{3}{5} \cdot \frac{5}{9} = \frac{15}{45} = \frac{1}{3}$ .

Therefore,

10. The probability of drawing two balls of the same color is:  $P(WW, GG) = P(WW) + P(GG) = \frac{1}{15} + \frac{1}{3} = \frac{1}{15} + \frac{5}{15} = \frac{6}{15} = \frac{2}{5} = 40\%$ .
11. The probability of drawing two balls of different colors can be determined by:  
 $P(RW, RG, WR, WG, GR, GW) = P(RW) + P(RG) + P(WR) + P(WG) + P(GR) + P(GW)$ .  
 Notice that the event “two balls of different colors” is the complement of the event “both ball of the same color”. Consequently,  $P(RW, RG, WR, WG, GR, GW) = 1 - P(RR, WW, GG) = 1 - 40\% = 60\%$ .

12. To draw a red and green balls, there are two possibilities: either drawing first a red ball then a green ball, or drawing first a green ball then a red ball. Consequently, the probability of drawing a red and green ball is  $P(\text{RG}, \text{GR}) = P(\text{RG}) + P(\text{GR}) = \frac{1}{15} + \frac{1}{15} = \frac{2}{15}$ .

**Example 8.24** One box contains three red balls and two blue balls. A second box contains three green balls and seven red balls. One ball is drawn from each box and the result is recorded. Let denote by R the outcome red ball, by B the outcome blue ball, and by G the outcome green ball. The tree diagram for this two-stage experiment is:



- The probability of drawing two red balls is:  $P(\text{RR}) = \frac{3}{5} \cdot \frac{7}{10} = \frac{21}{50} = 42\%$ .
- The probability of drawing one red ball and one green ball is:  $P(\text{RG}) = \frac{3}{5} \cdot \frac{3}{10} = \frac{9}{50} = 18\%$ .
- The probability of drawing one blue ball and one red ball is:  $P(\text{BR}) = \frac{2}{5} \cdot \frac{7}{10} = \frac{14}{50} = 28\%$ .
- The probability of drawing one blue ball and one green ball is:  $P(\text{BG}) = \frac{2}{5} \cdot \frac{3}{10} = \frac{6}{50} = 12\%$ .