

Solutions to Midterm Exam Emat 233

February, 2006

Problem 1. Consider the surface given by the formula

$$1 = \cos(zx)e^{z+y}$$

(a) Find the equation of the tangent plane at the point $(\pi, 0, 0)$

Solution: Let $f(x, y, z) = \cos(zx)e^{z+y}$. Then the surface given by $1 = \cos(zx)e^{z+y}$ is the level surface $\{(x, y, z) : f(x, y, z) = 1\}$. To find the tangent plane at a point we first need to find a normal vector to the surface at that point, which is given by the gradient

$$\vec{\nabla}f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Computing the partial derivatives (using the chain rule, the product rule and the fact that $e^{z+y} = e^ze^y$), we get:

$$\begin{aligned}\frac{\partial f}{\partial x} &= -z \sin(zx)e^{z+y} \\ \frac{\partial f}{\partial y} &= \cos(zx)e^{z+y} \\ \frac{\partial f}{\partial z} &= -x \sin(zx)e^{z+y} + \cos(zx)e^{z+y}.\end{aligned}$$

Plugging in $x = \pi, y = 0, z = 0$ gives:

$$\vec{\nabla}f(\pi, 0, 0) = -0 \sin(0)e^0\mathbf{i} + \cos(0)e^0\mathbf{j} + (-\pi \sin(0)e^0 + \cos(0)e^0)\mathbf{k} = 0\mathbf{i} + 1\mathbf{j} + (-0 + 1)\mathbf{k} = \mathbf{j} + \mathbf{k}.$$

This is a normal vector to the plane. Therefore any vector in the plane, of the form $(x - \pi, y - 0, z - 0)$ must be perpendicular to this vector, i.e.

$$0 = (\mathbf{j} + \mathbf{k}) \cdot ((x - \pi)\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 0(x - \pi) + y + z.$$

This gives the equation of the plane:

$$y + z = 0.$$

(b) Find normal line in symmetric or in parametric form passing through the point $(\pi, 0, 0)$.

Solution: Since we already know a normal vector at that point, namely $\vec{\nabla}f(\pi, 0, 0) = \mathbf{j} + \mathbf{k} = (0, 1, 1)$, all we need to do is give the equation of the line passing through $(\pi, 0, 0)$ in the direction of this vector. The parametric equation is:

$$\vec{r}(t) = (\pi, 0, 0) + t(0, 1, 1),$$

or in terms of coordinates: $x = \pi, y = t, z = t$.

To get the symmetric equations we eliminate the t so that we have $x = \pi, y = z$.

Problem 2 Let $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + t\mathbf{k}$ be the position vector of a moving particle.

(a) Find the velocity vector $\mathbf{v}(t)$ and the acceleration vector $\mathbf{a}(t)$ at any t .

Solution: The velocity is the derivative of the position vector:

$$\mathbf{v}(t) = \mathbf{r}'(t) = -3\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j} + 1\mathbf{k},$$

and the acceleration is the second derivative:

$$\mathbf{a}(t) = \mathbf{r}''(t) = -3 \cos(t)\mathbf{i} - 2 \sin(t)\mathbf{j}.$$

(b) Find the tangential component a_T of the acceleration vector $\mathbf{a}(t)$ at any t .

Solution: From the formula sheet, we know that

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{\|\mathbf{v}(t)\|}.$$

Computing the dot product, we get:

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = (-3 \sin(t))(-3 \cos(t)) + (2 \cos(t))(-2 \sin(t)) + 0 = 5 \sin(t) \cos(t).$$

For the speed we have

$$\|\mathbf{v}(t)\| = \sqrt{(-3 \sin(t))^2 + (2 \cos(t))^2 + 1} = \sqrt{9 \sin^2(t) + 4 \cos^2(t) + 1} = \sqrt{5 \sin^2(t) + 4 + 1} = \sqrt{5} \sqrt{\sin^2(t) + 1}.$$

Dividing, we get

$$a_T = \frac{5 \sin(t) \cos(t)}{\sqrt{5} \sqrt{\sin^2(t) + 1}} = \frac{\sqrt{5} \sin(t) \cos(t)}{\sqrt{\sin^2(t) + 1}}.$$

(c) At what point(s) does the particle pass through the xy -plane?

Solution: The xy -plane is given by the equation $z = 0$. Therefore the curve intersects it when $z = t = 0$, which gives $x = 3 \cos(0) = 3$ and $y = 2 \sin(0) = 0$. The point of intersection is thus $(3, 0, 0)$.

Problem 3. Answer the following two questions for the function

$$f(x, y) = e^{x^2+y^2}.$$

(a) Find the directional derivative of the function at the point $(1, 1)$ in the direction of the vector $-\mathbf{i} + \mathbf{j}$.

Solution: The directional derivative of the function f in the direction of the unit vector \mathbf{u} is given by

$$D_{\mathbf{u}}f = \vec{\nabla}f \cdot \mathbf{u}.$$

Computing the gradient (again using the chain rule) we get:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xe^{x^2+y^2}, \\ \frac{\partial f}{\partial y} &= 2ye^{x^2+y^2}, \end{aligned}$$

so at the point $(1, 1)$, $\vec{\nabla}f(1, 1) = 2e^2\mathbf{i} + 2e^2\mathbf{j}$.

Now we find the unit vector \mathbf{u} in the direction of $-\mathbf{i} + \mathbf{j}$:

$$\mathbf{u} = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{(-1)^2 + 1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

Taking the dot product gives:

$$D_{\mathbf{u}}f = (2e^2\mathbf{i} + 2e^2\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = -\frac{2e^2}{\sqrt{2}} + \frac{2e^2}{\sqrt{2}} = 0.$$

Note that this means that \mathbf{u} is perpendicular to the gradient at that point, hence is tangent to the level curve of f (which is a circle centered at the origin).

(b) Find the direction along which the function f increases most rapidly at the point $(1, 1)$, and find the maximum rate.

Solution: We know that the function increases most rapidly in the direction of the gradient, i.e. when \mathbf{u} is parallel to $\vec{\nabla}f$, and in that case the maximum rate of increase is

$$D_{\mathbf{u}}f = \vec{\nabla}f \cdot \frac{\vec{\nabla}f}{\|\vec{\nabla}f\|} = \|\vec{\nabla}f\|.$$

At the point $(1, 1)$, the direction of maximum increase is given by $\vec{\nabla}f(1, 1) = 2e^2\mathbf{i} + 2e^2\mathbf{j}$ (or if we want a unit vector,

$$\mathbf{u} = \frac{2e^2\mathbf{i} + 2e^2\mathbf{j}}{\|2e^2\mathbf{i} + 2e^2\mathbf{j}\|} = \frac{2e^2\mathbf{i} + 2e^2\mathbf{j}}{\sqrt{4e^4 + 4e^4}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j},$$

which is at an angle of $\pi/4$ from the horizontal. The maximum rate of increase is

$$\|2e^2\mathbf{i} + 2e^2\mathbf{j}\| = \sqrt{4e^4 + 4e^4} = 2e^2\sqrt{2}.$$

Problem 4. (a) Let $f(x, y, z) = x^2 - 2y + zx$; find $\text{grad}(f)$ and $\text{curl}(\text{grad}(f))$. Can you find also $\text{curl}(f)$ and $\text{div}(f)$? Explain.

Solution:

$$\text{grad}(f) = \vec{\nabla}f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = (2x + z)\mathbf{i} + (-2)\mathbf{j} + x\mathbf{k}.$$

$$\begin{aligned} \text{curl}(\text{grad}(f)) = \vec{\nabla} \times \vec{\nabla}f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x + z) & -2 & x \end{vmatrix} = \left(\frac{\partial}{\partial y}x - \frac{\partial}{\partial z}(-2)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial z}(2x + z)\right)\mathbf{j} + \left(\frac{\partial}{\partial y}x - \frac{\partial}{\partial z}(-2)\right)\mathbf{k} \\ &= (0 - 0)\mathbf{i} - (1 - 1)\mathbf{j} + (0 - 0)\mathbf{k} = \mathbf{0} \end{aligned}$$

In fact, this will be true (i.e. $\text{curl}(\text{grad}(f)) = \mathbf{0}$) for any function f with continuous second-order partial derivatives.

Since f is a scalar function, not a vector field, we cannot define $\text{curl}(f)$ and $\text{div}(f)$. The curl and divergence are only defined for vector fields.

(b) Let $\mathbf{F}(x, y, z) = xz\mathbf{i} + y^2\mathbf{j} + e^z\mathbf{k}$; find $\text{div}(\mathbf{F})$. Can you find $\text{grad}(\mathbf{F})$? Explain.

Solution:

$$\text{div}(\mathbf{F}) = \vec{\nabla} \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial(xz)}{\partial x} + \frac{\partial(y^2)}{\partial y} + \frac{\partial(e^z)}{\partial z} = z + 2y + e^z.$$

Since \mathbf{F} is a vector field, we cannot define $\text{grad}(\mathbf{F})$. The gradient is only defined for a scalar function.