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**Applied Ordinary Differential Equations**
**ENGR 213 - Section F**
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**Exam II (B)**

(1) (6 points) Solve the homogeneous ODE

$$x^2y'' + 3xy' + 5y = 0.$$

**Solution:** The characteristic equation of this homogeneous Cauchy-Euler ODE is $m(m-1) + 3m + 5 = 0$ or $m^2 + 2m + 5 = 0$. It has complex roots $r = -1 \pm 2i$. Hence the general solution of the ODE on $x > 0$ is

$$y(x) = c_1 x^{-1} \cos(2 \ln x) + c_2 x^{-1} \sin(2 \ln x), \quad c_{1,2} = \text{constants}.$$

(2) (14 points) Solve the initial value problem

$$y'' - 6y' + 9y = x, \quad y(0) = 0,\ y'(0) = 1.$$

**Solution:** Consider first the associated homogeneous ODE: $y'' - 6y' + 9y = 0$ with the characteristic equation $r^2 - 6r + 9 = 0$ with $r = 3$ as double root. Hence

$$y_c(x) = c_1 e^{3x} + c_2 x e^{3x}, \quad c_{1,2} = \text{constants}.$$

We now look for a particular solution $y_p$ to the non-homogenous ODE. We'll use here the method of undetermined coefficients by setting $y_p(x) = Ax + B$. As $y_p(x) = A$ and $y_p''(x) = 0$, we deduce that $-6A + 9Ax + 9B = x \quad \Rightarrow \quad A = 1/9, -6A + 9B = 0$ thus $B = 2A/3 = 2/27$ and $y_p(x) = \dfrac{x}{9} + \dfrac{2}{27}$.

Thus

$$y_{general}(x) = c_1 e^{3x} + c_2 x e^{3x} + \frac{x}{9} + \frac{2}{27}, \quad c_{1,2} = \text{constants}.$$

We'll now use the initial conditions to find $c_{1,2}$. As $y(0) = 0$, we have $c_1 + 2/27 = 0 \ \Rightarrow \ c_1 = -2/27$. Evaluating $y'(x) = c_1(3e^{3x}) + c_2(e^{3x} + 3xe^{3x}) + 1/9$, thus $y'(0) = 3c_1 + c_2 + 1/9 = 1$, implying $c_2 = 10/9$.

Therefore the solution of the IVP is

$$y(x) = -\frac{2}{27} e^{3x} + \frac{10}{9} x e^{3x} + \frac{x}{9} + \frac{2}{27}.$$

(3) (10 points) Use the variation of parameters to solve the differential equation

$$y'' + y = \sin^2 x.$$

**Solution:** The complementary part of the solution follows from $r^2+1=0 \Rightarrow r=\pm i$ and is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

Considering $y_1(x) = \cos x,\ y_2(x) = \sin x$, the Wronskian is $W(x) = 1 \neq 0$ for all real $x$'s. To find the complementary solution we calculate $W_1(x) = \det\begin{pmatrix} 0 & \sin x \\ \sin^2 x & \cos x\end{pmatrix} = -\sin^3 x$ and $W_2(x) = \det\begin{pmatrix} \cos x & 0 \\ -\sin x & \sin^2 x\end{pmatrix} = \sin^2 x \cos x$.

The method of variation of parameters gives $y_p(x) = y_1(x)u_1(x) + y_2(x)u_2(x)$, where $u_1'(x) = W_1(x)/W(x)$ and $u_2'(x) = W_2(x)/W(x)$.

Integrating (by taking $u = \sin x\ \ du = \cos x\, dx$) and taking the constant of integration to be zero, we have

$$u_2(x) = \int (\sin^2 x \cos x)\, dx = \int u^2\, du = \frac{u^3}{3} = \frac{\sin^3 x}{3}.$$

On the other hand,

$$u_1(x) = -\int \sin^3 x\, dx = -\int \sin x\,(1-\cos^2 x)\, dx = \int (1-u^2)\, du = u - \frac{u^3}{3} = \cos x - \frac{\cos^3 x}{3},$$

where above we used the fundamental identity of trigonometry ($\sin^2 x + \cos^2 x = 1$) and the substitution $u = \cos x,\ du = -\sin x\, dx$.

Consequently,

$$y_p(x) = \cos x \cdot \left(\cos x - \frac{\cos^3 x}{3}\right) + \sin x \cdot \frac{\sin^3 x}{3}$$

and

$$y(x) = c_1 \cos x + c_2 \sin x - \frac{\cos^4 x}{3} + \cos^2 x + \frac{\sin^4 x}{3}, \quad c_{1,2} = \text{arbitrary constants.}$$

(4) (10 points) A mass weighing 64 pounds stretches a spring 0.32 foot. Determine the equation of motion if the mass is initially released from a point 6 inches above the equilibrium position with a downward velocity of 5 ft/s. What is the instantaneous velocity at the first time when the mass passes through the equilibrium position?

**Solution:** The equation of motion is $mx'' + kx = 0$, where $m = 64/32$ slug and $k = 64/0.32 = 200$ ft/lb. Thus

$$x'' + 100x = 0 \quad \Rightarrow \quad x(t) = c_1 \cos 10t + c_2 \sin 10t,\ c_{1,2} = \text{constants.}$$

To determine the constants, use the initial conditions. As $x(0) = -1/2$ ft, $c_1 = -1/2$. Additionally, $x'(t) = -10c_1 \sin 10t + 10c_2 \cos 10t$, thus $x'(0) = 5$ ft/sec and $10c_2 = 5$.

Consequently, $x(t) = -\frac{1}{2}\,\cos 10t + \frac{1}{2}\,\sin 10t$ and $x'(t) = 5\,\sin 10t + 5\,\cos 10t$. To find the time when the mass passes through the equilibrium position, set $x(t) = 0$. Note that this implies $\tan(10t) = 1$ whose first positive solution is for $10t = \pi/4$. So

$$x'(\pi/40) = 5\,\frac{\sqrt{2}}{2} + 5\,\frac{\sqrt{2}}{2} = 5\sqrt{2}\ \text{ft/sec}$$

is the answer needed.