FACULTY OF ENGINEERING AND COMPUTER SCIENCS
CONCORDIA UNIVERSITY
APPLIED ADVANCED CALCULUS (ENGR 233) FINAL EXAMINATION
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Special instructions: Do all problems
No calculators are allowed

PROBLEM No. 1 (10 MARKS)

(a) Find the equation of the plane which contains the three points (2, 1, 0), (3, 4, 0), and (1, 1, 1)

(b) Find the parametric equations of the line through (3, 4, 0), which is perpendicular to the plane found in part (a).

PROBLEM No. 2 (10 MARKS)

(a) Let $w = \sqrt{x^3 + y} + e^{xz}$, and let $x = 2t$, $y = t^2$, and $z = t^{-1}$. Use the multivariate chain rule to find $\frac{dw}{dt}$ when $t = 1$.

(b) Suppose that the temperature distribution of the plane is given by $T(x,y) = 5x^2 + e^{xy}$. If an ant is sitting at the point (2,3), in which unit direction should it move in order to cool off as fast as possible? What is the directional derivative of T in that direction?

PROBLEM No. 3 (10 MARKS). Find point(s) on the surface $z = 10 - x^2 - y^2$ at which the tangent plane is parallel to the plane $x + \frac{3}{2}y + \frac{1}{2}z + d = 0$, where d is a constant.

PROBLEM No. 4 (10 MARKS). Find the moment of inertia about the y-axis of the lamina that has the given shape and density.

bounded by $x = 0$, $y = x$, $y = 1$; $\rho(x, y) = \sqrt{1 + y^4}$

PROBLEM No. 5 (10 MARKS). For the scalar function

$$f(x, y, z) = e^{x^2} \cos z + z^4 \sin y$$

compute the following quantities if they **make sense**. If not, explain why.

(a) **grad** f (b) div(**grad** f) (c) **grad**(div f) (d) **curl**(**grad** f) (e) **grad**(**grad** f)

PROBLEM No. 6 (10 MARKS). Compute the line integral

$$\int_C -y\,dx + x\,dy$$

for the following curves

(a) C is the curve from (3,0) to (−3,0) lying along the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in the xy plane, $y \geq 0$.

(b) C is the closed curve in the xy plane described, in the counterclockwise sense, by $r = 2\cos\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

PROBLEM No. 7 (10 MARKS). Prove that the line integral

$$I = \int_C \left(1 + e^{-y}\right)dx - \left(xe^{-y} + 4y\right)dy$$

is independent of the path, and then evaluate I when C is any path from (1, 0) to (2, 1).

PROBLEM No. 8 (10 MARKS). Find the outward flux of the radial vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the boundary of domain in R^3 given by two inequalities $x^2 + y^2 + z^2 \leq 2$ and $z \geq x^2 + y^2$.

PROBLEM No. 9 (10 MARKS). D is the region of three-dimensional space that is given by the inequalities $x^2+y^2+z^2 \leq 1$, $4z^2 \leq x^2+y^2+z^2$ and $z \geq 0$. Find the volume of D.

PROBLEM No. 10 (10 MARKS). A field of force: $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$ exists within a region of space.

(a) Solve $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where the closed contour "C" is the intersection of $x^2 + y^2 = 1$ and $z = y^2$ using Stokes Theorem. Show all of your work and justify your answer.

(b) If the object were moved along the closed path "C" 16 times, how much more work would be done than if the object were moved along the closed path just once? Justify your answer.

$\cos\theta = \vec{a} \bullet \vec{b} / (\|\mathbf{a}\| \|\mathbf{b}\|)$

$\text{comp}_{\mathbf{b}}\mathbf{a} = \|\mathbf{a}\| \cos\theta = \mathbf{a} \cdot \hat{\mathbf{b}}$

$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\vec{\mathbf{a}} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}$

Area of a parallelogram = $\| \mathbf{a} \times \mathbf{b} \|$

Volume of a parallelepiped = $| \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) |$

Equation of a line: $\vec{r} = \vec{r}_2 + t(\vec{r}_2 - \vec{r}_1) = \vec{r}_2 + t\vec{a}$

Equation of a plane: $a\,x + b\,y + cz + d = 0$

also: $\left[(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)\right] \bullet (\vec{r} - \vec{r}_1) = 0$

$$\frac{d\vec{r}(s)}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

Length of a curve : $s = \int_{t_1}^{t_2} |\vec{r}\,'(t)|\, dt$

$$\kappa = \left\|\frac{d\vec{\mathbf{T}}}{ds}\right\| = \left\|\frac{d^2\vec{\mathbf{r}}}{ds^2}\right\| = \frac{|\vec{\mathbf{T}}'|}{|\vec{\mathbf{r}}'|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

$$\vec{\mathbf{a}}(t) = \kappa v^2 \hat{\mathbf{N}} + \frac{dv}{dt}\hat{\mathbf{T}} = a_N \hat{\mathbf{N}} + a_T \hat{\mathbf{T}}$$

$$\hat{\mathbf{N}} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$$

$$\hat{\mathbf{T}} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

The Binormal $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$

$$a_T = \frac{dv}{dt} = \frac{\|\mathbf{v} \bullet \mathbf{a}\|}{\|\mathbf{v}\|} \quad \& \quad a_N = kv^2 = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \& \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$D_u(F) = \nabla F \bullet \hat{u}$, $\hat{u}$ = unit vector

Equation of Tangent Plane: $\vec{n}_o \bullet \left(\vec{r} - \vec{r}_o\right) = 0,\ \vec{n}_o = \nabla F$ at P

$$W = \int_C \vec{F} \bullet d\vec{r}$$

equation of normal line to a surface: $\vec{n}_o \times \left(\vec{r} - \vec{r}_o\right) = 0,\ \vec{n}_o = \nabla F$ at P

$$\int_C F(x,y)ds = \int_a^b F\big(f(t), g(t)\big)\sqrt{[f']^2 + [g']^2}\ dt = \int_a^b F\big(x, f(x)\big)\sqrt{1 + [f']^2}\ dx$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S (curl\ \vec{F}) \bullet \hat{n}dS$$

$$\oiint_S (\vec{F} \bullet \hat{n})dS = \iiint_D (div\ \vec{F})dV$$

$$\oint_C [Pdx + Qdy] = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy$$

$$\tilde{x} = \frac{\iiint_D x\rho(x,y,z)dV}{m},$$

$$m = \iiint_D \rho(x,y,z)dV$$

$$I_x = \iiint_D (y^2 + z^2)\rho(x,y,z)dV;$$

$x = r\cos\theta$, $y = r\sin\theta$; $z = z$; $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$$

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$,

$\rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}(y/x)$, $\phi = \tan^{-1}(\sqrt{x^2 + y^2}/z)$

$dV = r\ dr\, d\theta\, dz$

$dV = \rho^2\sin\phi\ d\rho d\phi\, d\theta$