Department of Mathematics, University of Toronto

MAT224H1S - Linear Algebra II
Winter 2016

Writing Assignment 5

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1. Recall from Writing Assignment 1 that a semimagic square is an $n \times n$ matrix with real entries in which the sum of the entries in every row and every column is the same.

Prove that an $n \times n$ matrix A is a semimagic square if and only if the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ (every entry is 1) is an eigenvector of both A and A^T with the same eigenvalue.

A is a semimagic square matrix

$\Rightarrow \sum_{j=1}^{n} a_{ij} = c$, for $i = 1, 2, 3, \dots, n$

& $\sum_{i=1}^{n} a_{ij} = c$, for $j = 1, 2, 3, \dots, n$

the sum of elements along any row $\sum_{i=1}^{n} a_{ij}$ or along any column $\sum_{j=1}^{n} a_{ij}$ is equal to a magic number such as $\underline{c}$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} \\ \sum_{j=1}^{n} a_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{bmatrix} = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

therefore, $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector to A ; eigenvalue is c.

consider $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$

Extra page for question 1, if necessary.

now,
$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{i1} \\ \sum_{i=1}^{n} a_{i2} \\ \vdots \\ \sum_{i=1}^{n} a_{in} \end{bmatrix} = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Therefore $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector to A^T, and eigenvalue is c.

2. Let $T : V \to V$ be a linear operator that has only two distinct eigenvalues λ_1 and λ_2. Prove that T is diagonalizable if and only if $V = E_{\lambda_1} \oplus E_{\lambda_2}$.

$T: V \longrightarrow V$ (linear operator)

two distinct eigenvalues λ_1 & λ_2

Prove: T is diagonalizable if and only if $V = E_{\lambda_1} \oplus E_{\lambda_2}$

Let's assume that T is diagonalizable, then T has a basis of eigenvectors with eigenvalues λ_1 & λ_2.

As T has only two distinct eigenvalues, it is clear that E_{λ_1} & E_{λ_2} together span V and their intersection is $\{0\}$.

So $V = E_{\lambda_1} \oplus E_{\lambda_2}$

Conversely, if we assume $V = E_{\lambda_1} \oplus E_{\lambda_2}$

then E_{λ_1} & E_{λ_2} together span V.

Thus means V has a basis consisting of eigenvectors of λ_1 & λ_2. T is clearly a diagonal matrix, with entries with respect to this basis,

λ_1, λ_2

Thus T is diaonalizable.

Extra page for question 2, if necessary.