Student Number:

YORK UNIVERSITY
FACULTY OF LIBERAL ARTS AND PROFESSIONAL STUDIES – DEPARTMENT OF ECONOMICS

AP/ECON 1540.03 SECTION O
Introductory Mathematics for Economists II

MIDTERM Solutions

INSTRUCTOR: Gordana Colby

Tuesday, February 9th, 2016
1 HOUR 15 MINS (1:00PM – 2:15PM)
Surnames A-K: CB 121
Surnames L-S: CLH H
Surnames T-Z: TEL 0006

NAME: ______________________ ______________________
LAST NAME FIRST NAME

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

INSTRUCTIONS:

1. **Make sure to print your NAME, STUDENT NUMBER and SIGN the sheet above. Write your student number at the top right of each page.**
2. You have **1 HOUR 15 MINUTES** to complete the exam.
3. Read the examination questions **VERY CAREFULLY**.
4. Answer **ALL** questions in the space provided on the exam – if you need more space, please use the back of the sheet for that question. **SHOW ALL YOUR WORK.** No marks will be given for unsubstantiated answers.
5. The examination consists of twelve (12) pages. Please ensure you have a complete examination.
6. The examination is out of 100 marks.

Formula

For a function $F(x,y) = c,\ y'' = -\frac{1}{(F_2')^3}\left(F_{11}''(F_2')^2 - 2F_{12}''F_1'F_2' + F_{22}''(F_1')^2\right)$

Student Number:

Question #1 (20 Marks)

A function f is given by:

$$f(x, y, z) = 3xyz + x^2y - xz^3$$

a) (6 Marks) Find all the first-order partial derivatives.

$f'_x = 3yz + 2xy - z^3$ — 2 Marks

$f'_y = 3xz + x^2$ — 2 Marks

$f'_z = 3xy - 3xz^2$ — 2 Marks

b) (14 Marks) Find all the second-order partial derivatives.

$f''_{xx} = 2y$ — 2 Marks

$f''_{xy} = f''_{yx} = 3z + 2x$ — 3 Marks

$f''_{xz} = f''_{zx} = 3y - 3z^2$ — 3 Marks

$f''_{yy} = 0$ — 2 Marks

$f''_{yz} = f''_{zy} = 3x$ — 2 Marks

$f''_{zz} = -6xz$ — 2 Marks

Question #2 (20 Marks)

Consider the following function: $z = 10(x+2)^2(y+3)^3$.

a) (10 Marks) Find the elasticity of *z* with respect to *x*.

$$El_x z = \frac{\partial z}{\partial x} \cdot \frac{x}{z}$$

$$\frac{\partial z}{\partial x} = 10(y+3)^3 \cdot 2 \cdot (x+2) \cdot (1) = 20(x+2)(y+3)^3$$

10mks $\therefore El_x z = 20\cancel{(x+2)}\cancel{(y+3)^3} \cdot \frac{x}{10(x+2)^{\cancel{2}}\cancel{(y+3)^3}}$

$$\therefore El_x z = \frac{2x}{x+2}$$

b) (10 Marks) Find the elasticity of *z* with respect to *y*.

$$El_y z = \frac{\partial z}{\partial y} \cdot \frac{y}{z}$$

$$\frac{\partial z}{\partial y} = 10(x+2)^2 \cdot 3(y+3)^2 (1) = 30(x+2)^2(y+3)^2$$

10mks

$$\therefore El_y z = 30(x+2)^2(y+3)^2 \cdot \frac{y}{10(x+2)^2(y+3)^3}$$

$$\therefore El_y z = \frac{3y}{y+3}$$

Student Number:

Question #3 (20 Marks)

The equation

$$x^3 \ln x + y^3 \ln y = 2z^3 \ln z$$

defines z as a differentiable function of x and y in a neighbourhood of the point $(x, y, z) = (e, e, e)$.

a) (8 Marks) Calculate $z_1'(e, e)$.

Differentiating both sides with respect to x, while holding y constant give

$$3x^2 \cdot \ln x + \frac{1}{x} x^3 = 6z^2 \ln z \; z_1' + \frac{1}{z} 2z^3 \cdot z_1'$$

$$3x^2 \cdot \ln x + x^2 = (6z^2 \ln z + 2z^2) z_1' \quad (*)$$

$$\therefore \; z_1' = \frac{3x^2 \cdot \ln x + x^2}{6z^2 \ln z + 2z^2}$$

8 mks

When $x = y = z = e$,

$$z_1' = \frac{3e^2 \cdot \ln e + e^2}{6e^2 \cdot \ln e + 2e^2} = \frac{4e^2}{8e^2} = \frac{1}{2}$$

b) (12 Marks) Calculate $z''_{11}(e, e)$.

Differentiating (*) a second time with respect to x yields:

$$6x + \ln x + \frac{1}{x} 3x^2 + 2x = \left(12z \ln z + \frac{1}{z} 6z^2 + 4z\right)(z_1')^2$$

$$+ z''_{11}\left(6z^2 \ln z + 2z^2\right)$$

$$\therefore z''_{11} = \frac{6x \cdot \ln x + 5x - (12z \ln z + 10z)(z_1')^2}{(6z^2 \ln z + 2z^2)}$$

When $x = y = z$, $z_1' = \frac{1}{2}$ and:

$$z''_{11} = \frac{6e \cdot \ln e + 5e - (12e \cdot \ln e + 10e)\left(\frac{1}{2}\right)^2}{(6e^2 \cdot \ln e + 2e^2)}$$

$$= \frac{6e + 5e - \left(22e \cdot \frac{1}{4}\right)}{8e^2}$$

12 mks

$$= \frac{11e - \left(22e \cdot \frac{1}{4}\right)}{8e^2}$$

$$= \frac{44e - 22e}{4} \cdot \frac{1}{8e^2}$$

$$\therefore z''_{11} = \frac{22e}{32e^2} = \frac{11}{16e}$$

Student Number:

Question #4 (20 Marks)

Consider the following production function:

$$Q(K,L) = 75(0.3K^{-0.4} + 0.7L^{-0.4})^{-2.5}$$

a) (8 Marks) Is this a homogeneous function and if so, what is the degree of homogeneity? (Be sure to prove your answer.)

$$Q = 75(0.3K^{-0.4} + 0.7L^{-0.4})^{-2.5}$$

$$= 75(0.3(tK)^{-0.4} + 0.7(tL)^{-0.4})^{-2.5}$$

$$= 75(t^{-0.4}(0.3K^{-0.4} + 0.7L^{-0.4}))^{-2.5}$$

$$= 75[(t^{-0.4})^{-2.5}(0.3K^{-0.4} + 0.7L^{-0.4})^{-2.5}]$$

$$= 75t(0.3K^{-0.4} + 0.7L^{-0.4})^{-2.5}$$

$$= t \cdot Q$$

$\therefore$ homogeneous of degree 1.

b) (12 Marks) Find the elasticity of substitution.

$$Q'_K = 75(-2.5)\left(0.3K^{-0.4}+0.7L^{-0.4}\right)^{-3.5}\cdot(0.3)(-0.4)K^{-1.4}$$

$$Q'_L = 75(-2.5)\left(0.3K^{-0.4}+0.7L^{-0.4}\right)^{-3.5}\cdot(0.7)(-0.4)L^{-1.4}$$

$$\therefore R_{KL} = \frac{Q'_L}{Q'_K}$$

$$= \frac{75(-2.5)\left(0.3K^{-0.4}+0.7L^{-0.4}\right)^{-3.5}\cdot(0.7)(-0.4)L^{-1.4}}{75(-2.5)\left(0.3K^{-0.4}+0.7L^{-0.4}\right)^{-3.5}(0.3)(-0.4)L^{-1.4}}$$

$$= \frac{0.7L^{-1.4}}{0.3K^{-1.4}} = 2.33\left(\frac{K}{L}\right)^{1.4}$$

$$\therefore \left(\frac{K}{L}\right)^{1.4} = \frac{1}{2.33}R_{KL}$$

$$\therefore \frac{K}{L} = \left(\frac{1}{2.33}\right)^{\frac{1}{1.4}}\cdot R_{KL}^{\frac{1}{1.4}}$$

$$\therefore \sigma_{KL} = \frac{1}{1.4} = \underline{\underline{0.71}}$$

<u>OR</u> $$\ln\left(\frac{K}{L}\right) = \frac{1}{1.4}\ln\left(\frac{1}{2.33}\right) + \frac{1}{1.4}\ln(R_{KL})$$

$$\sigma_{KL} = \frac{\partial\ln(K/L)}{\partial\ln(R_{KL})} = \frac{1}{1.4} = \underline{\underline{0.71}}$$

10

Student Number:

Question #5 (20 Marks)

a) (12 marks) The following system of equations defines $u = u(x, y)$ and $v = v(x, y)$ as differentiable functions of x and y around the point $P(x, y, u, v) = (1, 1, 1, 2)$:

$$u^\alpha + v^\beta = 2^\beta x + y^3$$
$$u^\alpha v^\beta - v^\beta = x - y$$

where α and β are positive constants. Differentiate the system. Then find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ at the point P.

derivative

$$\alpha u^{\alpha-1} du + \beta v^{\beta-1} dv = 2^\beta dx + 3y^2 dy$$

$$\alpha u^{\alpha-1} v^\beta du + \beta v^{\beta-1} u^\alpha dv - \beta v^{\beta-1} dv = dx - dy$$

At point P, these equations become:

$$\alpha (1)^{\alpha-1} du + \beta 2^{\beta-1} dv = 2^\beta dx + 3(1)^2 dy$$

$$\alpha du + \beta 2^{\beta-1} dv = 2^\beta dx + 3 dy \quad ①$$

and

$$\alpha (1)^{\alpha-1} (2)^\beta du + \beta (2)^{\beta-1} (1)^\alpha dv - \beta (2)^{\beta-1} dv = dx - dy$$

$$\alpha 2^\beta du + \beta 2^{\beta-1} dv - \beta 2^{\beta-1} dv = dx - dy$$

$$\alpha 2^\beta du = dx - dy$$

$$du = \frac{2^{-\beta}}{\alpha} dx - \frac{2^{-\beta}}{\alpha} dy \quad ②$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2^{-\beta}}{\alpha} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-2^{-\beta}}{\alpha}$$

7

Substitute ② into ① to yield:

$$\alpha\left(\frac{2^{-\beta}}{\alpha}\,dx - \frac{2^{-\beta}}{\alpha}\,dy\right) + \beta\,2^{\beta-1}\,dV = 2^{\beta}\,dx + 3\,dy$$

$$2^{-\beta}\,dx - 2^{-\beta}\,dy + \beta\,2^{\beta-1}\,dV = 2^{\beta}\,dx + 3\,dy$$

$$\beta\,2^{\beta-1}\,dV = \left(2^{\beta} - 2^{-\beta}\right)dx + \left(3 + 2^{-\beta}\right)dy$$

$$\therefore\ dV = \frac{\left(2^{\beta} - 2^{-\beta}\right)}{\beta\,2^{\beta-1}}\,dx + \frac{\left(3 + 2^{-\beta}\right)}{\beta\,2^{\beta-1}}\,dy$$

$$\therefore\ \frac{\partial V}{\partial x} = \frac{\left(2^{\beta} - 2^{-\beta}\right)}{\beta\,2^{\beta-1}} \quad \text{and} \quad \frac{\partial V}{\partial y} = \frac{\left(2^{-\beta} + 3\right)}{\beta\,2^{\beta-1}}$$

b) (8 Marks) For the function $u(x, y)$ in part a), find the approximation to $u(0.99, 1.01)$.

$$u(0.99, 1.01) \simeq u(1,1) + \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$\simeq 1 + \frac{2^{-\beta}}{\alpha}(-0.01) - \frac{2^{-\beta}}{\alpha}(0.01)$$

$$\simeq 1 - \frac{2^{-\beta}}{100\alpha} - \frac{2^{-\beta}}{100\alpha}$$

$$\simeq 1 - 2\left(\frac{2^{-\beta}}{100\alpha}\right)$$

$$\simeq 1 - \frac{2^{-\beta}}{50\alpha}$$