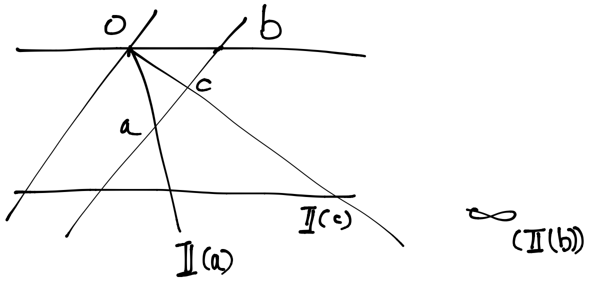
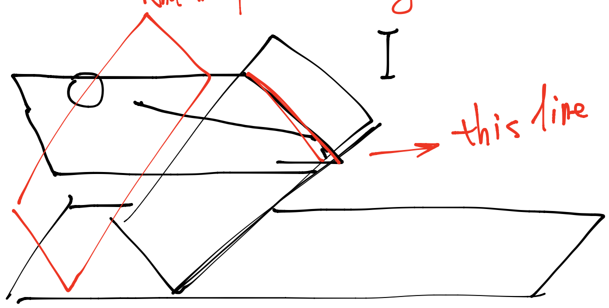
Lecture 8
Projective Geometry

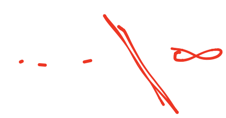
Projection basically 1-1 and onto, but 2 exceptions

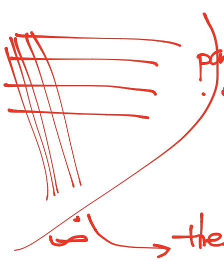


And this plane containing O



this line is projected to ∞





parallel : no common pts

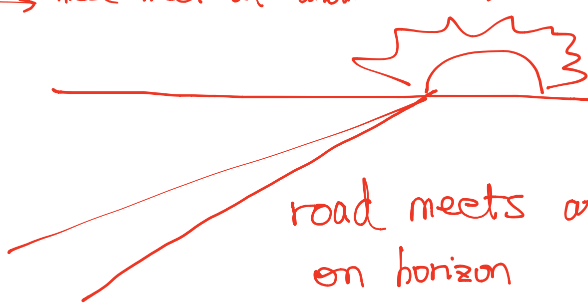
∞

but in projective geo
they meet at ∞

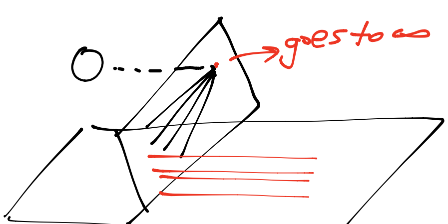
different !

∞ these meet at another ∞

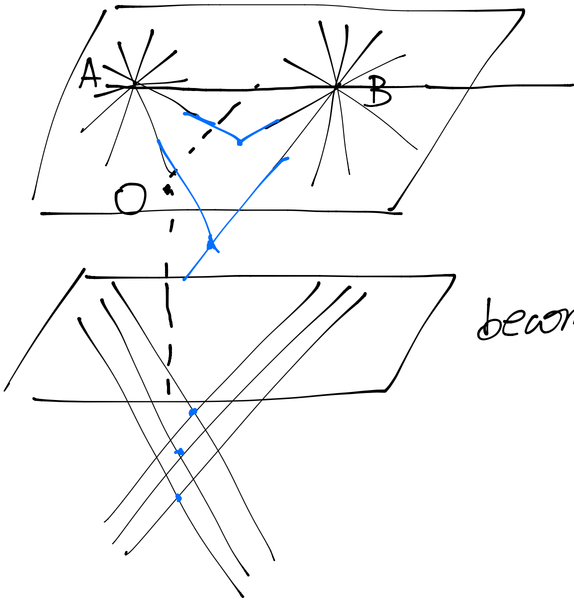
e.g.



road meets at 1 pt
on horizon



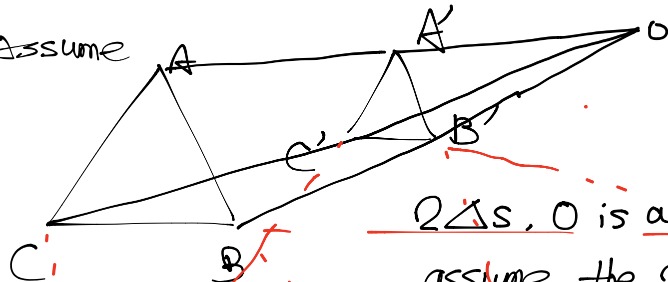
lines passing 1 (∞) pt go to parallel lines



parallel to floor

become 2 families of parallel lines

Assume



2 Δs, O is a pt

assume the 3 pairs of similar sides meet at 3 pts

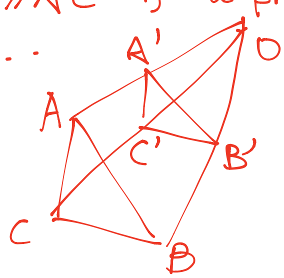
And these 3 pts on the same line.

~~Statement~~ Invariant after projection b/c line is still line.

· $AC \parallel A'C'$ if a projection sends their intersection pt to ∞.

. . . . . .

Now



know $AC \parallel A'C'$, $AB \parallel A'B'$

prove $BC \parallel B'C'$

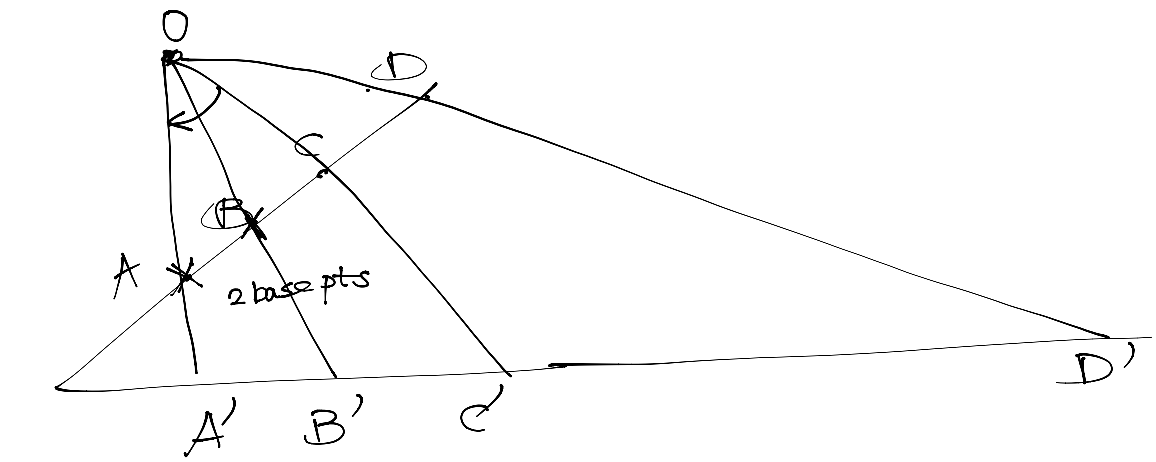
$\Delta ACO \sim \Delta A'C'O$

$\Delta ABO \sim \Delta A'B'O$

$\Rightarrow \Delta BCO \sim \Delta B'C'O$

$\Rightarrow BC \parallel B'C'$

projection preserves our problem

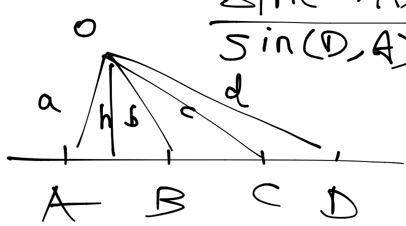


$$\frac{C-A}{D-A} : \frac{C-B}{D-B} = (A,B,C,D)$$

True for A', B', C', D'.

projection: length does not survive
quotient does not survive
quotient of quotient does

$$\frac{\sin(C,A)}{\sin(D,A)} : \frac{\sin(C,B)}{\sin(D,B)}$$ holds since α not changed!



$$\frac{S_{\triangle OAC}}{S_{\triangle OAD}} : \frac{S_{\triangle OBC}}{S_{\triangle OBD}}$$

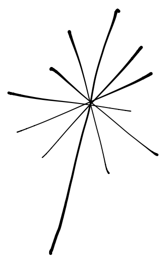
b/c $S_{\triangle OAC} = \frac{1}{2} ac \sin(C,A)$

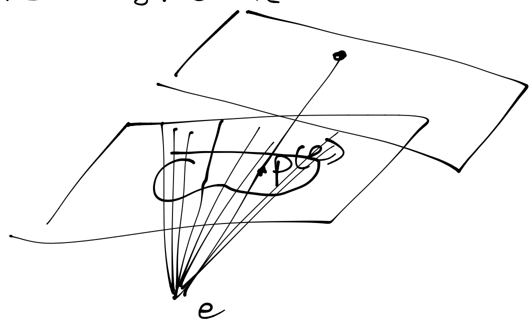
$$\frac{\frac{1}{2}h(C-A)}{\frac{1}{2}h(D-A)} : \frac{\frac{1}{2}h(C-B)}{\frac{1}{2}h(D-B)}$$

then we get a desired result

$\mathbb{R}P^n \longleftrightarrow \mathbb{R}^{n+1}$

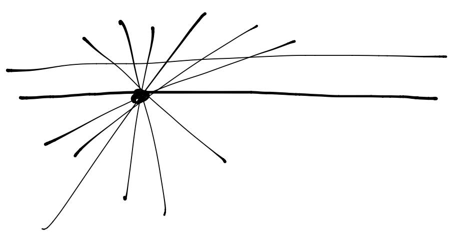
a point in $\mathbb{R}P^2$ is a line through O in $\mathbb{R}^3$



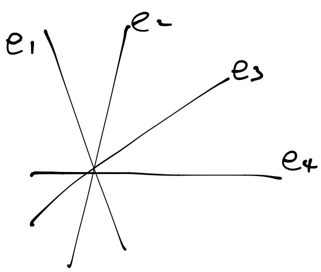


$X \in \mathbb{R}^3$

$x \sim y$ if $x = \lambda y$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$



$\mathbb{R}P^1$ is like a set of all line on $\mathbb{R}^2$ passing through 1 pt

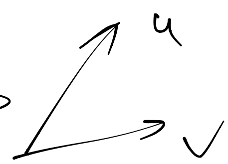


Linear Trans

$\Longrightarrow$

Ae_1
Ae_2
Ae_3
Ae_4

Choose basis



$e_1 = a_{11}u + a_{21}v$

$e_2 = a_{12}u + a_{22}v$

$\det(e_1, e_2) = \det\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$\frac{\det(e_3, e_1)}{\det(e_4, e_1)} : \frac{\det(e_3, e_2)}{\det(e_4, e_2)}$$

if e_1, e_2 ... multiple any λ_i, it will be canceled so don't worry.

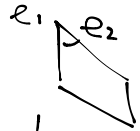
if chang basis , $u = c_{11}f + c_{21}g$

$v = c_{12}f + c_{22}g$

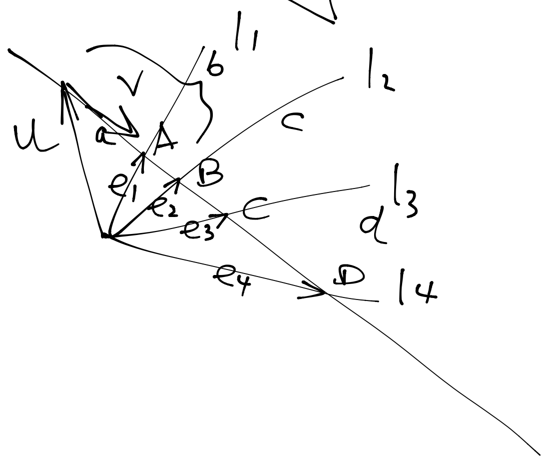
$\det\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$

orthonormal: $(e_1 \cdot e_1) = 1$

$(e_2 \cdot e_2) = 1$



$\det(e_1, e_2) =$ Area of ▱



$e_1 = u + av$

$e_2 = u + bv$

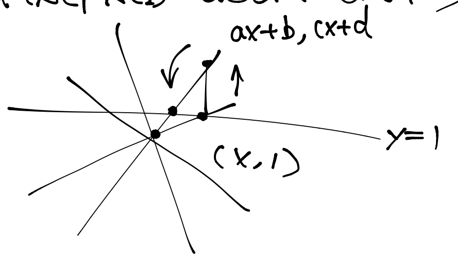
$e_3 = u + cv$

$e_4 = u + dv$

$$\det(e_3\ e_1) = \begin{vmatrix} 1 & 1 \\ c & a \end{vmatrix} = a - c$$

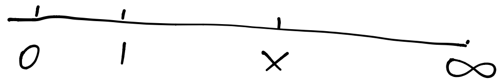
$$\frac{\det(Ae_3, Ae_1)}{\det(Ae_4, Ae_1)} : \frac{\det(e_3, e_2)}{\det(e_4, e_2)}$$

$\det(Ae_1, Ae_2) = \det A \det(e_1 e_2)$ cancel out A



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ cx+d \end{pmatrix}$$ some point

$\frac{ax+b}{cx+d}$ fractional linear transformation

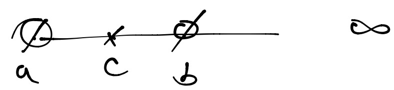


4 pts on line
find cross ratio
base pts $0, \infty$.

$(0, \infty, x, 1)$

$$\frac{x-0}{1-0} : \frac{\infty - x}{\infty - 1} = x$$

what if cross-ratio $= -1$



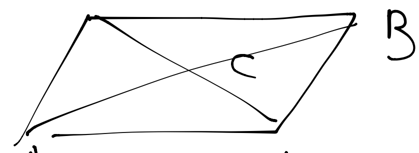
$$\frac{c-a}{\infty-a} : \frac{c-b}{\infty-b} = \frac{c-a}{c-b} : 1 = -1$$

so c is exactly in the mid of a and b.

$(A, B, C, D) = -1$

$\Rightarrow$ take any of them $= \infty$,
then one of the rest is exactly the mid of the other two.

Vetex
something
4-gon
complete 4-gon



$(A, B, C, \infty) = -1$