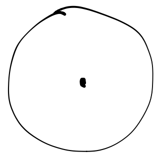
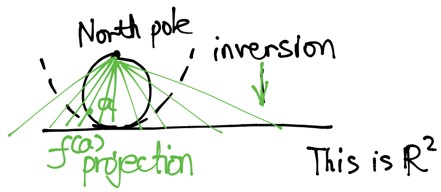
Lecture 7

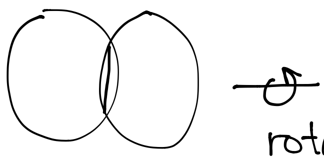
Mobius transformation

Consider $\mathbb{R}^2 \cup \{\infty\}$





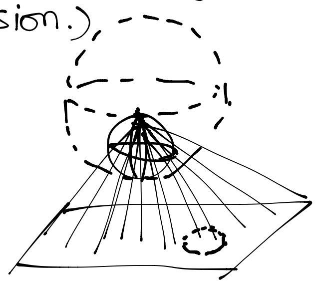
Think in $\mathbb{R}^3$
go 1-1 to a plane



rotate
we get 2 spheres intersecting each other

intersection of 2 spheres is a circle.

Projection: circle projects to a circle on plane (actually in this case, it's an inversion.)

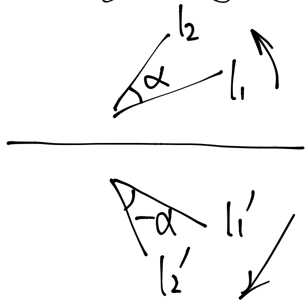


$f: \mathbb{R}^2 \cup \{\infty\} \longrightarrow \mathbb{R}^2 \cup \{\infty\}$

2 properties
1. ⊙ goes to ⊙ (equal figure goes to equal figure)
2. preserves angles

any compositions will have the same properties

Let symmetry of angle α has negative value $-\alpha$



Complex variables background

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0 \quad \text{i.e. not proportional}$$

$a, b, c, d \in \mathbb{C}$

This formula is important

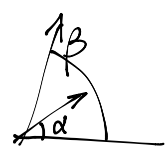
$z \to az$

$|az| = |a||z|$ assume $|a| = 1$

$\arg az = \arg a + \arg z$

$\alpha + \arg z = \beta$

$\arg a = \alpha$



a: ① rotation by argument ② times scalar

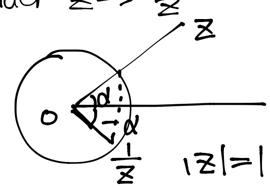
$\Rightarrow \boxed{z \to az + b}$

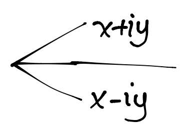
geometric meaning:
if $|a| = 1$,
then it's Euclidean

$z \to z + b$

<u>Shift</u> by b

Consider $z \to \frac{1}{z}$





$\left|\frac{1}{z}\right| = \frac{1}{|z|}$ b/c $\frac{1}{|z|} \cdot |z| = 1$

Think $z \to \frac{1}{z}$

$\overline{\left(\frac{1}{z}\right)} = \arg z$

$z \to \frac{1}{z}$ is actually
is composition of (inversion + symmetry) ✓

⊗ can be proved that it stands for all types of transformation

$$f(z)=\frac{az+b}{cz+d}=\frac{\frac{a}{c}(cz+d)-\frac{ad}{c}+b}{cz+d}$$

$$=\frac{a}{c}+\frac{\frac{ad}{c}+b}{cz+d}$$

$$=A+\frac{B}{cz+d}$$

$$=A+\frac{B}{c(z+\frac{d}{c})}$$

$z\to V=z+\frac{d}{c}$ $\boxed{\text{shift}}$

$V\to W=\frac{1}{V}$ i.e. $z\to\frac{1}{z+\frac{d}{c}}$ $\boxed{\text{inversion + symmetry proved above}}$

$W\to\frac{B}{c}\frac{1}{V}$ i.e. $z\to\frac{B}{c(z+\frac{d}{c})}$ $\boxed{\text{scalar}}$

. - - -

Any transformation has this property.(be composition of these 3 types)

Property 1: given any different pts $A_1,B_1,C_1\in\mathbb{R}^2\cup\{\infty\}$ and 3 different pts A_2,B_2,C_2

$\exists\, f(z)=\frac{az+b}{cz+d}$ s.t. $f(A_1)=A_2$, $f(B_1)=B_2$, $f(C_1)=C_2$

Plug in A_1,A_2,B_1,B_2,C_1,C_2, solve for a,b,c,d done. cannot do for 4 pts

Property 2: Given 4 pts x_1,x_2,x_3,x_4.

consider $[x_1,x_2,x_3,x_4]=$

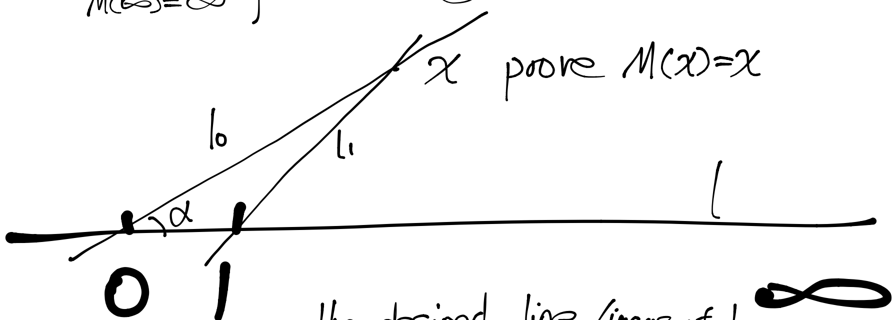
$$\frac{x_3-x_1}{x_4-x_1}:\frac{x_3-x_2}{x_4-x_2}$$

cross-ratio

if $f(z)=\frac{az+b}{cz+d}$ $\Rightarrow[x_1,x_2,x_3,x_4]=[f(x_1),f(x_2),f(x_3),f(x_4)]$

Thm: Any Mobius transformation fixes 3 pts is just identity.

Sps $\left.\begin{array}{l} M(0)=0 \\ M(1)=1 \\ M(\infty)=\infty \end{array}\right\} \Rightarrow M$ is idenity



prove $M(x)=x$

the desired line / image of l_0 goes to l, but α should be preserves (origin goes to origin, and the angle preservation of Mobius)

l_1 goes to itself.

another e.g.

$\exists f(z)=\dfrac{az+b}{cz+d}$ (proved above)

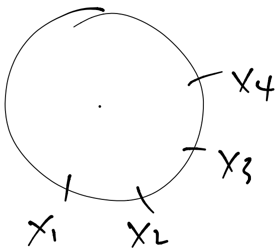
$M(0)=A$
$M(1)=B$
$M(\infty)=C$

$f\circ M=$ identity

$M=f^{-1}$

i.e. $M(0)=A$, $f(A)=0$
$M(1)=B$, $f(B)=1$
$M(\infty)=C$, $f(C)=\infty$

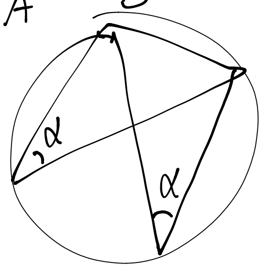
Start from $x_1, \cdots, x_4 \in \mathbb{R}$, then $[x_1, \cdots, x_4] \subset \mathbb{R}$
Take 4 points on a circle, $[\cdot\cdot]$ cross ratio is real.

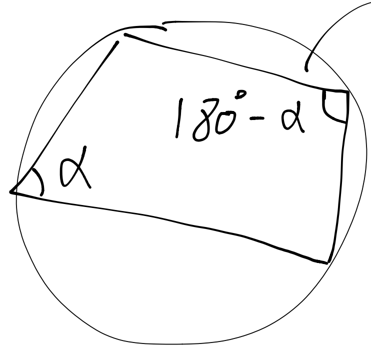


Let $x_1, \cdots, x_3$ be mapped to $f(x_1)$ $f(x_2)$ $f(x_3)$ on a real line, then $x_4 \to f(x_4)$ must be real (proved above) $\Longrightarrow$ cross ratio real.

$$\underbrace{\frac{x_3-x_1}{x_4-x_1}}_{A} : \underbrace{\frac{x_3-x_2}{x_4-x_2}}_{B} \quad \text{say} \ = a \in \mathbb{R} \Rightarrow$$

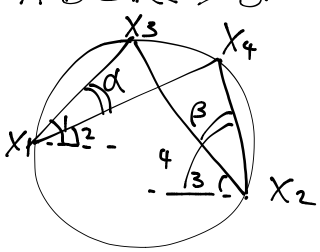
$\arg a = 0$ if $a>0$
or $\arg = 180°$ if $a<0$





actually it's $\alpha - 180°$

$A : B \in \mathbb{R} \Longleftrightarrow \arg A - \arg B = 0$ or $180°$

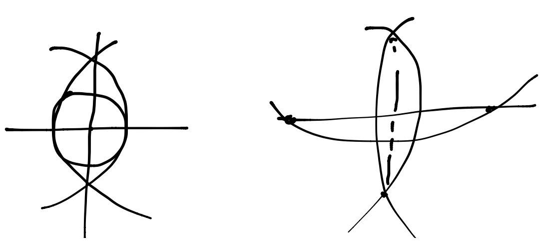


$$\angle 1 - \angle 2 = \alpha = \arg \frac{x_3 - x_1}{x_4 - x_1}$$

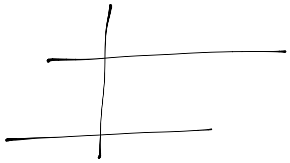
$$\angle 3 - \angle 4 = \beta = \arg \frac{x_3 - x_2}{x_4 - x_2}$$

Construction with ruler & compasses

Perpendicular line



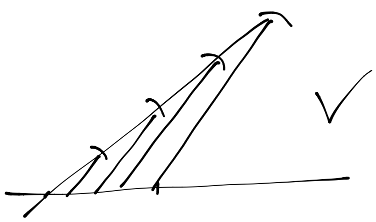
so parallel line can be constructed



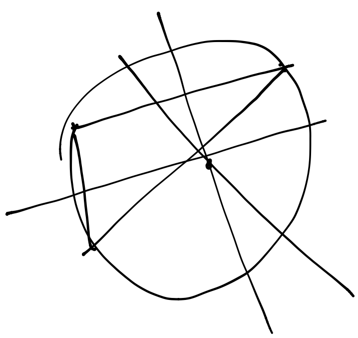
middle point:

bisector

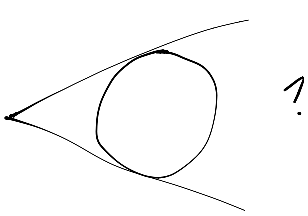
trisect a segment



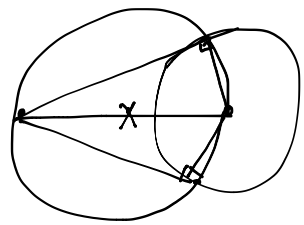
a circle passing 3 pts.



tangent line (natural but NOT ALLOWED!)

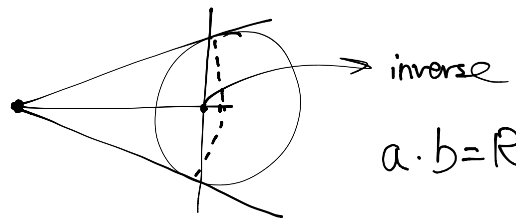


First find center



tangent lines constructed

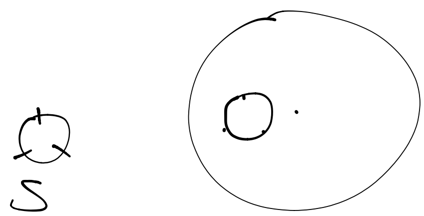
Find inverse point

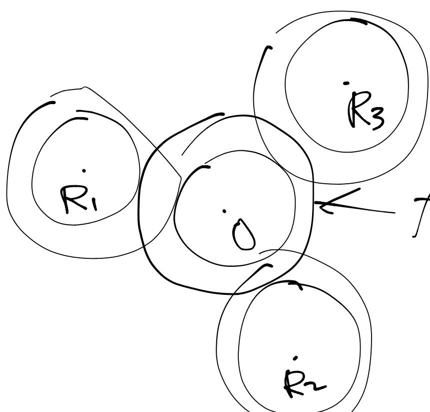


$a \cdot b = R^2$ by similar Δs

or backwards (inside $\rightarrow$ outside)

inversion (pick pt then construct)



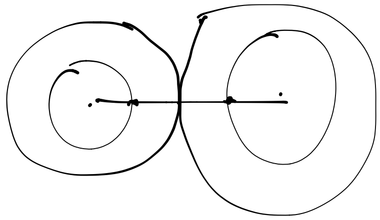


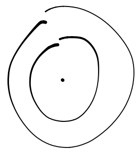
find basically find
the center. 'O

Consider,
in a new question,
for R_1+x, R_2+x, R_3+x,
the new center still is O but
with $R-x$.

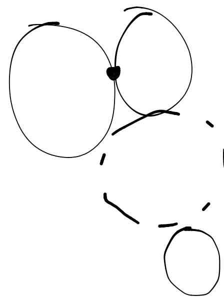
Now if 2 of them are tangent

(1)



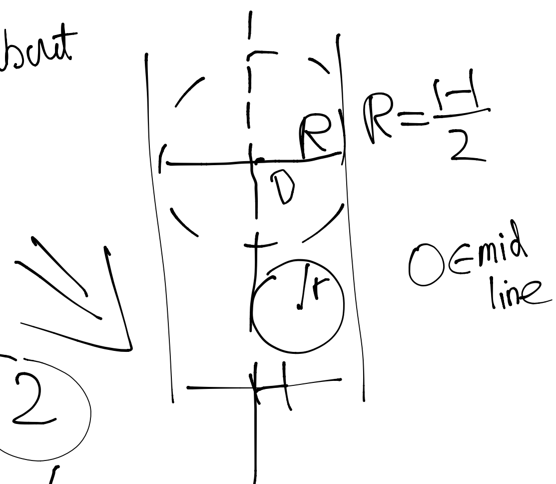


(2)

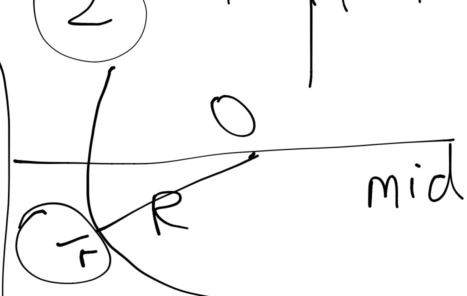


do inversion about that pt

now

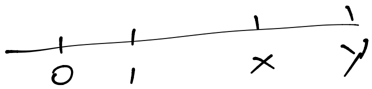


(2)



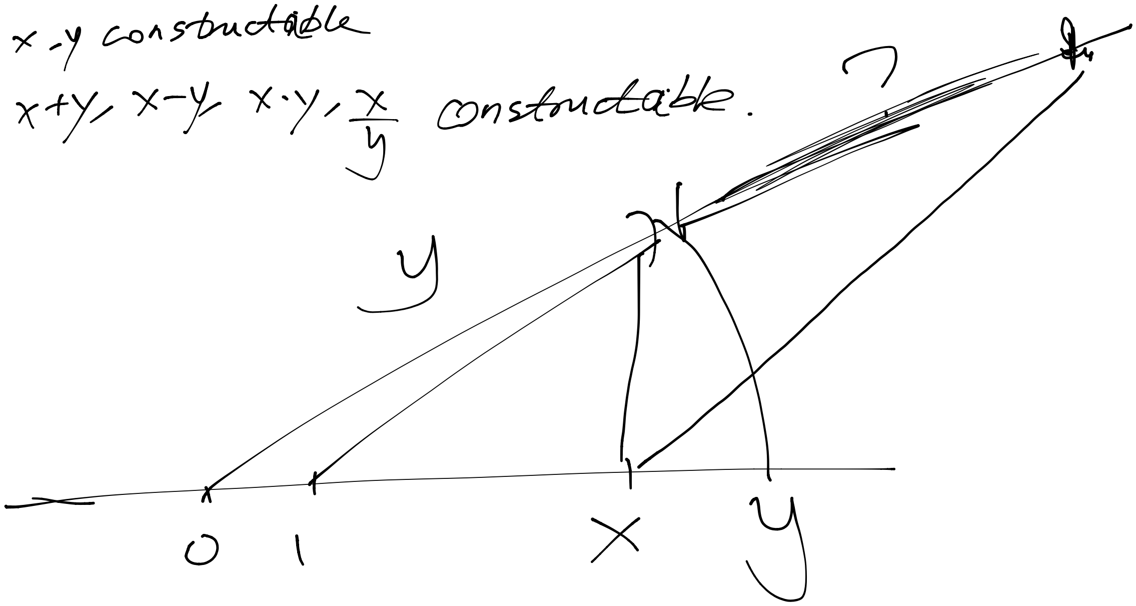
(3) As long as we know where ○ is, the real ○ is the same point, then the rest part is trivial.

so draw ○ at small ball center with r+R (radius) intersects with mid line, that's ○ so ○○ tangent to small circle.



x, y constructable

$x+y,\ x-y,\ x\cdot y,\ \frac{x}{y}$ constructable.

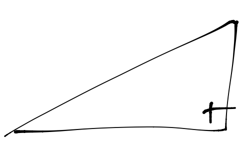


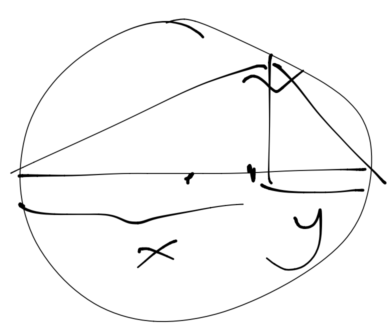
$$\frac{1}{x} = \frac{y}{xy}$$

Similarly for $\frac{x}{y}$

$\sqrt{x^2+y^2}$

$\sqrt{x}$





$a_n x^n + \cdots + a_1 x^1 + a_0 = 0$

$a_i \in \mathbb{Z}$

can find $\mathbb{R}$ solution?

Sps $\frac{p}{q} = r$ is a solution,

plug it in

$\underbrace{a_n p^n}_{\text{has to be divisible by } q.} + \underbrace{a_{n-1} p^{n-1} q + \cdots a_0 q^n}_{\text{divisible by } q} = 0$

i.e. $a_n \vdots q$

$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$

trisector?

Why can't?

$\cos\alpha \Rightarrow \cos\frac{\alpha}{3}$

$\cos 3\alpha =$ given

find $\cos\alpha$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$4x^3 - 3x = \cos 3\alpha$

let $\alpha = 20°$

$4x^3 - 3x = \frac{1}{2}$

$8x^3 - 6x = 1$

$y^3 - 3y - 1 = 0$

no sol'n (using $\sqrt{\ }$)