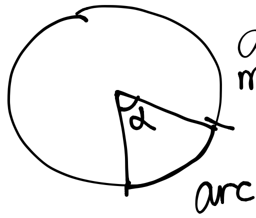
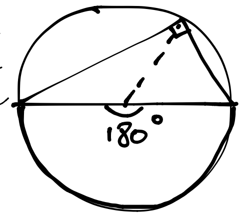
Lecture 6

Inversion 反演

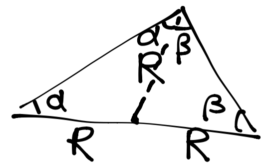
Start with simple theorems



α is the measure of arc

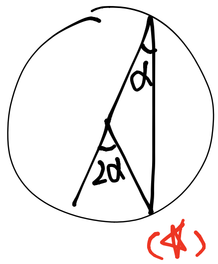


Why 90°?

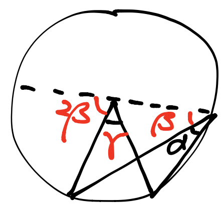


$$2(\alpha+\beta) = 180^\circ$$

$$\alpha+\beta = 90^\circ$$



$\Longrightarrow$

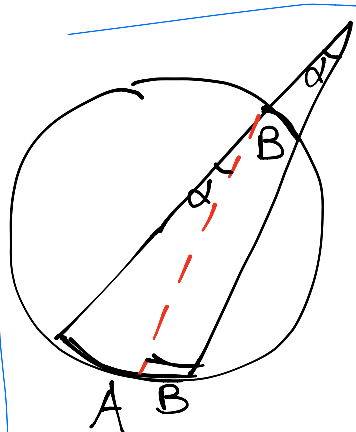


Know. $\alpha+\beta = \frac{1}{2}(2\beta+\gamma)$

so $\alpha = \frac{1}{2}\gamma$

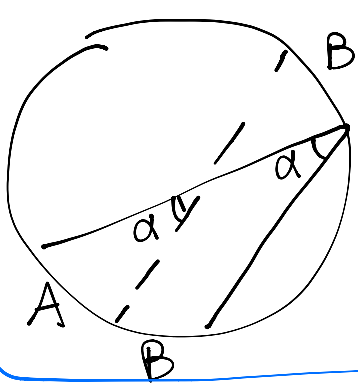
$\gamma = 2\alpha$

so in this case (*) holds

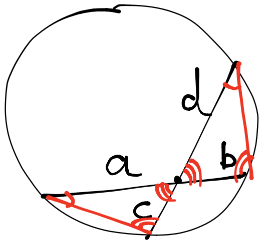


$$\alpha = \frac{A-B}{2}$$

2 cases



$$\alpha = \frac{A+B}{2}$$

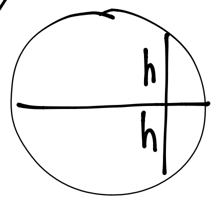


$ab=cd$

(proved by similar triangles)

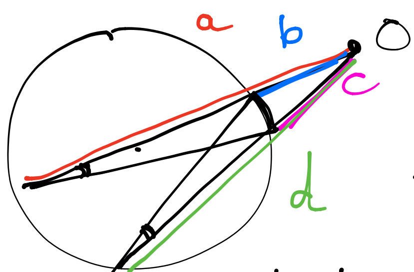
$\frac{a}{c}=\frac{d}{b} \Rightarrow ab=cd$

$\Downarrow$



$ab=cd=h^2$

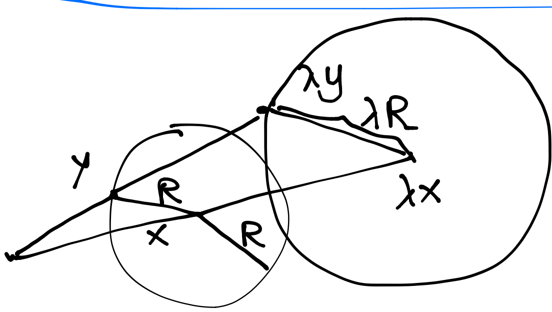
2 cases



$ab=cd=h^2$, h again is the distance from O to the tangency point

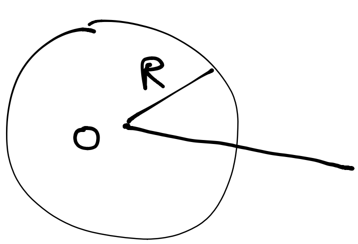
$\frac{b}{d}=\frac{c}{a}$ (proof same) (not appeared here)

$\triangle$s same b/c they have the same arc



circle λ times bigger as well

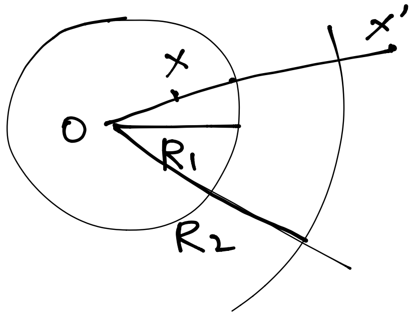
Inversion needs 2 data:
center O
radius R



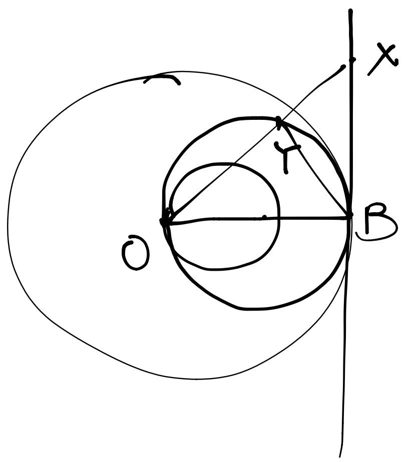
$OX \cdot OY = R^2$

OX↓ OY↑ product fixed

if we apply inversion twice ⇒ identity



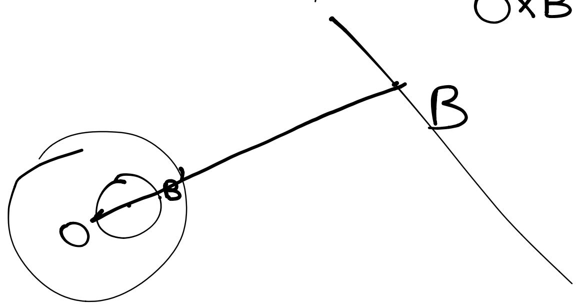
$$OX\cdot OX' = R_1^2$$
$$OX\cdot OX'' = R_2^2$$
$$\frac{OX'}{OX''} = \left(\frac{R_1}{R_2}\right)^2$$
$$OX' = OX''\left(\frac{R_1}{R_2}\right)^2$$



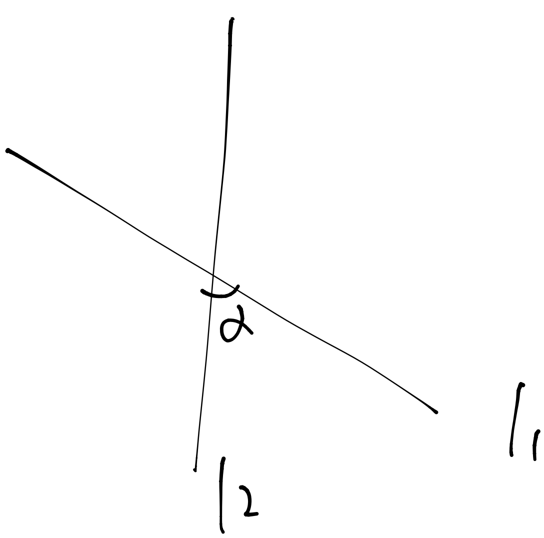
Claim:
image will be exactly a circle.

$$OX\cdot OY = R^2$$
$$\frac{OY}{R} = \frac{R}{OX}$$

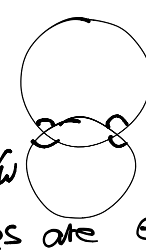
$OXB \sim OYB$ (check it)



Farther B is, closer B' is. (To pt O)

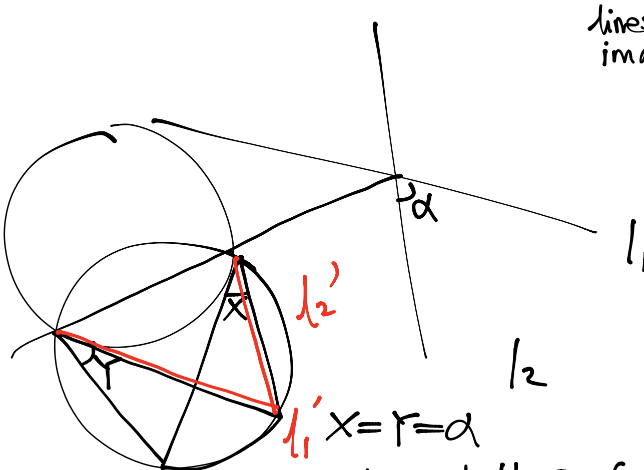


inversion



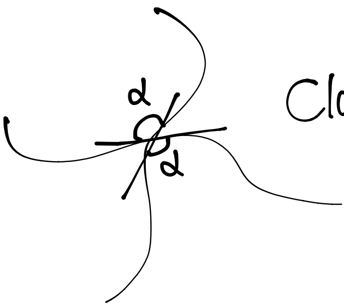
Claim: angle b/w 2 circles are equal

lines have 1 pt in common
images (circles) have 2 pts in common
(one of them is the origin)



$x = y = \alpha$

tangent lines of circles parallel to original lines



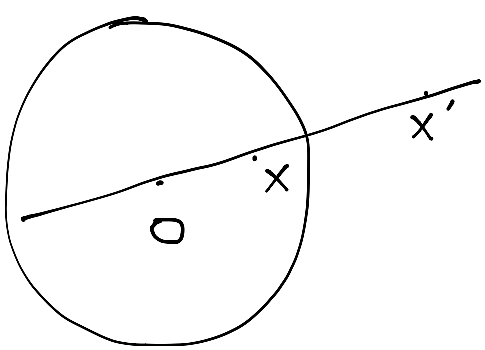
Claim: inversion preserves b/w any two curves

angle b/w curves
= angle b/w tangent lines.

tangent lines $\xrightarrow{\text{inversion}}$ tangent circles

so angles are preserved

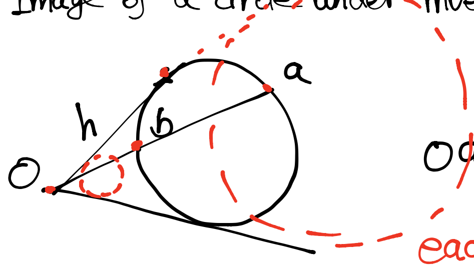
if line passes origin



so the image of inversion is

- a line if the line passes through origin
- a circle if the line does not pass through origin

Claim: Image of a circle under inversion is always an inversion

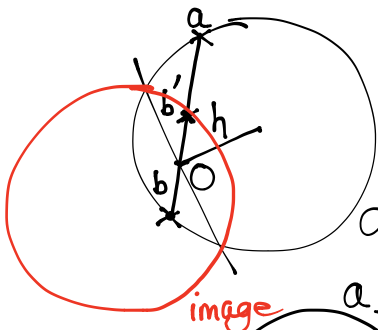


$oa \cdot ob = h^2$ (distance to tangency)

each pt goes to itself

So circle goes to itself

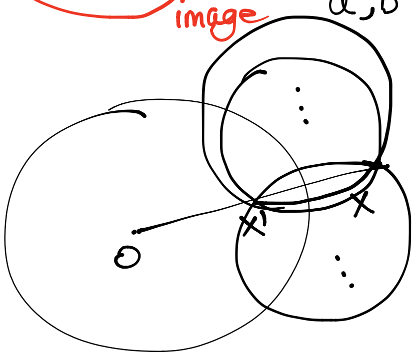
if I make circle λ times bigger, image will be $(\frac{1}{\lambda})$ times smaller



$ab = h^2$

a, b don't belong to 1 ray.

have to choose b's semetric pt. b'

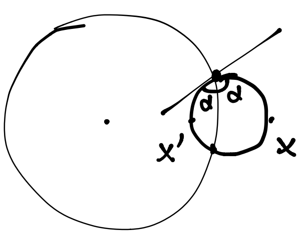


Two points (inversion)

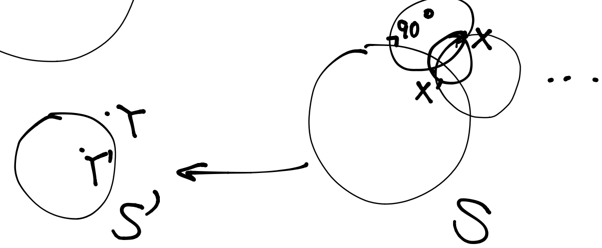
All the circles passing through these 2 pts will be $\perp$ the original circle.

perpendicular to

↓ e.g.

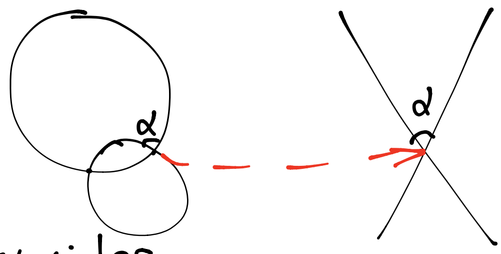


$180° = 2\alpha \quad \alpha = 90°$

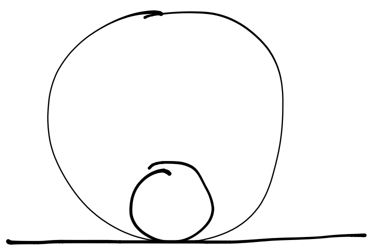


inversion preserves the symmetry

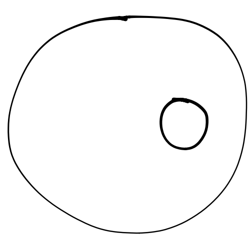
(Lost in the pdf notes)
1 h 52 min in recording



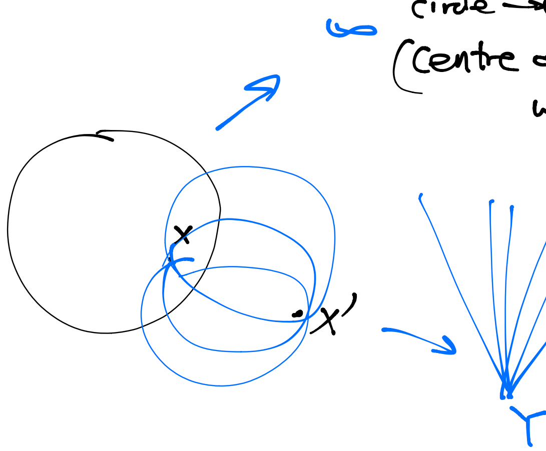
if two circles tangent to each other



just any 2 not connected circles

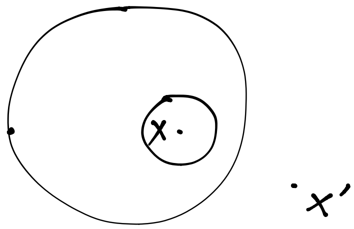


$x \to \infty$
$x' \to Y$
circle → circle centered at Y
(Centre of image will be the centre of inverse image of x')

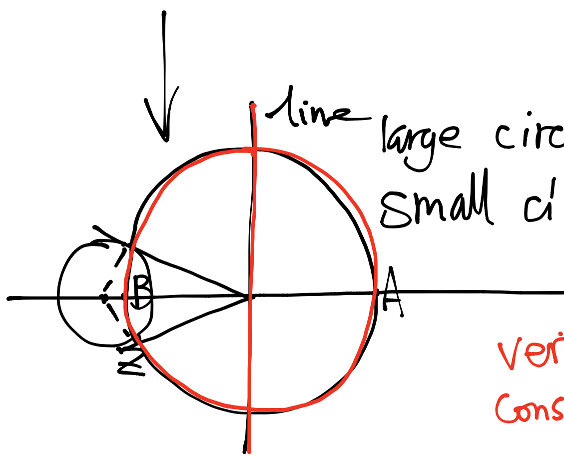


circles pass X, X' become lines pass Y.

back to

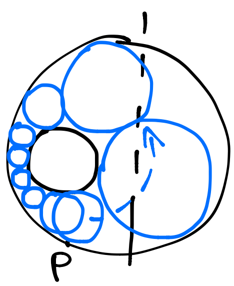


X and X' are sym, S_1 & S_2



large circle ⇒ a line
small circle ⇒ a circle

vertical line orthogonal to ⊙Z
Constructed circle (AB) orthogonal to ⊙Z.



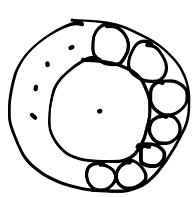
every circle is tangent to the 2 black circles & previous circle.

Suppose we do this 2014 times (we are back to the first)
then we get two fully matched/covered circles.

(Not matter where P we choose to be)

Sps we have 2 concentric circles.

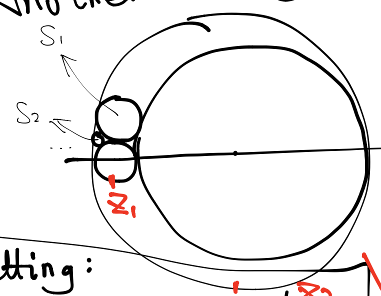
Say Q, after 2014 steps, we go to Q.



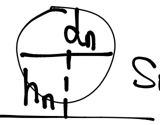
(obvious for concentric)

For non concentric, use inversion.

Another ancient Greek problem



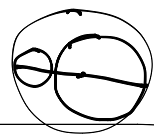
... S_n

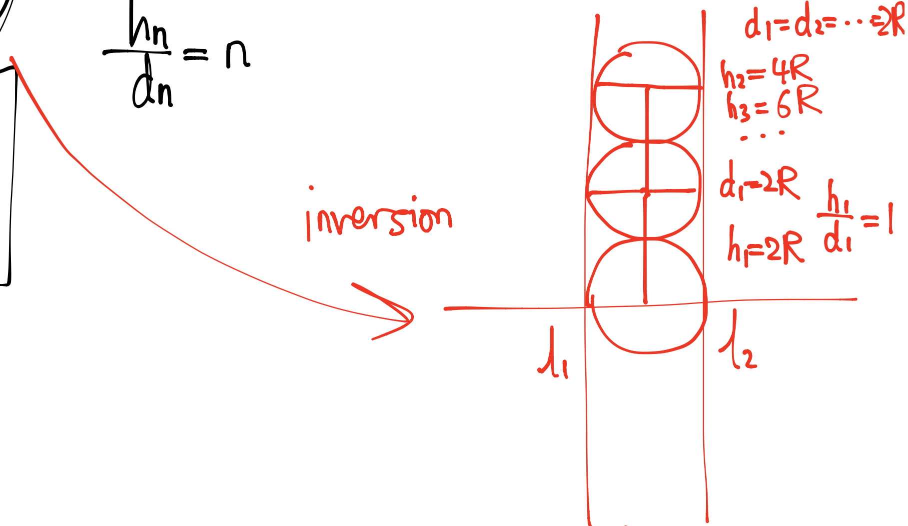


S_n take it out:

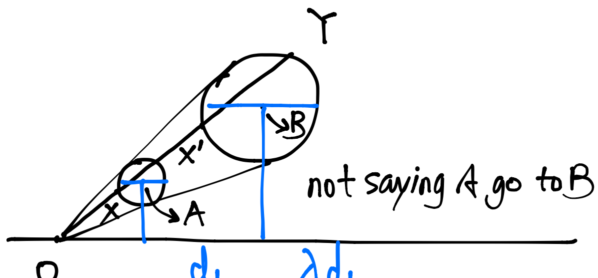
$$\frac{h_n}{d_n} = n$$

Setting:
We have 3 original Circles





So $\frac{h_n}{d_n} = \frac{2nR}{2R} = n$

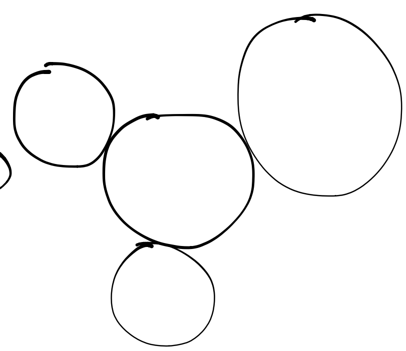


not saying A go to B

$$\frac{d_1}{h_1} = \frac{\lambda d_1}{\lambda h_1}$$

Apollonian circles

(skipped?)



Recall: $ax^2 + bx + c = 0$

$x^2 + px + q = 0$

Now assume $x = (x_1, \cdots, x_n)$

$b = (b_1, \cdots, b_n)$

$a\langle x, x\rangle + \langle b, x\rangle + c = 0$

$$\langle x, x\rangle + \langle p, x\rangle + q = 0$$

$$\langle x+\tfrac{p}{2}, x+\tfrac{p}{2}\rangle = \langle x, x\rangle + \langle p, x\rangle + \langle \tfrac{p}{2}, \tfrac{p}{2}\rangle + q - \langle x+\tfrac{p}{2}, x+\tfrac{p}{2}\rangle$$

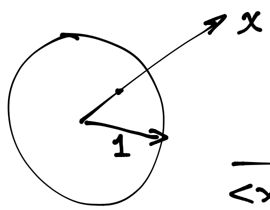
$$= \frac{\langle p, p\rangle}{4} - q$$

$$\langle x+\tfrac{p}{2}, x+\tfrac{p}{2}\rangle$$

$$\langle x-(-\tfrac{p}{2}), x-(-\tfrac{p}{2})\rangle = \frac{\langle p, p\rangle}{4} - q$$

$$-\frac{p}{2} \text{ of } R = \sqrt{\frac{\langle p, p\rangle}{4} - q}$$

"So we'll have a sphere, a plane, and an empty set" ?



$$\frac{x}{\langle x, x\rangle} \xRightarrow{\text{inversion}} y$$

$$\frac{y}{\langle y, y\rangle} \Rightarrow x$$

$$a\frac{\langle y, y\rangle}{\langle y, y\rangle^2} + \langle b, \frac{y}{\langle y, y\rangle}\rangle + c = 0$$

$$= a + \langle b, y\rangle + c\langle y, y\rangle = 0$$