Lecture 5

Convex geometry

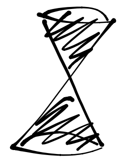
Finite convex polytopes (simple)

A closed half space in R^n

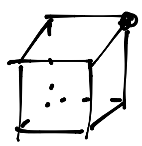
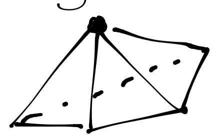
$$\left\{ \begin{matrix} (x_1, \cdots, x_n) \in \mathbb{R}^n \\ a_1x_1 + \cdots + a_nx_n \leq C \end{matrix} \right\}$$

Closed convex figure it is intersection of some number of half spaces.

Bounded Polytopes (finite of intersection



In $\mathbb{R}^n$ simple polytope is such polytopes that in each vertex we have exactly n edges meeting.

 ✓  ✗

With any polytopes, $P \in \mathbb{R}^n$ we associate a set numbers

f_0: number of vertices

f_1: number of edges

f_2: number of 2-dim faces

...

f_n: number of n-dim faces

Euler relation: $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n = 1$

$f_0 - f_1 + f_2 - f_3 = 1$ $f_0 - f_1 + f_2 = 2$

Denh-Semmervile relations (holds only for SIMPLE POLYTOPES)

$f(t) = f_0 + f_1 t + \cdots + f_n t^n$

$h(t) = f(t-1)$

$h(t) = f_0 + f_1(t-1) + \cdots + f_n(t-1)^n = h_0 + h_1 t + \cdots + h_n t^n$

$h_0 = f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^n f_n$

D-S Thm: For simple polytopes

① $h_i = h_{n-i}$

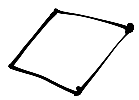
② $h_1 \geq 1$

Proof: L is a linear fundamental on $\mathbb{R}^n$, $L:\mathbb{R}^n \to \mathbb{R}$ it is linear

$L(\lambda v+\mu w)=\lambda L(v)+\mu L(w)$ $\qquad$ $L(\cdots)=x_1L(\vec{e_1})+\cdots+x_nL(\vec{e_n})$

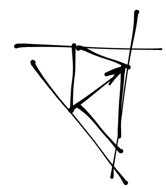
$L(x_1,\cdots,x_n)=a_1x_1+\cdots+a_nx_n=\langle a, x\rangle$

Linear function L is generic w.r.t. polytope P iff it takes different values on all vertices of P.



$\langle v_i, a\rangle=\langle v_j, a\rangle$

$\langle v_{i-j}, a\rangle=0$

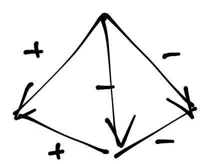


P, L

index of vertex i w.r.t. functional L

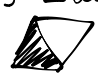
$\parallel$

# of edges point out of v s.t. L decreases along the edges.



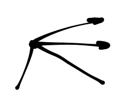
Lemma: For every L and P the number $\tilde{h}_i$ of vertices of index i is equal to h_i,
(in particular it doesn't depend on L)

vertex v maximize? for face F.
If L attains its maximum on the face F at vertex V.



$f_m \,/\!/\, (F, v)$, assume F n-dim

$\binom{i}{m}$ $\qquad$ assume v, sps v has index i,



$$\sum_{i\geq m}\binom{i}{m}\tilde{h}_i$$

$$f_i=\sum_{i\geq m}\binom{i}{m}\tilde{h}_i$$

$$f(t)=\sum_{i=0}^n f_i t^i=\sum_{i=0}^n\sum_{i\geq m}\binom{i}{m}\tilde{h}_i=\sum_{0\leq i\leq m\leq n}\binom{i}{m}\tilde{h}_i t_i=\sum_{i=0}^n\left(\sum_{m=0}^n\binom{i}{m}t^m\right)\tilde{h}_i=\sum_{i=0}\tilde{h}_i(1+t)^i$$

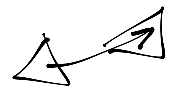
$$h(t)=f(t-1)=\sum_{i=0}^n\tilde{h}_i t^i \qquad h_i=\tilde{h}_i$$

A subset δ of $\mathbb{R}^n$ is called convex if along with any 2 pts $a, b \in \delta$, δ contains all pts in the segment joining a, b.

$\{X \mid \langle a, x\rangle \leq c\}$

$\bigcap$ convex sets is convex

Not true union

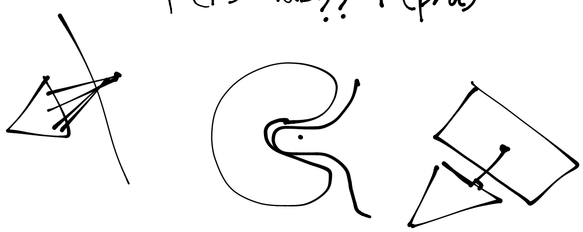


Separation Theorem:

C-C set P and pt a $a \notin P$, then $\exists$ a hyperplane that separates P from a.

Pf: take $F: P \to \mathbb{R}$

$F(P) = \text{idis}?? + (p, a)$

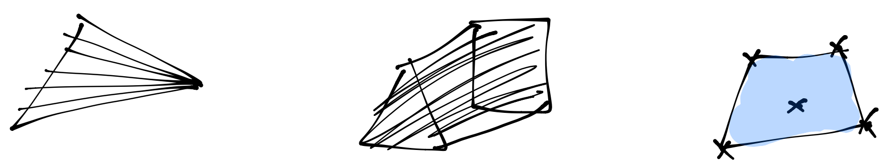


$\delta = \bigcap_{i \in \lambda} L_0$

Missed a lot

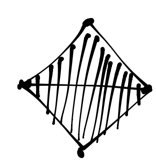
Convex Hull

For a set μ, convex hull of μ is the smallest convex set that contains μ.



Claims: P belongs to the convex null of S iff there exists points $x_1, \cdots, x_m \in S$ $\lambda_i \geq 0$, $\lambda_1 + \cdots + \lambda_m = 1$ (*)

N - all points that can be represented in the form (*)



$N = S$

$S \ni x_1, \cdots, x_n$, $c.h\, S \supset N$, $c.h\, S \ni \lambda_n x_n + \cdots + \lambda_1 x_1$

$\lambda_1 x_1 + \lambda_2 x_2$

$\lambda_1 + \lambda_2 = 1$ $\quad \lambda_1 x_1 + (1 - \lambda_1) x_2$

$x_1, \cdots, x_m$

$\lambda_1 x_1 + \cdots + \lambda_m x_m \in c\,h\, S$

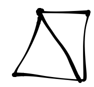
$\lambda_1 x_1 + \cdots + \lambda_m x_m + \lambda_{m+1} x_{m+1}$

$\lambda_1 + \cdots + \lambda_m = 1 - \lambda_{m+1}$ $\quad (1 - \lambda_{m+1})\left(\frac{\lambda_1}{1 - \lambda_{m+1}} x_1 + \cdots + \frac{\lambda_m}{1 - \lambda_{m+1}} x_m\right) + x_{m+1}\lambda_{m+1}$

Carathéodory Lemma

$S \subset \mathbb{R}^d$, any $p \in c.h\, S$
can be written as $\lambda_1 x_1 + \cdots + \lambda_{d+1} x_{d+1}$ for some $x_1, \cdots x_{d+1} \in S$

$x = \lambda_1 x_1 + \cdots + \lambda_{m+1} x_{m+1}$, $m > d$



$x_2 - x_1, x_3 - x_1, \cdots, x_{m+1} - x_1$

$\mu_2 (x_2 - x_1) + \cdots + \mu_{m+1} (x_{m+1} - x_1) = 0$

$\mu_1 + \cdots + \mu_{m+1} = 0$

$(\mu_1 x_1 + \cdots + \mu_{m+1} \lambda_{m+1}) = 0$

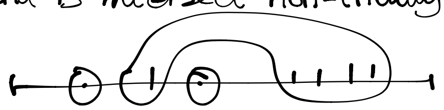
$\mu_1 = -\mu_2 - \cdots - \mu_{m+1}$

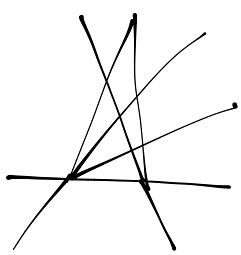
$\xi (\mu_1 x_1 + \cdots + \mu_{m+1} x_{m+1}) = 0$

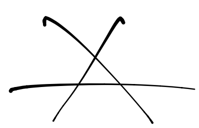
$\Rightarrow x = (\lambda_1 - \xi \mu_1) x_1 + (\lambda_2 - \xi \mu_2) x_2 + \cdots + (\lambda_{m+1} - \xi \mu_{m+1}) x_{m+1}$

Helly's Theorem:
Assume we have a collection of convex sets in $\mathbb{R}^d$ s.t. any $d+1$ intersect then all sets intersect.

Radon's lemma :
If S is a finite set of at least $d+2$ pts in $\mathbb{R}^d$, then the set S can be partitioned as a union of two subsets, A & B s.t. the convex hulls of A and B intersect non-trivially.







$A_1, \cdots, A_n$

$A_i \cap A_j \cap A_k \neq \emptyset$

G_1, G_2, G_3, G_4

$G_1 \cap G_2 \cap G_3 = B_4$

$G_1 \cap G_2 \cap G_4 = B_3$

$G_2 \cap G_3 \cap G_4 = B_1$

. . . .