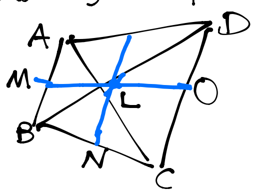
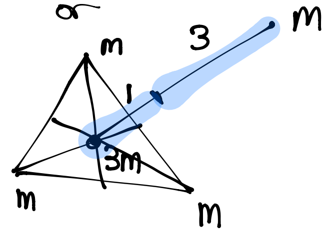
Lecture 4

Start from quiz problem



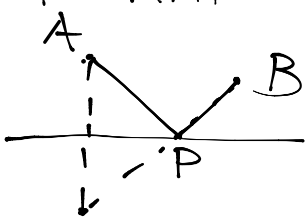
M: 2m
O: 2m
So 4m at the mid point of MO

or



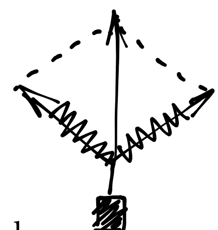
Extremal Problems in Geometry:

For example $AP+PB=\min$



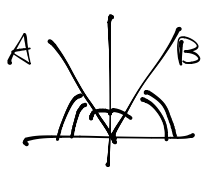
Format's principle

Force → stable . for 2-spring situation

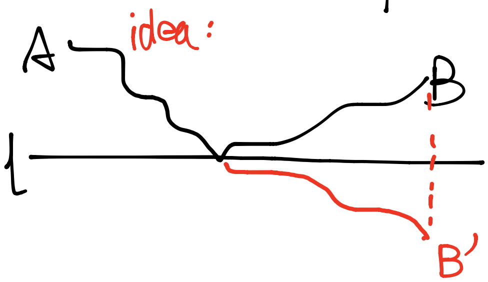


If either spring takes more force, the system becomes unstable.

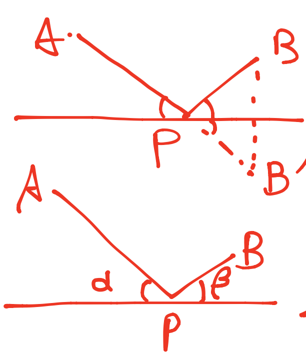
So seemingly



This is a situation for equal angles?



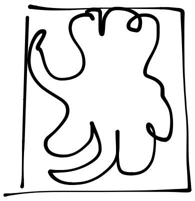
Consider this path A → B. find shortest path

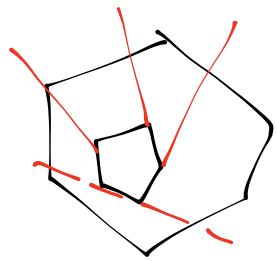


reflect B,
then draw straight line

$\angle\alpha=\angle\beta$

Two figures, one inside the other. Both are convex the the outside one has larger perimeter, and this is not true for convex figure.





Prove by triangle inequality (cut it into several Δs)

Consider function with many parameters

$F(X_1(t), \cdots, X_n(t))$

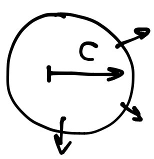
derivative?

$$\frac{d}{dt} F(x_1(t), \cdots, x_n(t)) = \underbrace{\left(\frac{\partial F(a)}{\partial x_1}, \cdots, \frac{\partial F(a)}{\partial x_n}\right)(x_1, \cdots, x_n)}$$

$a = (x_1(t_0), \cdots, x_n(t_0))$ $\qquad$ $\langle \text{gradient } F, x(t) \rangle$

$F(x, y, z)$ = distance from (x, y, z) to origin

$\sqrt{x^2+y^2+z^2} = C$



gradient always orthogonal to the level surface

$$F(x(t), y(t), z(t)) = C$$
$$\Downarrow$$
$$\frac{\partial F}{\partial t} = 0$$

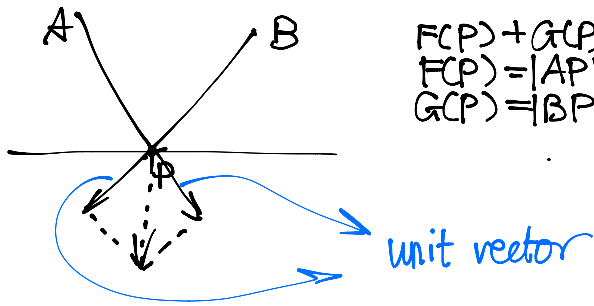
gradient always points out of the circle (orthogonal) & of unit length

fastness rate of going outside.

e.g.

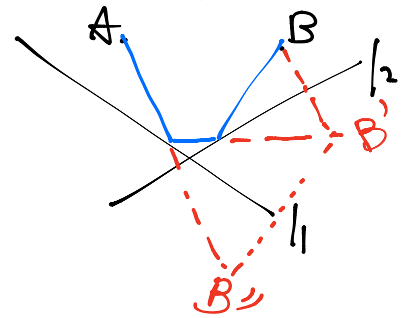
$\frac{d}{dt} F(x(t), y(t), z(t))$ $\qquad$ $\text{grad}(F+G) = \text{grad}\, F + \text{grad}\, G$

$\langle \text{grad}\, F \cdot \vec{x} \rangle = 0$



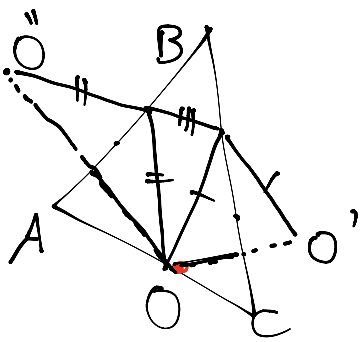
$F(P)+G(P)$
$F(P)=|AP|$
$G(P)=|BP|$

More complicated problem:



Reflect along l2 first

If two points located in different half planes, just draw straight line to connect them, if two points located in the same half plane, use reflection.



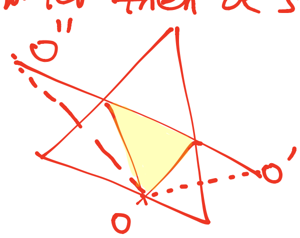
Find the inscribed △ with smallest perimeter.

Subdivide it to 2 problems

$B', C', A' \longrightarrow A'B'+B'C'+C'A'=\min$
Fix B' anywhere on AC, find min
Then we deal with B''s location

Let B' be O (rename it)
So perimeter is $\underline{+ + \# + \#\!\#}$ but O, O', O'' are all fixed.

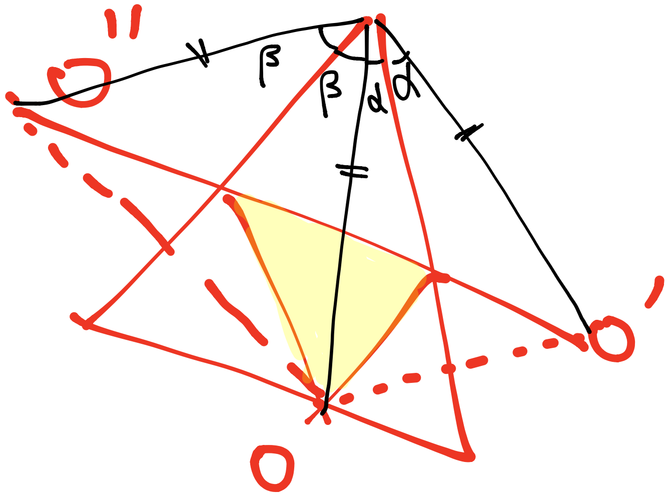
Nothing is shorter then a straight line.
So

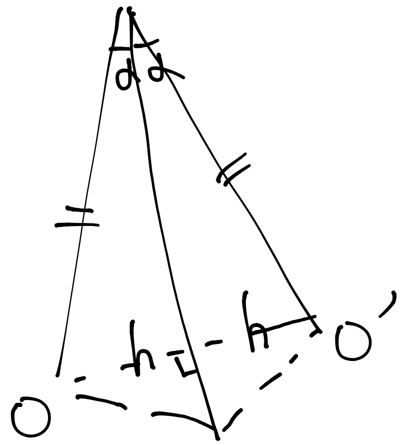


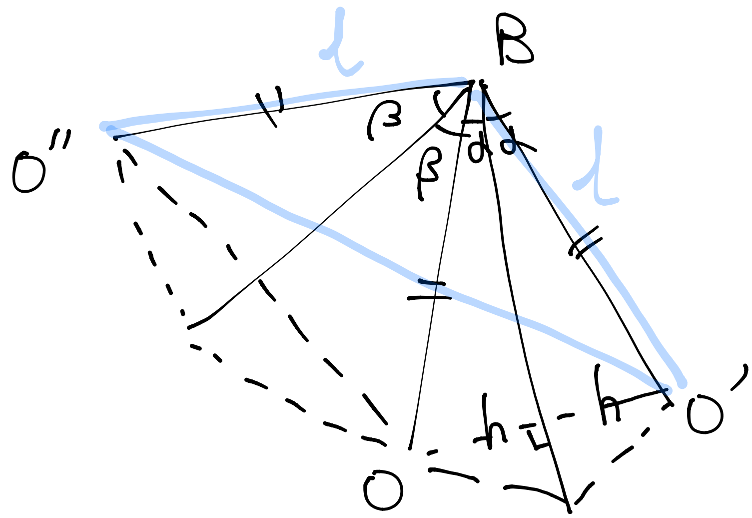
This is the solution to the situation
If B is fixed.

(*)

No, what if B moves?





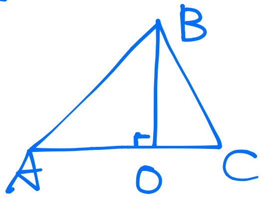


$\angle O''BO' = 2\beta + 2\alpha = 2(\beta + \alpha) = 2\angle B$

$O''B = OB = O'B = l$

For $\triangle O''BO'$

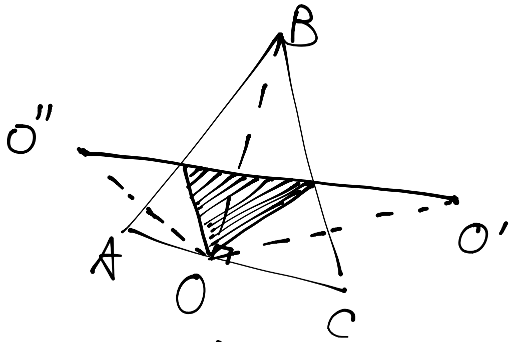
know 2 sides, the angle so $O'O''$ is shorter if l is shorter $\Rightarrow$ l is original distance from B to AC, so OB is the height!



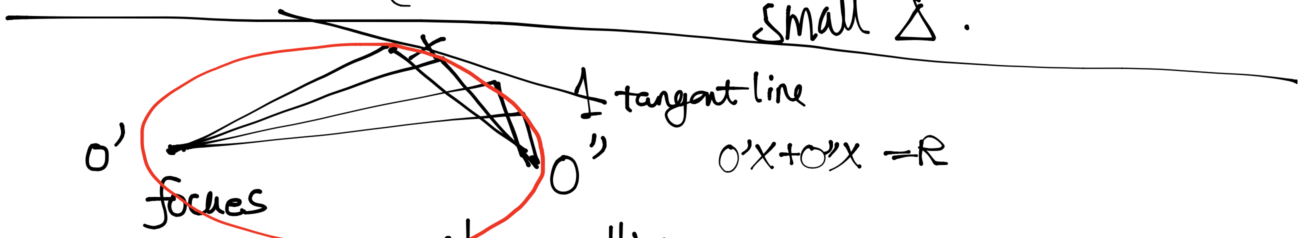
then combine the result ⊛ we are done.

Such triangle is unique.

Check:



The heights of big △ are 3 bisectors of small △.

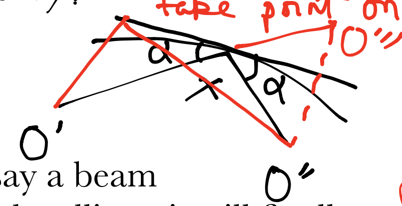


⊥ tangent line

$O'X + O''X = R$

repeat ⇒ ellipse

Claim: the line has property.
tangent



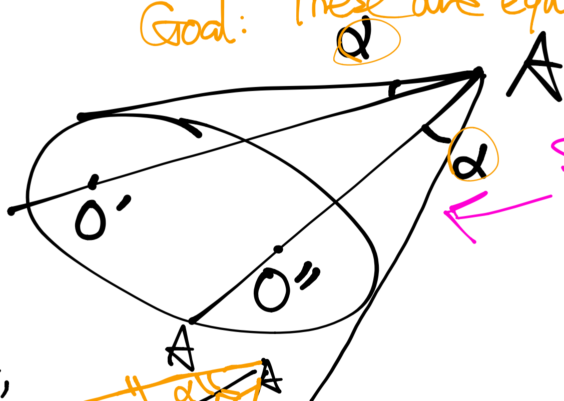
oh, only at the tangent point the sum $O'X + O''X = \min$

so reflection of, say O'

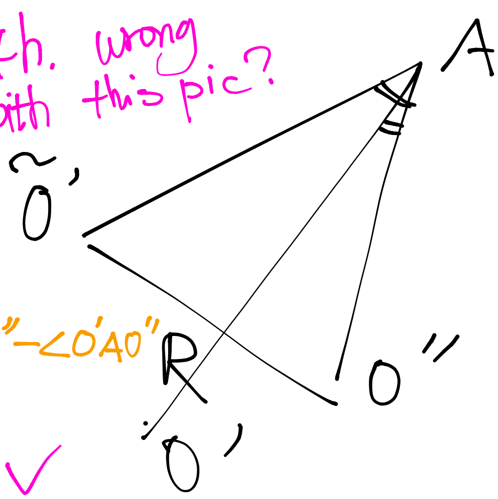
so done

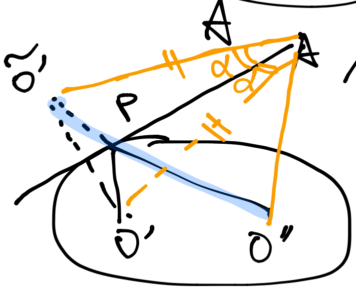
Starting from one focus, say a beam reflects all the way inside the ellipse, it will finally pass through the other focus.

Goal: These are equal



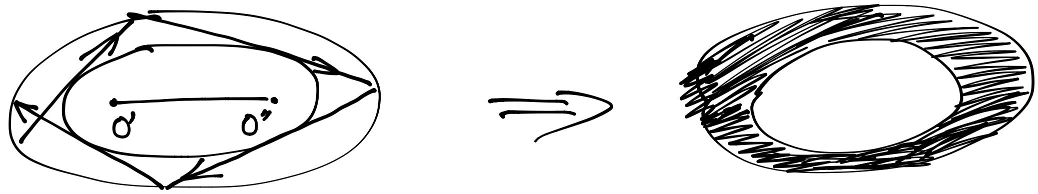
sth. wrong with this pic?





$2\alpha = \angle\tilde{O}'AO'' - \angle O'AO''$

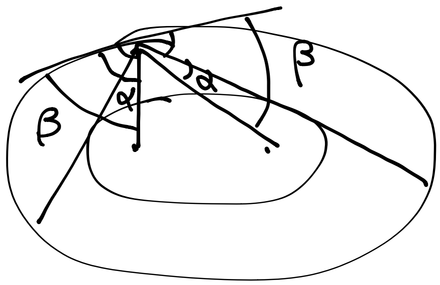
$O''\tilde{O}' = R$ ✓



forever tangent

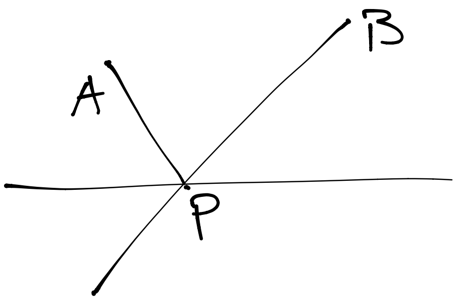
This is a very special case, two ellipses.

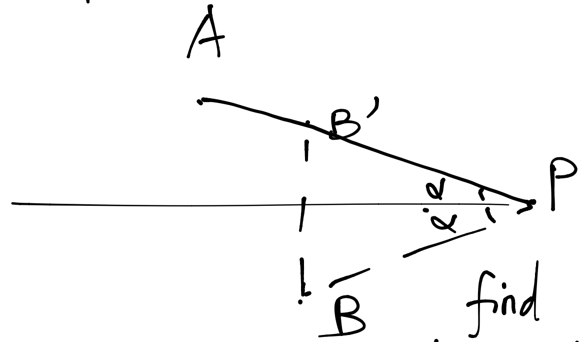
Idea :



Not to prove tangent, will start another way around. Say two tangents, then they have same corresponding angles, so that's a proved reflection.

For hyperbola, does all the properties above still hold?



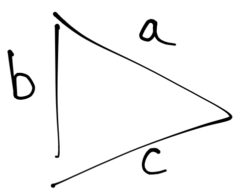


find

$|AP - PB|$

$\to$ max

Why?



$a + b > c$

$a + c > b$

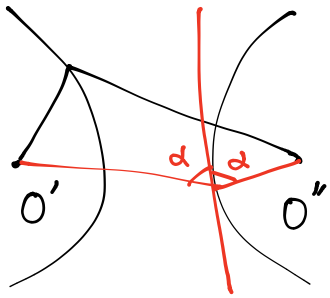
$a > c - b$

$a > b - c$

$a > |c - b|$

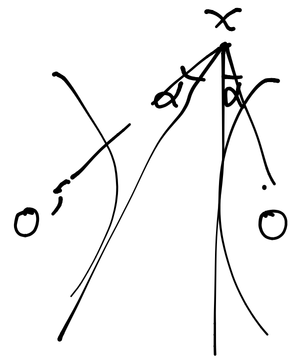
equality $\Longleftrightarrow$ on the same line

2 focuses



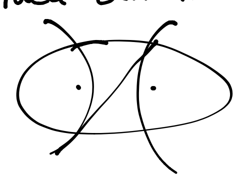
$|O'x - O''x| = R$

for the line, x on it has
$|O'x - O''x| \le R$



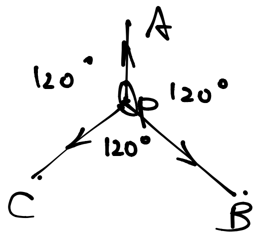
$\angle\alpha = \angle\alpha$

idea: construct :



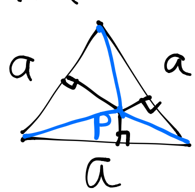
finally it will be tangent to the other branch.

Given A, B, C. Find P s.t. $PA + PB + PC = \min$



P: where 3 angles $= 120°$.

Tool:

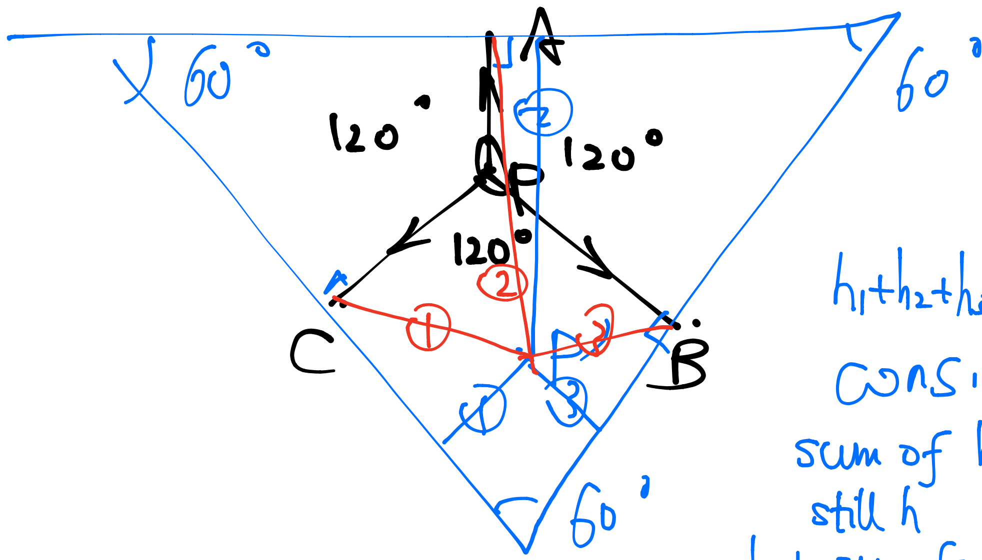


$h_1 + h_2 + h_3 = h$ of Δ always!

$\frac{1}{2}ah_1 + \frac{1}{2}ah_2 + \frac{1}{2}ah_3 = \frac{1}{2}a(h)$

So $h_1 + h_2 + h_3 = h$

if P is out of Δ
then $h_1 + h_2 + h_3 > h$
b/c the Δs cover the original Δ.



$h_1+h_2+h_3=h$

consider:

sum of heights still h

but sum of distances to points

$1 > 1$

$2 > 2$

$3 > 3$

$1+2+3 > h$