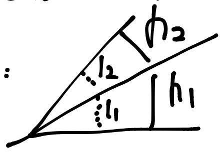
Lecture 3

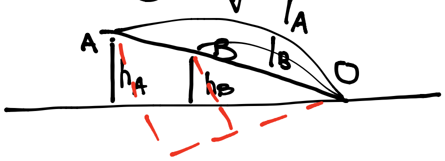
Recall: (Last time)

Tool:



$$\frac{h_1}{h_2} = \alpha = \frac{l_1}{l_2} = \cdots$$ similarity of Δs

intuitively ⇓



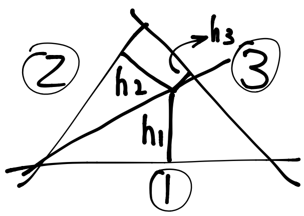
$$\frac{h_A}{h_B} = \frac{l_A}{l_B} = \alpha$$

两点共线的结论

There are infinitely many lines passing O

⇓ ×3

三点共线的充要条件



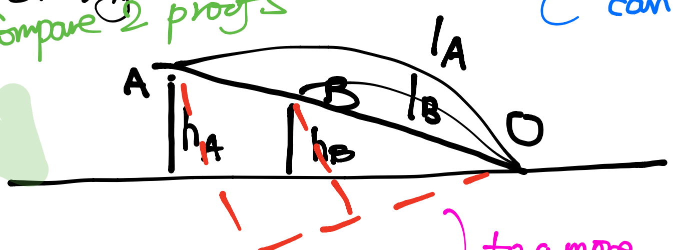
$\frac{h_1}{h_2} = \alpha_1$, $\frac{h_2}{h_3} = \alpha_2$, $\frac{h_3}{h_1} = \alpha_3$

Whether $h_1 h_2 h_3$ pass 1 point? iff $\alpha_1\alpha_2\alpha_3 = 1$

Similarly

Compare 2 proofs

C can be on any line passing pt O.

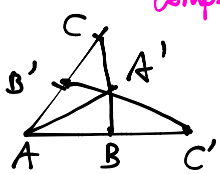


$\frac{h_A}{h_B} = \alpha_1$, $\frac{h_B}{h_C} = \alpha_2$, $\frac{h_C}{h_A} = \alpha_3$

they coincide $\Longleftrightarrow \alpha_1\alpha_2\alpha_3 = 1$

to a more compliated case 3 pts

Note: $\frac{C'A}{C'B} = \alpha_1$



$\frac{C'A}{C'B} = \alpha_1 = \frac{h_A}{h_B}$, $\frac{B'C}{B'A} = \alpha_2 = \frac{h_C}{h_A}$, $\frac{A'B}{A'C} = \alpha_3 = \frac{h_B}{h_C}$

$\alpha_1\alpha_2\alpha_3 = 1 \Longleftrightarrow B', A', C'$ on one line

The necessary/sufficient condition for two points on the same line:

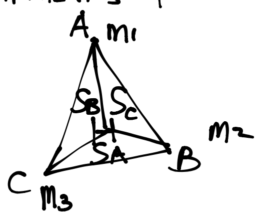
ratio of distances from two opposite vertices to the line = segment ratio on the line

BY SIMILAR TRIANGLE

So if we want to prove 3 pts on the same line then we should have 3 those conditions.

Assume we have a $\triangle$ & H, How to find COM exactly on H.

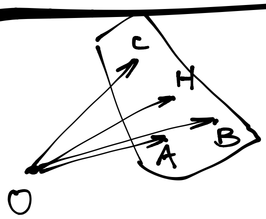
$m_1+m_2+m_3=1$



Recall using area to prove Ceva.

$$S_A+S_B+S_C=S$$

Claim $\underbrace{m_B=\dfrac{S_B}{S},\ m_C=\dfrac{S_C}{S},\ m_A=\dfrac{S_A}{S}}$



$$\vec{H}=m_1\vec{A}+m_2\vec{B}+m_3\vec{C}$$

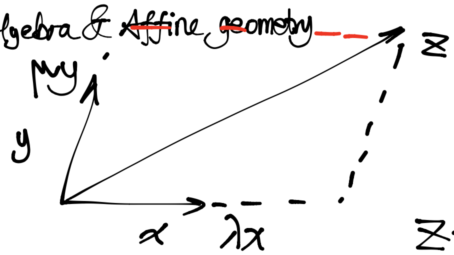
$$m_1+m_2+m_3=1$$

actually this is equivalent to cramer's rule

e.g. solve for m_1, use cramer's rule.

$$m_1=\frac{\det(H,B,C)}{\det(A,B,C)}$$

Algebra & ~~Affine~~ geometry



$$z=\lambda x+\mu y$$

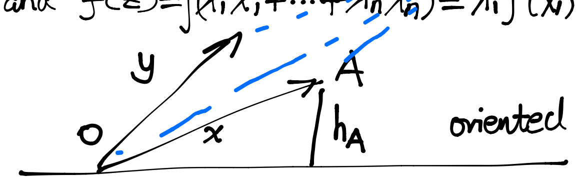
basis

x, y can be any nonzero vector, we just need to change coordinates λ, μ.

linear function: $f(\lambda a+\mu b)=\lambda f(a)+\mu f(b)$

so $f(z)=f(\lambda x+\mu y)=\lambda f(x)+\mu f(y)$

and $f(z)=f(\lambda_1 x_1+\cdots+\lambda_n x_n)=\lambda_1 f(x_1)+\cdots+\lambda_n f(x_n)$

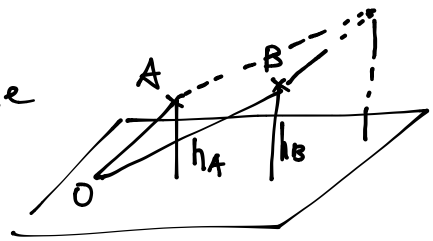


oriented distance +

Claim $h(x+y)=h(x)+h(y) \longrightarrow$ linear function

Distance is a linear function if it is oriented.

Plane

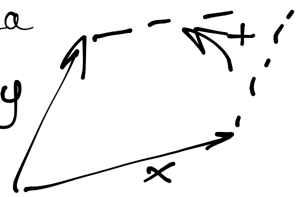


+ half plane so oriented.

So $S(x,y) = -S(y,x)$

Still holds as a linear function

Area



Consider oriented area (with + direction positive)

$S(x,y)$

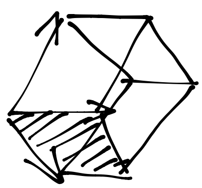
is also linear

$S(x, \lambda y_1 + \mu y_2) = \lambda S(x, y_1) + \mu S(x, y_2)$

$S(x,y) =$ length x · height of y

Volume

Vol (x, y, z)



Vol $(x,y,z) =$ length x · height of y · height of z.

$x = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ $y = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$ $\det(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}$

$x = a_{11}e_1 + a_{12}e_2$

$S(x,y) = S(x, a_{12}e_1 + a_{22}e_2) = a_{12}S(x, e_1) + a_{22}S(x, e_2)$

$= a_{12}a_{21}S(e_1, e_1) + a_{12}a_{21}S(e_2 e_1)$

$= -a_{12}a_{21} + a_{22}a_{11}$ proved

$S(e_1, e_1) = 0$ $S(e_1, e_2) = 1$

$S(e_2, e_1) = -1$ $S(e_2, e_2) = 0$

Proof of

Cramer's Rule:

given x, y, z, find λ, μ s.t. $\lambda x + \mu y = z$

original: $\det(x,y)$

consider $\det(z,y) = \det(\lambda x + \mu y, y) = \lambda \det(x,y) + \mu \underbrace{\det(y,y)}_{=0}$

So $\dfrac{\det(z,y)}{\det(x,y)} = \lambda$

consider $\det(x,z) = \mu \det(x,y)$

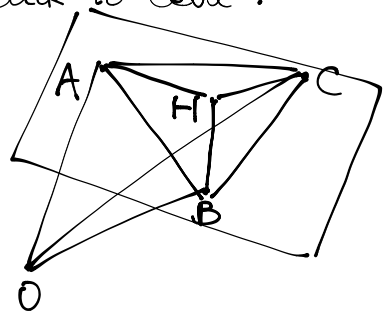
so $\dfrac{\det(x,z)}{\det(x,y)} = \mu$

For 3dim space, x, y, z, $w = \lambda x + \mu y + \gamma z$

$\dfrac{\det(w,y,z)}{\det(x,y,z)} = \lambda$ due to the same reason ...

and $\dfrac{\det(x,w,z)}{\det(x,y,z)} = \mu$, $\dfrac{\det(x,y,w)}{\det(x,y,z)} = \gamma$

Back to Ceva :



$\lambda + \mu + \gamma = 1$

$\lambda A + \mu B + \gamma C = H$

$(1,0,0) = A$

$(0,1,0) = B$

$(0,0,1) = C$

$$\lambda = \frac{\det(H,B,C)}{\det(A,B,C)} = \frac{S(H,B,C)}{S(A,B,C)}$$