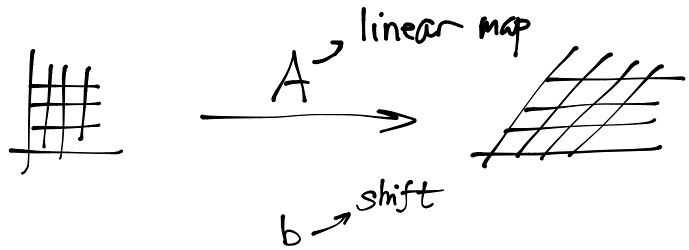
# Lecture 2 something about affine geometry

Geometrical plane can be represented by a 2-dim vector space, only need to fix one point as the origin.




$A(a+b)=Aa+Ab$

$A(\lambda a)=\lambda Aa$

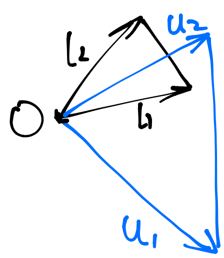
$X \to AX+b$ (In this case, circle & ellipse are equal)

Sps we have vector $x$ and $\lambda x$ then after linear transformation,
(note that they are on the same line)
$A\lambda x=\lambda Ax$, their ratio is still $\lambda$. However this fails on lines that aren't colinear.

circle, length, angle, aren't preserved.

But things like Ceva still exist in affine geometry.
Why? b/c $\triangle$ after linear Trans is still a $\triangle$.

Claim: all triangles in affine geometry are equal.

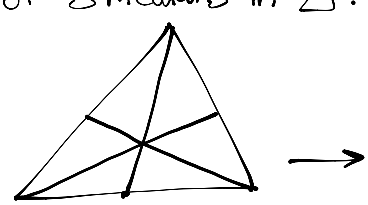



$$\begin{cases} Al_1=u_1 \\ Al_2=u_2 \end{cases}$$

But, ratio preserved: namely, $\frac{a_1}{a_2}=\frac{a_1'}{a_2'}$ ... $\Rightarrow 1=\frac{a_1b_1c_1}{a_2b_2c_2}$

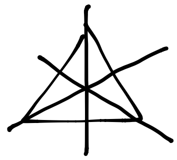
Same for Menelaus.

For 3 medians in $\triangle$.



No matter what we get, still 3 medians in a triangle.

For equalateral triangle



bisector = median = height.

* every $\Delta$ = equalateral $\Delta$, then every $\Delta$ has 3 of its median coincide at the same point.

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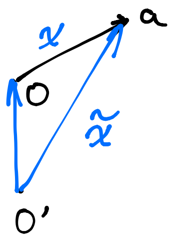
Vector combination is independent of the choice of origin.




But for single vector, here is what will happen if changing origin:

$$\tilde{x} = x + O'O$$

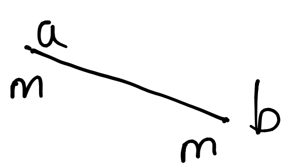
new vector = old vector + O'O

Some linear combination is well-defined if their vector sum is 1.
(note that we cannot sum up two pts, but two vectors)

Uniqueness of centre of masses

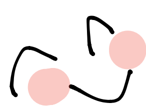
$$\frac{m_1}{\Sigma m_i} a_1 + \cdots + \frac{m_n}{\Sigma m_i} a_n$$ (what?)

- e.g.

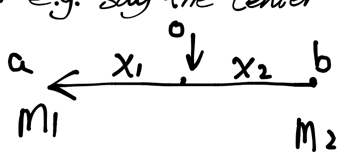



$$\frac{m}{2m} a + \frac{m}{2m} b = \frac{a+b}{2}$$

center of masses



- another e.g. say the center is




$$\frac{m_1}{m_1+m_2} a + \frac{m_2}{m_1+m_2} b$$

$$x_1 m_1 = -x_2 m_2 \quad \text{(opposite sign)}$$

$$x_1 m_1 + x_2 m_2 = 0$$

$$\frac{x_1 m_1}{m_1+m_2} + \frac{x_2 m_2}{m_1+m_2} = 0$$

• $\underbrace{a_1, \cdots, a_k;\ m_1, \cdots}_{A}$ $\underbrace{a_{k+1}, \cdots, a_N;\ m_N}_{B}$

Two ways $\begin{cases} \text{① each point, center of masses} \\ \text{② divide into two groups, center of masses} \end{cases}$

② : $M_1 = m_1 + \cdots + m_k$

$M_2 = m_{k+1} + \cdots + m_N$

$$A = \frac{1}{m_1 + \cdots + m_k}(m_1 a_1 + \cdots + m_k a_k)$$

$$B = \frac{1}{m_{k+1} + \cdots + m_N}(m_{k+1} a_{k+1} + \cdots + m_N a_N)$$

$$\frac{A \cdot M_1 + B M_2}{M_1 + M_2} = \frac{m_1 a_1 + \cdots + m_k a_k + m_{k+1} a_{k+1} + \cdots + m_N a_N}{m_1 + \cdots + m_N}$$

So basically

2 methods are identical

uniqueness proved.

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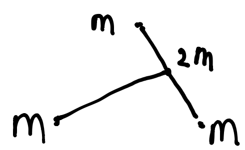
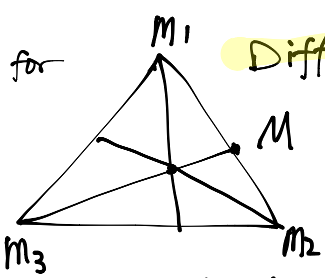



! the center of mass belongs to a median !

similarly, it belongs to the second & the third median!

(This is the equal mass situation)

for




Different Masses

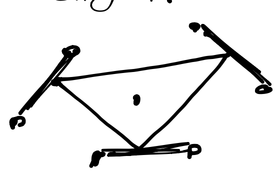
similarly, $\frac{a_1}{a_2} = \frac{m_2}{m_1}$ ... on the first Cevian

. — — — second — . .

. — — — third . . .

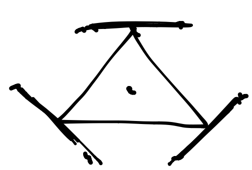
3 Cevian's common point is the center of mass

• For sixgon.

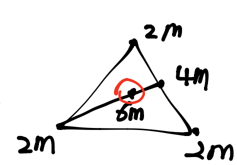


Think in this way

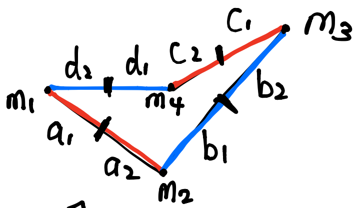
or



if equal mass on each vertix :



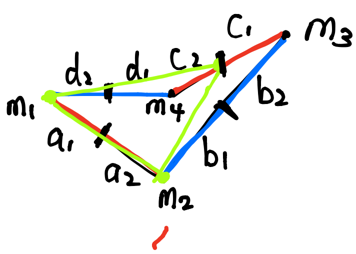
Consider 4-gon




no opposite points,
but —— sides (unlike △)

Ceva: $\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} \cdot \frac{d_1}{d_2} = 1$ (precondition) if this is true we can do the following

put 4 masses into 4 vertices.




We convert $m_3$ and $m_4$ to a merged mass, then we have a center of mass problem in a triangle now.

So all four planes will pass through the center of mass

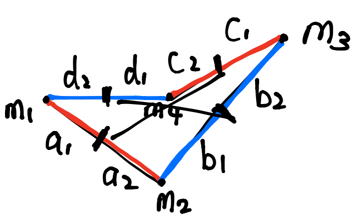
$\frac{a_1}{a_2} = \frac{m_2}{m_1}$ if $m_1$ is known, then $m_2 = \frac{a_1}{a_2}$

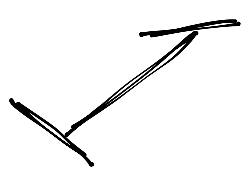
$\frac{b_1}{b_2} = \frac{m_3}{m_2}$, $m_3 = \frac{b_1}{b_2} \cdot \frac{a_1}{a_2}$

similarly $m_4 = \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2}$

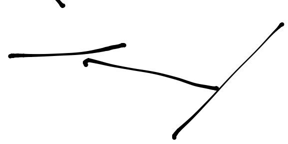
and back to $m_1 = \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} \cdot \frac{d_1}{d_2} = 1$

Menelaus: Assume $\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} \cdot \frac{d_1}{d_2} = 1$, then 4 points on the same plane.



on the same line

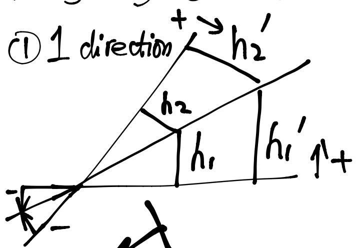


on the same line

Hence on the same plane.

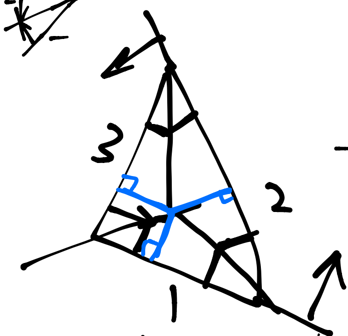
Proofs of $\hat{C}$eva & Menelaus in different way

① 1 direction




$h_2' = \lambda h_2, \quad h_1' = \lambda h_1$

$\frac{h_1}{h_2} = c$




$\frac{h_1}{h_3} = \alpha_1, \quad \frac{h_2}{h_1} = \alpha_2, \quad \frac{h_3}{h_2} = \alpha_3$ (bisectors)

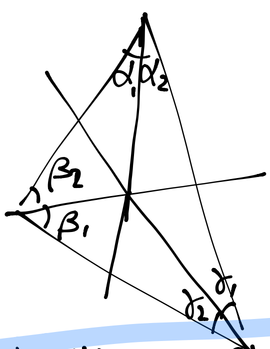
each $\alpha$ fixes 1 certain line.

When will they meet at the same point?

When they are all $\alpha_1 = \alpha_2 = \alpha_3 = 1$

② The other direction is trivial. (like $\hat{c}$eva)

Statement: $\alpha_1 = \alpha_2 = \alpha_3 = 1 \Rightarrow$ 3 bisectors pass through the same point.

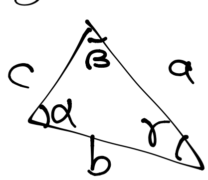



Claim: $\frac{h_1}{h_2} = \frac{\sin\alpha_1}{\sin\alpha_2}$ (by trignometry)

then we can rewrite $\hat{C}$eva:

$\frac{\sin\alpha_1}{\sin\alpha_2} \cdot \frac{\sin\beta_1}{\sin\beta_2} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} = 1 \iff$ 3 lines pass 1 point.

Why this true?




$\boxed{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}} = \frac{\sin\gamma}{c}$ Sine theorem

we only prove

& $S = \frac{1}{2} bc\sin\alpha = \frac{1}{2} ac\sin\beta$

$\Rightarrow b\sin\alpha = a\sin\beta$

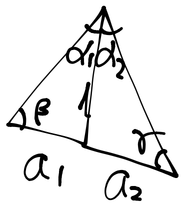
$\Rightarrow \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$

Recall:

First condition for Ceva/Menelaus : $\frac{a_1}{a_2} \cdots = 1$

Second ... $\frac{\sin\alpha_1}{\sin\alpha_2} \cdots = 1$

Similarities?




$$\frac{\sin\alpha_1}{a_1} = \frac{\sin\beta}{l}$$

$$\frac{\sin\alpha_2}{a_2} = \frac{\sin\gamma}{l}$$

$$\frac{\frac{\sin\alpha_1}{a_1}}{\frac{\sin\alpha_2}{a_2}} = \frac{\sin\beta}{\sin\gamma}$$

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{a_1}{a_2} \cdot \frac{\sin\beta}{\sin\gamma}$$

Similarly, $\frac{\sin\beta_1}{\sin\beta_2} = \frac{b_1}{b_2} \cdot \frac{\sin\gamma}{\sin\alpha}$

$$\frac{\sin\gamma_1}{\sin\gamma_2} = \frac{c_1}{c_2} \cdot \frac{\sin\alpha}{\sin\beta}$$

Next time: for menelaus and why this new form of condition is better.