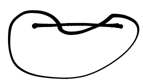
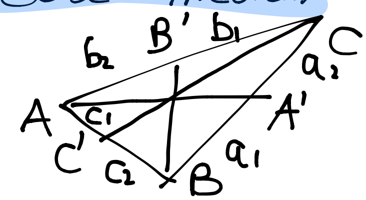
Lecture 1

In affine geometry ⬭ = ○
"angle" survived in affine geometry.
because it does not have "length".

Convex geometry.  not convex

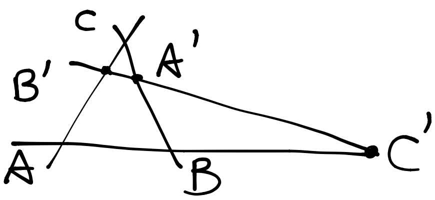
the line between any 2 pts has some part out of this figure

Čeva Theorem

① 

$$\frac{a_1}{a_2}\cdot\frac{b_1}{b_2}\cdot\frac{c_1}{c_2}=1$$

i.e. $a_1 b_1 c_1 = a_2 b_2 c_2$

② 

sufficient for three points on a line

$$\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{B'C}{B'A}=1$$

Remarks:

proportion stays the same but "length" doesn't, if we change the unit scale/direction.

$\boxed{\text{Ratio (proportion)}}$ is the same. (think about Čeva)

AB is +
BA is −

For ① : $\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}\cdot\frac{AC'}{C'B}=1$

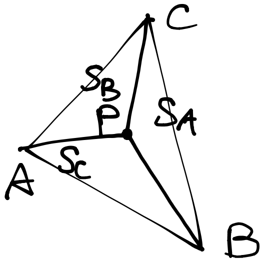
To change the way you measure segments, the formula would be different (like ②)

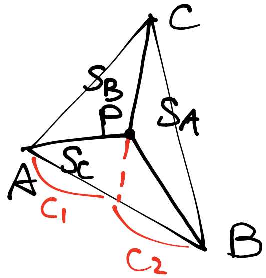
For ② :

sign change :

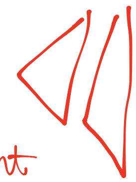
Čeva thm ⟹ Menelaus' thm

Proof of Ĉeva's Theorem:

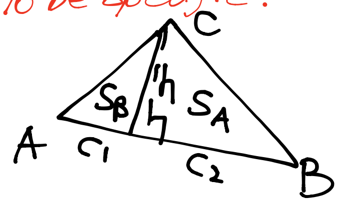




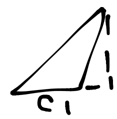
$\frac{c_1}{c_2} = \frac{S_B}{S_A}$

(why? b/c have same height  , ...)

To be specific:

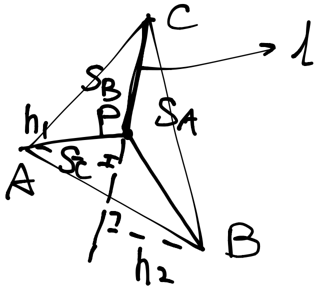


$\frac{S_B}{S_A} = \frac{c_1}{c_2}$



"same height"

$S_B = \frac{1}{2} c_1 h$, $S_A = \frac{1}{2} c_2 h$ $\qquad \frac{S_B}{S_A} = \frac{c_1}{c_2}$

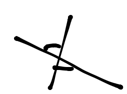


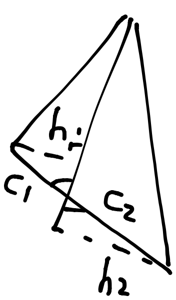
still $\frac{S_B}{S_A} = \frac{c_1}{c_2}$

same base l, but different heights h_1, h_2.

So $\frac{S_B}{S_A} = \frac{\frac{1}{2} l h_1}{\frac{1}{2} l h_2} = \frac{h_1}{h_2}$

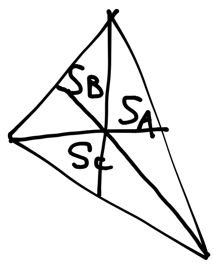
want $\frac{h_1}{h_2} = \frac{c_1}{c_2}$ (but how?)

by similar triangles (right angle and )



Therefore $\frac{S_B}{S_A} = \frac{c_1}{c_2}$

$$\frac{S_B}{S_A} = \frac{h_1}{h_2} = \frac{c_1}{c_2}$$



Similarly $\frac{b_1}{b_2} = \frac{S_C}{S_A}$, $\frac{a_1}{a_2} = \frac{S_B}{S_C}$

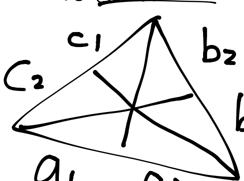
Therefore ....

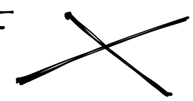
(if 3 lines pass 1 point, then ___ × ___ × ___ = 1

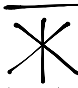
What about the converse?
If ..... = 1, then .... concurrent
Yes, it is true.

- if you know $a_1 + a_2 = A$ (sum of two, proportion of two)
  $\frac{a_1}{a_2} = X$ Then the two can be constructed uniquely.
  why? (equation solving)

Back to <u>converse</u>:



given
$a_1 b_1 c_1 = a_2 b_2 c_2$
idea: construct  then check 

whether divide ___ into a_1 & a_2 correctly
(if does, then  pass the same point)

say now, ___ is divided into a_1' and a_2'.

$a_1' + a_2' = a_1 + a_2$

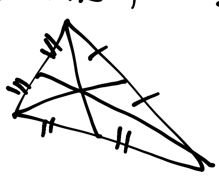
$\frac{a_1'}{a_2'} = \frac{a_1}{a_2}$

by Ĉeva ← $a_1' b_1 c_1 = a_2' b_2 c_2$ and $a_1 b_1 c_1 = a_2 b_2 c_2$ (by given)

so $\frac{a_1'}{a_1} = \frac{a_2'}{a_2}$ proved

Ancient Greeks know some particular case of Ĉeva's.

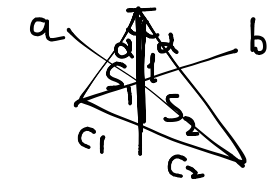
- 3 medians pass 1 point. proved by Archimedes?
  (centre of mass method)
  (be proved in next class)



$a_1 = a_2$
$b_1 = b_2$
$c_1 = c_2$

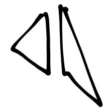
- 3 bisectors pass 1 point. (2 ways to prove)

Method I:

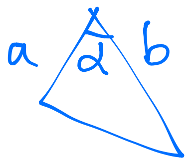


$\frac{a}{b} = \frac{c_1}{c_2}$ (have to use this lemma)

proof of the "lemma"

$\frac{c_1}{c_2} =$  $\frac{S_1}{S_2}$

Note:



$S = \frac{1}{2} ab \sin\alpha$

so

$$\left.\begin{aligned} S_1 &= \tfrac{1}{2} a l \sin\alpha \\ S_2 &= \tfrac{1}{2} b l \sin\alpha \end{aligned}\right\} \Rightarrow \frac{S_1}{S_2} = \frac{a}{b} = \frac{c_1}{c_2}$$

lemma proved

Back to the question:

So $\frac{a_1}{a_2} = \frac{b}{c}$, $\frac{b_1}{b_2} = \frac{c}{a}$, $\frac{c_1}{c_2} = \frac{a}{b}$

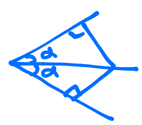
then $\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$

then by the converse of Ceva, they pass the same point.

∎

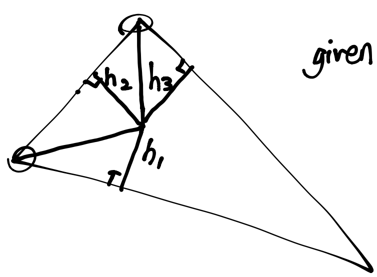
Method II: (Greek's method)

look at "bisector"



This is a bisector

Converse ✓

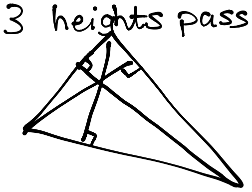


given 2 bisectors

$h_1 = h_2 = h_3$

Thus it's a bisector as well by previous results.

- 3 heights pass 1 point.



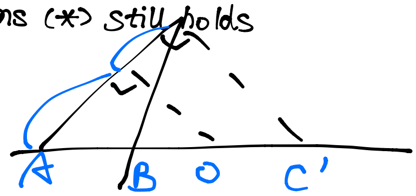
Tool: 

real line ordered points A, B, C

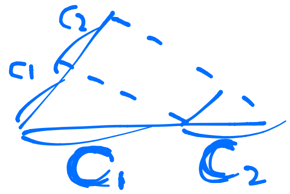
$\frac{OA}{OB} = \frac{OB}{OC} = \frac{OC}{OA} = 1$ (*)

Claim:

all proportions (*) still holds



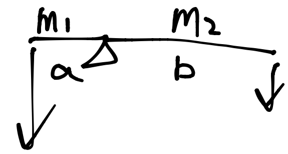
(projection)



$\frac{c_1}{c_2} = \frac{C_1}{C_2}$ why? (similar triangles)

Now we look at Archimedes' methods (General idea)
(Center of mass)

First



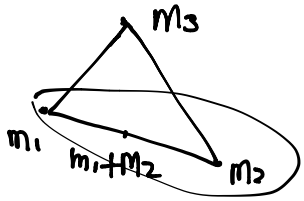
$am_1 = bm_2$
$\frac{a}{b} = \frac{m_2}{m_1}$

This is a case of 2 points what about many points on a $\triangle$ or irregular shapes.

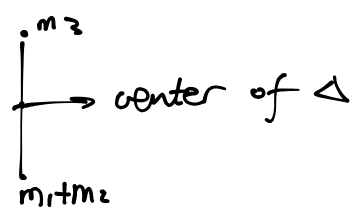
mass of right part

mass of this point

e.g.

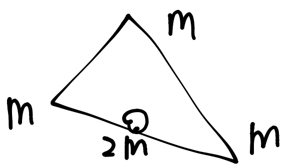


"3 mass centers"
2 parts
① m_1 & m_2 (m_1+m_2 is their centre of mass)
② m_3
Now

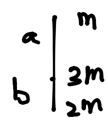


also we can do m_2, m_1+m_3 or m_1, m_2+m_3
(Centre of mass properties, next time)

Sps

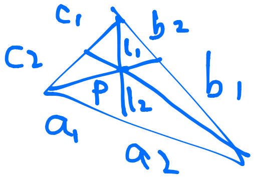


$\Longrightarrow$



$ma = 2mb$
$\frac{a}{b} = \frac{2m}{m} = 2$

3 medians pass 1 point?
Why because centre of mass is on each of median.



$a_1b_1c_1 = a_2b_2c_2$
in what proportion does P divide l_1/l_2

use centre of mass to prove

Claim: $\frac{l_1}{l_2} = \frac{b_2}{b_1} + \frac{c_1}{c_2}$!

Note that if we have 3 medians,

$\frac{b_2}{b_1} = \frac{c_1}{c_2} = 1$, then $\frac{l_1}{l_2} = 2$

$a_1 m_1 = a_2 m_2$

similarly $b_1 m_2 = b_2 m_3$

$c_1 m_3 = c_2 m_1$

centre of mass should be on these 3 lines.

$(m_1+m_2) l_2 = m_3 l_1$

$\frac{l_1}{l_2} = \frac{m_1+m_2}{m_3} = \frac{c_1}{c_2} + \frac{b_2}{b_1}$