UNIVERSITÉ D'OTTAWA • UNIVERSITY OF OTTAWA

Sciences • Science

Mathématiques et statistique
Mathematics and Statistics

MAT 2355 Final Exam

December 6, 2002. Duration: 3 hours

Family Name:______________________

First Name:______________________

Student number:______________________

1	
2	
3	
4	
5	
6	
(Bonus) 7	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. Calculators are allowed, but may only be used for computations and not for storage of data or text. The use of cell phones is not permitted.
3. The correct answer requires reasonable justification written legibly and logically –you must convince me that you know why your solution is correct.
4. Each question is worth an equal number of points. Questions 1 to 6 should be attempted by everyone. Question 7 is a bonus question, and should not be tried until all parts of questions 1-6 have been completed and checked.
5. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
6. Good luck! Bonne chance!

1. Let $P = (0, 0, 1)$, $Q = (1, 1, 0)$ and $R = (0, 2, 0)$ be 3 points in $\mathbf{R}^3$ and let H be the plane containing P, Q and R.

a) Find $b \in \mathbf{R}$ and $0 \neq a \in \mathbf{R}^3$ so that $H = \{v \in \mathbf{R}^3 \mid a \cdot v = b\}$.

b) Find $v_0 \neq v_1 \in \mathbf{R}^3$ so that $H = \{v \in \mathbf{R}^3 \mid ||v - v_0|| = ||v - v_1||\}$.

c) Show by using the formula that the reflection in H interchanges the points v_0 and v_1 you found in (b).

d) Show that the set of all points equidistant from P, Q and R is a line in $\mathbf{R}^3$.

2. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x, y) = \frac{1}{5}(3x + 4y, 4x - 3y + 5).$$

a) Show that f is an orientation reversing isometry of $\mathbf{R}^2$.

b) Find a line $L \subset \mathbf{R}^2$, and a vector w, which is parallel to L, so that f is reflection in L followed by translation by w.

c) Check that f is the same as the isometry which is translation by w followed by reflection in L.

3. Let $C_1 = \{v \in \mathbf{S}^2 \mid (-1,0,1) \cdot v = 0\}$ and $C_2 = \{v \in \mathbf{S}^2 \mid (0,1,-1) \cdot v = 0\}$ be great circles on $\mathbf{S}^2$.

a) Find $C_1 \cap C_2$.

b) Find formulae for the reflections R_{C_1} and R_{C_2} in C_1 and C_2 respectively.

c) Use your answers from (b) to find $A \in \mathbf{O}(3)$ so that $R_{C_2} \circ R_{C_1} = Av$.

d) Find a point $a \in \mathbf{S}^2$ and an angle $\theta \in [0, 2\pi)$ such that $R_{C_2} \circ R_{C_1}$ is the rotation by θ about the point $a \in \mathbf{S}^2$. (Use the formula $R_{a,\theta}(v) = (v \cdot a)(1 - \cos\theta)a + (\cos\theta)v + (\sin\theta)(a \times v)$ to check your answer.)

4. Let g be defined by $g(v) = Av$ where $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$.

a) Show that g is a isometry of $\mathbf{R}^3$.

b) Show that $g(\mathbf{S}^2) = \mathbf{S}^2$.

c) Determine the isometry type of g, giving sufficient detail to determine g.

d) Determine the isometry type of g^2, giving sufficient detail to determine g^2.

5. The latitude and longitude of Ottawa and Valencia are given below:

	Latitude	Longitude
Ottawa	$45°24' \, N$	$75°43' \, W$
Valencia	$39°28' \, N$	$0°23' \, W$

a) Find the (ϕ, θ) coordinates of Ottawa and Valencia from the given data. Express your answer in radians.

b) Find the shortest distance from Ottawa to Valencia. Assume that the earth is a sphere of radius 6380 km.

c) Find the bearing of Ottawa from Valencia. Express your answer in radians.

d) If you were to start at Valencia and maintain the bearing found in (c), where would you end up?

(Give 3 significant figures in your answers. i.e. in the form $d_1.d_2d_3 \times 10^N$, where $1 \le d_1 \le 9, 0 \le d_2, d_3 \le 9$ and N is an integer.)

6. If $N = (0, 0, 1)$, let $f : \mathbf{S}^2 - \{N\} \to H = \{(x, y, 0) \mid x, y \in \mathbf{R}\}$ be the stereographic projection, and $g : H \to \mathbf{S}^2 - \{N\}$ its inverse.

Recall that a subset $C \subset \mathbf{R}^3$ is a *circle* if there is a plane K, a point $k \in K$ and $0 < r \in \mathbf{R}$ such that $C = K \cap \{v \in \mathbf{R}^3 \mid ||v - k|| = r\}$.

a) Show that if C is any circle on $\mathbf{S}^2$ containing N then $f(C - \{N\})$ is a line in H, and conversely that if L is a line in H, then $g(L)$ lies in a circle on $\mathbf{S}^2$ containing N.

b) Give an example to show that the first statement in (a) is false if the circle on $\mathbf{S}^2$ does not contain N. Can you given a simple description of $f(C)$ if $N \notin C$?

c) What property does f share with the Mercator projection?

d) Is the image by f of a curve of constant bearing on the sphere always a line in H? If so, give a proof, if not, give an example.

7. (Bonus: Do not attempt this question until you have completed all parts of questions 1-6.)

We say that 4 points P, Q, R, S in $\mathbf{R}^3$ are *independent* if $\{P-Q, P-R, P-S\}$ is linearly independent.

Suppose that P, Q, R, S in $\mathbf{R}^3$ are independent.

a) Show that an isometry f of $\mathbf{R}^3$ is uniquely determined by $f(P), f(Q)$, $f(R)$ and $f(S)$.

b) Show carefully that an isometry of $\mathbf{R}^3$ which fixes 3 points P, Q and R is either the identity, or it is reflection in the plane through P, Q and R.

c) What kinds of isometries of $\mathbf{R}^3$ fix exactly 2 points of $\{P, Q, R, S\}$? (no proofs are necessary)

d) Give an isometry of $\mathbf{R}^3$ that is not a translation and which does not fix any points.