01/21 V

Linear Combinations

Let $\vec{v_1}, \vec{v_2}, \vec{v_3} \ldots \vec{v_k}$ be vectors.

A vector of the form

$$s_1\vec{v_1} + s_2\vec{v_2} + s_3\vec{v_3} + \ldots + s_k\vec{v_k}$$

where s_i are scalars, is called a *linear combination* of $\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}$

E.g. $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a linear combination of $\vec{v_1}$ and $\vec{v_2}$

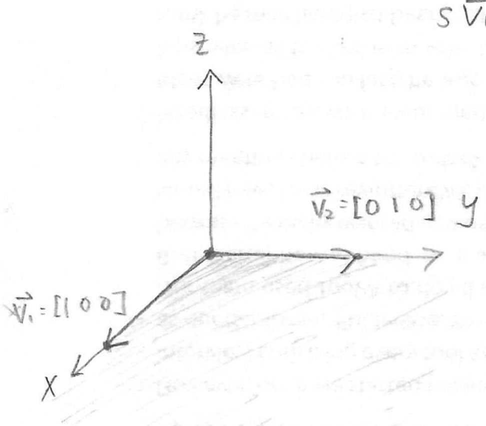
$\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\vec{v_1} + 3\vec{v_2}$

E.g. $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2\vec{v_1} + (1)\vec{v_2}$ = a linear combination of $\vec{v_1}$ and $\vec{v_2}$

Non-E.g. $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0\vec{v_1} + 0\vec{v_2} + {?}\vec{v_3}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not a linear combination of $\vec{v_1}$ and $\vec{v_2}$

$$s\vec{v_1} + t\vec{v_2} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$$



x [2 1 0]

WW. 2.10 Find the point of intersection between

L_1 $x = -t$ L_2 $x = 7 + 2s$

$y = 12 + 2t$ $y = 2 - 2s$

$z = 11 + 3t$ $z = 10 + 4s$

Note that in $\mathbb{R}^3$, 2 lines in general do not intersect, but these have been carefully chosen so they do.

$$-t = 7 + 2s$$
$$12 + 2t = 2 - 2s$$
$$11 + 3t = 10 + 4s$$

$2s + t = -7 \Rightarrow t = -7 - 2s$ $t = -7 - 2(-2) = -3$

$2t + 2s = -10$ $\Rightarrow 2(-7-2s) + 2s = -10$

$3t - 4s = -1$ $-2s = 4$

$s = -2$

$\Rightarrow 3(-3) - 4(-2) = -1$ ✓

$(s, t) = (-2, -3)$ $[3, 6, 2]$

Find the point where the line $x = -3 + 2t$

$y = 4 + 2t$

$z = -3 + 3t$

intersects the plane $-x + 2y + z = 2$

$$-(-3+2t) + 2(4+2t) + (-3+3t) = 2$$
$$3 - 2t + 8 + 4t - 3 + 3t = 2$$
$$5t = -6$$
$$t = -6/5$$

$x = -3 + 2(-6/5)$ $y = 4 + 2(-6/5)$ $z = -3 + 3(-6/5)$

$= -3 - 12/5$ $= 1.6$ $= -3 - 18/5$

$= -5.4$ $= -6.6$

$[-5.4, 1.6, -6.6]$

Span

$\text{span}\{v_1, v_2, \ldots v_k\}$

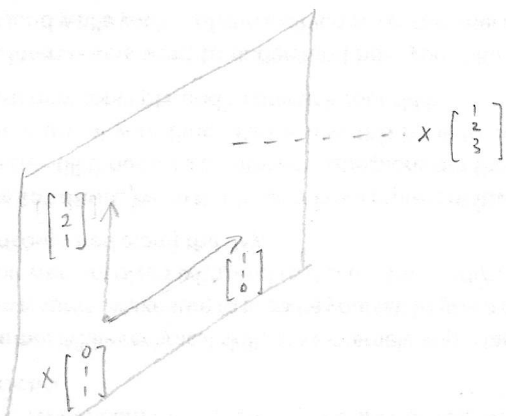
: the set of all linear combinations of $v_1, v_2 \ldots v_k$

From previous page's example,

$\text{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ is the plane P

$\begin{bmatrix}0\\1\\1\end{bmatrix}$ is in the span $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

$\begin{bmatrix}1\\2\\3\end{bmatrix}$ is not.



Rmk

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Are $\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$ linear combinations of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$?

- Suppose that $\begin{bmatrix}0\\1\\1\end{bmatrix} = s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\2\\1\end{bmatrix}$

$$\Rightarrow \left\{\begin{matrix} s + t \\ s + 2t \\ t \end{matrix}\right\} \begin{matrix} A \\ B \\ C \end{matrix} \quad \begin{bmatrix}0\\1\\1\end{bmatrix}$$

Put C into A ; $s + (1) = 0$ $\quad$ $t = 1$

$s = -1$ $\quad$ $s = -1$

Put C and s into B : $(-1) + 2(1) = 1$

$\begin{bmatrix}0\\1\\1\end{bmatrix} = (-1)\begin{bmatrix}1\\1\\0\end{bmatrix} + (1)\begin{bmatrix}1\\2\\1\end{bmatrix}$, so it is a linear combination of the two.

- Suppose that $\begin{bmatrix}1\\2\\3\end{bmatrix} = s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\2\\1\end{bmatrix}$

$$\begin{aligned} s + t &= 1 \quad A \\ s + 2t &= 2 \quad B \\ t &= 3 \quad C \end{aligned}$$

C $\Rightarrow$ A, $s + (3) = 1$

$s = -2$

$\Rightarrow$ B $(-2) + 2(3) = 4$

$4 \neq 2$ $\therefore$ contradiction

No solution; $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is not a linear combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$

Not on the plane : see spans, next pg

WW3 4 $\vec{u} = \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} -7 \\ 5 \\ h \end{bmatrix}$

For what value(s) of h can $\vec{w}$ be expressed as a linear combination of $\vec{u}$ and $\vec{v}$?

$$x[-1, 3, -5] + y[4, -4, 3] = [-7, 5, h]$$

$$-x + 4y = -7 \quad \text{A}$$
$$3x - 4y = 5 \quad \text{B}$$
$$-5x + 3y = h \quad \text{C}$$

A + B : $2x = -2, \quad x = -1$ D

D → A $-(-1) + 4y = 7, \quad y = 3/2$ E

Check B $3(-1) - 4(3/2) = -9$ X

A − B : $-4x + 8y = -12$

$$-x + 2y = -3$$
$$x = 3 + 2y \quad \text{F}$$

F → A $-3 + 2y = -7$

$$y = -2 \quad \rightarrow \text{F}$$
$$x = 3 + 2(-2)$$
$$x = -1 \quad \text{G}$$

Check: $-(-1) + 4(-2) = -7$ ✓

$3(-1) - 4(-2) = 5$ ✓ $(x, y) = (-1, -2)$

$$-5(-1) + 3(-2) = h$$
$$5 - 6 = h$$
$$h = -1$$

WN. 3. 1) $\vec{u} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Are these linear combinations? (of $\vec{u}$ and $\vec{v}$)

$$s\vec{u} + t\vec{v} = s[-2, 2, -1] + t[1, 1, -2]$$

$$= \begin{bmatrix} -2s \\ 2s \\ -s \end{bmatrix} + \begin{bmatrix} t \\ t \\ -2t \end{bmatrix} = \begin{bmatrix} t-2s \\ t+2s \\ -2t-s \end{bmatrix}$$

a) $\begin{bmatrix} -3 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} t-2s \\ t+2s \\ -2t-s \end{bmatrix}$

$5 = t + 2s$

$t = 5 - 2s$

$t = 5 - 2(2)$

$= 1$

$(5-2s) - 2s = -3$

$5 - 4s = -3$

$s = 2$

$-2(1) - (2)$

$= -4$

$(s, t) = (2, 1)$

b) $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} t-2s \\ t+2s \\ -2t-s \end{bmatrix}$

$0 = -2t - s$

$s = -2t$

$s = -2(-1/3)$

$= 2/3$

$t - 2(-2t) = 1$

$-3t = 1$

$t = -1/3$

$(-1/3) + 2(2/3) = 3$

$4/3 - 1/3 = 1$

Contradiction

Not a linear combination.

c) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} t-2s \\ t+2s \\ -2t-s \end{bmatrix}$

$2 = t - 2s$

$t = 2 + 2s$

$t = 2 + 2(-3/4)$

$= 1/2$

$(2+2s) + 2s = -1$

$4s = -3$

$s = -3/4$

$3 = -2(1/2) - (-3/4)$

$= -1 + 3/4$

Contradiction

Not a linear combination

d) $\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} t-2s \\ t+2s \\ -2t-s \end{bmatrix}$

$4 = t - 2s$

$t = 4 + 2s$

$t = 4 + 2(-1)$

$= 2$

$0 = (4+2s) + 2s$

$s = -1$

$-3 = -2(2) - (-1)$

$= -4 + 1$

$(s, t) = -1, 2$