Final Exam 2009

CVG2181, Winter 2010

Instructor: Dr. Majid Mohammadian
majid.mohammadian@uottawa.ca

Problem 1, 15 points

Find the distance traveled from t=1 to t=10 based on the following data

t, min	1	2	3.25	4.5	6	7	8	9	9.5	10
v, m/s	5	6	5.5	7	8.5	8	6	7	7	5

(a) Using the trapezoidal method (b) using the best combination of the trapezoidal and Simpson's 1/3 and 3/8 methods.

Solution:

(a) Trapezoidal method:

$$I = (2-1)\frac{5+6}{2} + (3.25-2)\frac{6+5.5}{2} + \bullet\bullet\bullet = 60.375\frac{\text{m}\cdot\text{min}}{\text{s}} \times \frac{60\,\text{s}}{\text{min}} = 3{,}622.5\,\text{m}$$

(b) Combination

$$I = (2-1)\frac{5+6}{2} + (4.5-2)\frac{6+4(5.5)+7}{6} + (6-4.5)\frac{7+8.5}{2}$$

$$+ (9-6)\frac{8.5+3(8+6)+7}{8} + (10-9)\frac{7+4(7)+5}{6} = 59.9375\frac{\text{m}\cdot\text{min}}{\text{s}} \times \frac{60\,\text{s}}{\text{min}} = 3{,}596.25\,\text{m}$$

Problem 2, 15 points

Use the Richardson extrapolation to estimate the first derivative of y=cos(x) by using $h_1=\pi/3$ and $h_2=\pi/6$.

Use the 2nd order central difference method for the first estimates.

The exact value is $-\sin(\pi/4)= -0.7071678$

Solution

The 2nd order central difference method: $$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

$$D(\pi/3) = \frac{-0.25882-0.965926}{2(1.047198)} = -0.58477$$

$$D(\pi/6) = \frac{0.258819-0.965926}{2(0.523599)} = -0.67524$$

$$D = \frac{4}{3}(-0.67524) - \frac{1}{3}(-0.58477) = -0.70539$$

Problem 3, 15 points

Evaluate the following integral using the five point Gauss method

$$I = \int_0^1 e^{x^2}\, dx$$

Parameters of the five point Gauss method:

I	t_i	c_i
0	-0.90618	0.236927
1	-0.53847	0.478629
2	0	0.568889
3	0.538469	0.478629
4	0.90618	0.236927

$$a = 0$$

$$b = 1$$

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} g(t)dt \text{ where } g(t) = f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) = e^{\frac{(t+1)^2}{4}}$$

$$\int_0^1 e^{x^2} dx = \frac{1}{2}\int_{-1}^{1} e^{\frac{(t+1)^2}{4}} dt = \frac{1}{2}\sum_{i=1}^{4} c_i g(t_i)$$

$$= \frac{1}{2}\begin{pmatrix} 0.2369269 e^{\frac{(1-0.906179)^2}{4}} + 0.4786287 e^{\frac{(1-0.538469)^2}{4}} + 0.568888889 e^{\frac{(1+0)^2}{4}} \\ +0.4786287 e^{\frac{(1+0.538469)^2}{4}} + 0.2369269 e^{\frac{(1+0.906179)^2}{4}} \end{pmatrix} = 1.46255$$

Problem 4, 20 points

Solve the following ODE from t=1.5 to 2=2.5

$$\frac{dy}{dt} = -\frac{2y}{1+t} \qquad y(0)=2$$

using the fourth order Adams method without any iterations on the corrector step (m=1) with a time step of 0.5. Do not use modifiers.
Use the Euler method to generate the needed point for starting the Adams method.

SOLUTION

$$\frac{dy}{dt} = \frac{-2y}{1+t} = f(t, y)$$

Euler method:$y_{i+1} = y_i + f(t_i, y_i)h$

$$y_{i-3}^m = y(0) = 2;\ f_{i-3}^m = -4$$

$$y_{i-2}^m = y(0.5) = y(0) + f(0,2)\left(0.5\right) = 2 - 4\left(0.5\right) = 0;\ f_{i-2}^m = 0$$

$$y_{i-1}^m = y(1) = y(0.5) + f(0.5,0)\left(0.5\right) = 0;\ f_{i-1}^m = 0$$

$$y_i^m = y(1.5) = y(1) + f(1,0)\left(0.5\right) = 0;\ f_i^m = 0$$

Fourth-Order Adams Method

- Uses the fourth-order Adams-Bashforth formula as the predictor:

$$y_{i+1}^{0} = y_i^m + h\left(\frac{55}{24} f_i^m - \frac{59}{24} f_{i-1}^m + \frac{37}{24} f_{i-2}^m - \frac{9}{24} f_{i-3}^m\right)$$

and the fourth-order Adams-Moulton formula as the corrector:

$$y_{i+1}^{j} = y_i^m + h\left(\frac{9}{24} f_{i+1}^{j-1} + \frac{19}{24} f_i^m - \frac{5}{24} f_{i-1}^m + \frac{1}{24} f_{i-2}^m\right) \qquad for\ j = 1, 2, \ldots, m$$

Or (simpler notations)

Predictor $$y_{i+1}^{p} = y_i^m + h\left(\frac{55}{24} f_i^m - \frac{59}{24} f_{i-1}^m + \frac{37}{24} f_{i-2}^m - \frac{9}{24} f_{i-3}^m\right)$$

Corrector $$y_{i+1}^{c} = y_i^m + h\left(\frac{9}{24} f_{i+1}^p + \frac{19}{24} f_i^m - \frac{5}{24} f_{i-1}^m + \frac{1}{24} f_{i-2}^m\right)$$

Using the above formulas:

PREDICTOR(t=2)

$$y_{i+1}^{p} = y_i^m + h(\frac{55}{24} f_i^m - \frac{59}{24} f_{i-1}^m + \frac{37}{24} f_{i-2}^m - \frac{9}{24} f_{i-3}^m) = 0 + 0.5\left(-\frac{9}{24}(-4)\right) = 0.75$$

$$f_{i+1}^{p} = \frac{-2\times 0.75}{2+1} = -0.5$$

CORRECTOR(t=2)

$$\mathrm{y}_{i+1}^{c} = y_i^m + h\left(\frac{9}{24} f_{i+1}^p + \frac{19}{24} f_i^m - \frac{5}{24} f_{i-1}^m + \frac{1}{24} f_{i-2}^m\right) = 0 + 0.5\left(\frac{9}{24}\left(-0.5\right)\right) = -0.09375$$

$$f_{i+1}^{c} = f_{i+1}^{m} = \frac{-2\left(-0.09375\right)}{2+1} = 0.760417$$

PREDICTOR (t=2.5)

$$y_{i+1}^{p} = y_i^m + h(\frac{55}{24} f_i^m - \frac{59}{24} f_{i-1}^m + \frac{37}{24} f_{i-2}^m - \frac{9}{24} f_{i-3}^m) =$$

$$-0.09375 + 0.5\left(\frac{55}{24}(0.760417) - \frac{59}{24}(0) + \frac{37}{24}(0) - \frac{9}{24}(0)\right) = -0.23633$$

$$f_{i+1}^{p} = \frac{-2\times(-0.23633)}{2.5+1} = 0.135045$$

CORRECTOR (t=2.5)

$$\mathrm{y}_{i+1}^{c} = y_i^m + h\left(\frac{9}{24} f_{i+1}^p + \frac{19}{24} f_i^m - \frac{5}{24} f_{i-1}^m + \frac{1}{24} f_{i-2}^m\right) =$$

$$-0.09375 + 0.5\left(\frac{9}{24}(0.135045) + \frac{19}{24}(0.760417) - \frac{5}{24}(0)\frac{5}{24}(0)\right) = -0.232569$$