

Concordia University--Department of Mathematics & Statistics

Course	Number	Section(s)
MATH	364 (and 626)	ALL
Examination	Date	Pages
Final	December 2012	2
Instructors	Course Examiner	
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Special Instructions: No materials, no calculators.		

1. (12 POINTS) Let S be a subset of R which is bounded above and denote $\alpha = \sup S$.
 - (a) Prove that for any $\varepsilon > 0$, there exists $s \in S$ such that $\alpha - \varepsilon < s \leq \alpha$.
 - (b) Prove that there exists a sequence $\{s_n\}_{n=1}^{\infty} \subset S$ such that $s_n \rightarrow \alpha$.
2. (12 POINTS)
 - (a) Prove that $f(x) = 2x + 5$ is uniformly continuous on R .
 - (b) Suppose that a function f is uniformly continuous on an interval D . Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in D . Prove that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.
 - (c) Prove that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1]$.
 - (d) Prove that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$.
3. (9 POINTS)
 - (a) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in the interval $[a, \infty)$. Prove that if x_n converges to x , then $x \geq a$.
 - (b) Suppose that $f : R \rightarrow R$ is continuous at c , and that $f(c) > 0$. Prove that there is a neighborhood U of c such that $f(x) > 0$ for all $x \in U$.
 - (c) Use the Intermediate Value Theorem to show that the function $f(x) = 3x^4 + x - 7$ has a root in the interval $[1, 2]$.
4. (9 POINTS)
 - (a) Suppose that S is an infinite subset of R . Prove that S contains a sequence of distinct points.
 - (b) Suppose that S is a subset of R which is unbounded above. Prove that S contains a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \rightarrow \infty$.
 - (c) Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is unbounded above. Prove that it has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \rightarrow \infty$.

5. (14 POINTS)

- (a) Suppose $\{x_n\}_{n=1}^{\infty}$ is a nonconvergent sequence. Prove that for any $x \in \mathcal{R}$ there exists $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\|x_{n_k} - x\| \geq \varepsilon$ for all k in N .
- (b) Also show that there are at least two subsequences of $\{x_n\}_{n=1}^{\infty}$ with distinct limits.

6. (11 POINTS) Use induction to prove that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

7. (12 POINTS) Let $f : \mathcal{R} \rightarrow \mathcal{R}$ be given by

$$f(x) = \begin{cases} x^4 \sin\left(\frac{5}{x^2+x^3}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f is differentiable at $x = 0$ and find $f'(0)$.

8. (9 POINTS) Find the Taylor polynomial $P_3(x)$ for the function $f(x) = x^{1/2}$ about the point $x = 1$ and state the formula for the remainder, $R_3(x)$.
9. (12 POINTS) Suppose that the function $f : \mathcal{R} \rightarrow \mathcal{R}$ is differentiable on \mathcal{R} . Prove that f is monotone increasing on \mathcal{R} if and only if $f'(x) \geq 0$ for all $x \in \mathcal{R}$.

BONUS: (10 POINTS)

Suppose $0 < b < a$. Prove that $\ln\left(\frac{a}{b}\right) \leq \frac{a-b}{b}$.