Department of Mathematics, University of Toronto
MAT223H1F - Linear Algebra I
Fall 2015

Tutorial Problems 1

Key terms and ideas:

Here are the key terms and ideas we are trying to demonstrate in this tutorial question set. The page number refers to the course text book.

1. System of linear equations and augmented matrices (p. 1-3).
2. Elementary row operations on a matrix (p. 5).
3. (Reduced) Row-echelon form and Gaussian algorithm (p. 10).
4. Rank of a matrix (p. 14).
5. Homogeneous linear system (p. 18).
6. Linear combinations (p.20).
7. Basic solutions (p.22-23).

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1. Consider the system of linear equations

$$\begin{aligned} x_1 - 3x_2 &= 2 \\ -2x_1 + 6x_2 &= -4 \end{aligned}$$

(a) Write the augmented matrix of the system, find all solutions, and express your answer in parametric form.

(b) Give two different particular solutions to the given system (your choice). Call your solutions $\mathbf{x}$ and $\mathbf{y}$. Show that $\dfrac{\mathbf{x}+\mathbf{y}}{2}$ is a solution to the system but $\mathbf{x}+\mathbf{y}$ is not.

(c) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are two solutions to the given system, then $a\mathbf{x}+b\mathbf{y}$ is also a solution if and only if $a+b=1$.

(d) Prove that, in general, a system of linear equations cannot have exactly two solutions.

Solution:

(a) First we give the augmented matrix of the linear system and row-reduce using elementary row operations.

$$\left[\begin{array}{cc|c} 1 & -3 & 2 \\ -2 & 6 & -4 \end{array}\right] \xrightarrow{2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

We see that this system has one parameter and infinite solutions. Let $x_2 = r$, then

$$x_1 - 3x_2 = 2 \Rightarrow x_1 = 2 + 3r.$$

Thus the solution to this system in parametric form is:

$$\begin{aligned} x_1 &= 3r + 2 \\ x_2 &= r. \end{aligned}$$

Notice that geometrically, we interpret this solution to be that the two lines in the linear equations are identical.

(b) To obtain two different solutions to this system we have to choose two different values for our parameter r. Let $\mathbf{x}$ be the solution obtained with $r = 0$, that is

$$\mathbf{x} = \begin{bmatrix} 3(0) + 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Similarly, let $\mathbf{y}$ be the solution obtained with $r = 1$, $\mathbf{y} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. We see that

$$\frac{\mathbf{x} + \mathbf{y}}{2} = \frac{1}{2}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix}.$$

To check that this is a solution to the system, we can substitute $x_1 = 7/2$ and $x_2 = 1/2$ into the left-hand side (LHS) of the system and see if we obtain the right-hand side (RHS). We see that

$$\begin{aligned} x_1 - 3x_2 &= 7/2 - 3/2 = 4/2 = 2 \\ -2x_1 + 6x_2 &= -14/2 + 6/2 = -7 + 3 = -4. \end{aligned}$$

Thus we verified that $\frac{\mathbf{x}+\mathbf{y}}{2}$ is a solution.

To see that $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ is not a solution, we again plug it into the system and see if the LHS equals the RHS.

$$\begin{aligned} x_1 - 3x_2 &= 7 - 3 = 4 \neq 2 \\ -2x_1 + 6x_2 &= -14 + 6 = -8 \neq -4. \end{aligned}$$

Since the LHS is not equal to the RHS, $\mathbf{x} + \mathbf{y}$ is not a solution to the system.

(c) Let us express the two solutions as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The assumption we will be working with is that $\mathbf{x}$ and $\mathbf{y}$ are two solutions to the given system (i.e. this is already given to us). There are two parts to the proof, first we will prove $a\mathbf{x} + b\mathbf{y}$ is a solution implies $a + b = 1$ (the ($\Rightarrow$) part), then we will show that $a + b = 1$ implies $a\mathbf{x} + b\mathbf{y}$ (the ($\Leftarrow$) part).

($\Rightarrow$)

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := a\mathbf{x} + b\mathbf{y}$ is a solution means that $z_1 = ax_1 + by_1$, $z_2 = ax_2 + by_2$ satisfy the system. So we have

$$(ax_1 + by_1) - 3(ax_2 + by_2) = 2 \qquad \text{and} \qquad -2(ax_1 + by_1) + 6(ax_2 + by_2) = -4.$$

Rearranging gives

$$a(x_1 - 3x_2) + b(y_1 - 3y_2) = 2 \qquad \text{and} \qquad a(-2x_1 + 6x_2) + b(-2y_1 + 6y_2) = -4.$$

Since $\mathbf{x}$, $\mathbf{y}$ both satisfy the system, we can substitute and get

$$a(2) + b(2) = 2 \qquad \text{and} \qquad a(-4) + b(-4) = -4, \tag{1}$$

which both reduce to Since $\mathbf{x}$, $\mathbf{y}$ both satisfy the system, we can substitute and get

$$a + b = 1 \qquad \text{and} \qquad a + b = 1.$$

Thus we can conclude that $a + b = 1$.

($\Leftarrow$)

Suppose $a + b = 1$ then

$$2(a + b) = 2 \qquad \text{and} \qquad -4(a + b) = -4,$$

then by reversing what we did from (**??**), we get that

$$(ax_1 + by_1) - 3(ax_2 + by_2) = 2 \qquad \text{and} \qquad -2(ax_1 + by_1) + 6(ax_2 + by_2) = -4.$$

which implies $\begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$ is a solution to the system as desired.

(d) Consider a general linear system

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= b_1 \\ c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n &= b_2 \\ &\vdots \\ c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n &= b_m \end{aligned}$$

Let $\mathbf{y}$ and $\mathbf{z}$ be two distinct solutions (distinct means that $\mathbf{y} \neq \mathbf{z}$). Then let $a = 1/2$, $b = 1/2$ and consider the vector $\mathbf{w} = a\mathbf{y} + b\mathbf{z}$. For any equation in the system, say the k-th one, we substitute the coordinates of $\mathbf{w}$ to see if the LHS equals the RHS. If they are equal for an arbitrary equation of the system, then we can conclude that $\mathbf{w}$ is a solution, and it is certainly not equal to $\mathbf{y}$ or $\mathbf{z}$ Thus a system cannot have exactly two solutions since if it did, we can always generate a third, distinct one.

So let us check the computations.

$$\begin{aligned} c_{k1}w_1 + c_{k2}w_2 + \cdots + c_{kn}w_n &= c_{k1}(ay_1 + bz_1) + c_{k2}(ay_2 + bz_2) + \cdots + c_{kn}(ay_n + bz_n) \\ &= a(c_{k1}y_1 + c_{k2}y_2 + \cdots + c_{kn}y_n) + b(c_{k1}z_1 + c_{k2}z_2 + \cdots + c_{kn}z_n) \\ &= a(b_k) + b(b_k) \\ &= 1/2(b_k) + 1/2(b_k) \\ &= b_k. \end{aligned}$$

We made no assumptions about which equation we are checking, which means that for any equation in the system, the LHS is equal to the RHS. Thus $\mathbf{w}$ is a third distinct solution to the system and a system cannot have exactly two solutions.

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2. Consider the system of linear equations

$$\begin{aligned} ax_1 + bx_2 &= c \\ bx_1 + cx_2 &= d \end{aligned}$$

where $a \neq 0$. Find conditions on the remaining constants b, c, d such that the system has a unique solution; infinitely many solutions; or no solutions.

Solution: Let us first convert this system into a matrix.

$$\left[\begin{array}{cc|c} a & b & c \\ b & c & d \end{array}\right].$$

Notice that $a \neq 0$, so we can divide $R1$ by a.

$$\left[\begin{array}{cc|c} 1 & b/a & c/a \\ b & c & d \end{array}\right].$$

We then add $-bR1$ to $R2$ to get

$$\left[\begin{array}{cc|c} 1 & b/a & c/a \\ 0 & c - b^2/a & d - bc/a \end{array}\right]$$

1. The system will have infinitely many solutions if $c - b^2/a = 0$ and $d - bc/a = 0$, that is $ac = b^2$ and $ad = bc$.
2. The system will have exactly one solution if $c - b^2/a \neq 0$, that is $ac \neq b^2$.
3. The system will have no solutions if $c - b^2/a = 0$ and $d - bc/a \neq 0$, that is $ac = b^2$ and $ad \neq bc$.

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3. A homogeneous system of linear equations has augmented matrix

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \end{array}\right].$$

Show that the system has a unique solution if and only if $a + b + c \neq 0$, and a, b, c are not all equal.

Solution: This is an if-and-only-if statement, thus there are two parts to prove.
($\Rightarrow$)
First let us add $R2$ and $R3$ to $R1$ so that we get

$$\left[\begin{array}{ccc|c} a+b+c & a+b+c & a+b+c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \end{array}\right]$$

If the system has a unique solution, then it cannot have any rows of zeros, thus $a+b+c \neq 0$. As $a+b+c \neq 0$, we can divide $R1$ by $a+b+c$ to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ c & a & b & 0 \\ b & c & a & 0 \end{array}\right]$$

Then adding $-cR1$ to $R2$ and $-bR1$ to $R3$ we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & a-c & b-c & 0 \\ 0 & c-b & a-b & 0 \end{array}\right]$$

Now we have to consider 2 cases:

- Suppose $b - c \neq 0$, i.e. $b \neq c$. Then we are done since we have shown that a, b, c are not all equal.
- Suppose $b - c = 0$ i.e. $b = c$. Then
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \end{array}\right]$$
Then $a - b \neq 0$ as the system has a unique solution, so $a \neq b$. Thus a, b, c are not all equal.

($\Leftarrow$)

Now we work with the assumption that a, b, c are not all equal and $a+b+c \neq 0$. Notice that $a+b+c \neq 0$ means we can divide by $a + b + c$. We will begin by row-reducing the augmented matrix.

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \end{array}\right] \xrightarrow{R2+R3+R1 \to R1} \left[\begin{array}{ccc|c} a+b+c & a+b+c & a+b+c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \end{array}\right]$$
$$\xrightarrow{1/(a+b+c)R1 \to R1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ c & a & b & 0 \\ b & c & a & 0 \end{array}\right]$$
$$\xrightarrow[-bR1+R3 \to R3]{-cR1+R2 \to R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & c-b & a-b & 0 \\ 0 & a-c & b-c & 0 \end{array}\right]$$

Again, we have to consider 2 cases:

- Suppose $b - c = 0$ i.e. $b = c$. Then
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \end{array}\right].$$
However as a, b, c are not all equal and $b = c$, then $a \neq b$. Thus $a - b \neq 0$ and we can divide $R2$, $R3$ by $a - b$ to get
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$
so we have the unique solution $x_1 = x_2 = x_3 = 0$.
- Suppose $b - c \neq 0$, i.e. $b \neq c$. Then we can swap $R2$ and $R3$ and divide both by $b - c$ to get
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & (a-b)/(c-b) & 0 \\ 0 & a-c & b-c & 0 \end{array}\right].$$
Then we add $-(a - c)R2$ to $R3$ to get
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & (a-b)/(c-b) & 0 \\ 0 & 0 & (b-c) - \frac{(a-c)(a-b)}{c-b} & 0 \end{array}\right].$$

Let us focus on $R3$.

$$\begin{aligned}(b-c) - \frac{(a-c)(a-b)}{c-b} &= (b-c) + \frac{(a-c)(a-b)}{b-c} \\ &= \frac{(b-c)^2 + (a-c)(a-b)}{b-c}\end{aligned}$$

Let us look at the numerator of $\frac{(b-c)^2+(a-c)(a-b)}{b-c}$

$$\begin{aligned}(b-c)^2 + (a-c)(a-b) &= a^2 + b^2 + c^2 - ab - ac - bc \\ &= \frac{1}{2}[(a-b)^2 + (c-a)^2 + (b-c)^2]\end{aligned}$$

Since $b \neq c$, then $(b-c)^2 > 0$ and so $(a-b)^2 + (c-a)^2 + (b-c)^2 > 0$ (remember that the square of a number is always nonnegative). Then $\frac{1}{2}[(a-b)^2 + (c-a)^2 + (b-c)^2]/(b-c) > 0$ and so we can divide by it to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & (a-b)/(c-b) & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

and we have the unique solution $x_1 = x_2 = x_3 = 0$.

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4 (a) Let A be an $m \times n$ matrix, prove that $0 \leq \text{rank}(A) \leq$ the minimum between m and n. In symbols, $0 \leq \text{rank}(A) \leq \min\{m, n\}$.

4 (b) Use part (a) and Theorem 2 on page 15 of the textbook to prove that if a consistent system of linear equations has more unknowns than equations, then the system has infinitely many solutions.

4 (c) Find an example to show that if we omit the word 'consistent' from the statement in part (b), the statement is no longer true.

Solution:

(a) A has m rows and n columns. Since rank is the number of leading ones in row-echelon form that the matrix A can be carried to by elementary row operations, then every row has at most one leading 1. Thus the number of leading ones is less than or equal to the number of rows, i.e. $\text{rank}(A) \leq m$. Similarly, every column has at most one leading 1 and so $\text{rank}(A) \leq n$. Therefore if $m \leq n$, then $\text{rank}(A) \leq m$, and if $n \leq m$ then $\text{rank}(A) \leq m$. Thus $\text{rank}(A) \leq \min\{m, n\}$.

(b) Let A be the augment matrix of a linear system, A has m rows and n columns corresponding to m equations in n variables. If a consistent system has more unknowns than equations, then $n > m$. This means that $\min\{m, n\} = m$. By part (a), $\text{rank}(A) \leq \min\{m, n\} = m < n$ and by Theorem 2 part (2) the system has infinitely many solutions.

(c) Consider the following counterexample:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

This system is not consistent (inconsistent) since otherwise it would imply $1 = 2$. Here $m = 2$ and $n = 3$. Let us look at its corresponding augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right].$$

If we subtract $R2$ from $R1$ then we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Thus its rank r is 1. So $r < n$ but as we see the system has no solutions.