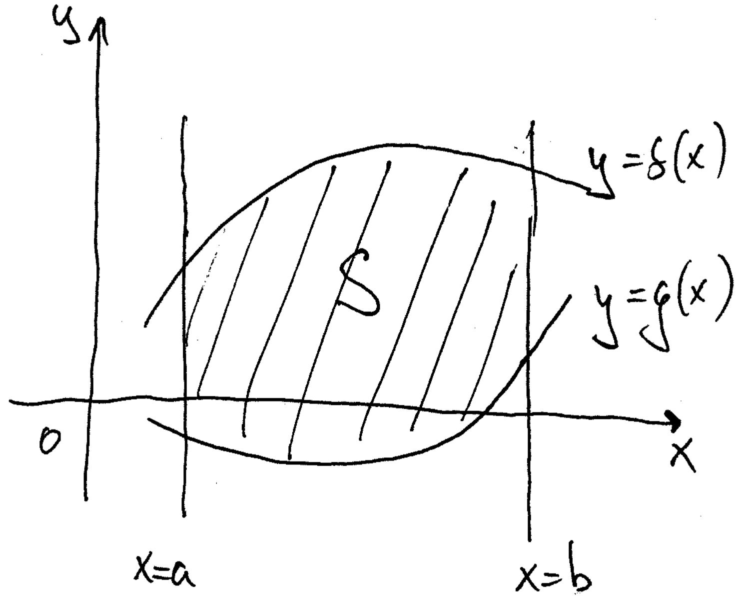
Areas between Curves

1-1

The region S lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$.



<u>Find the area</u> of the region S.

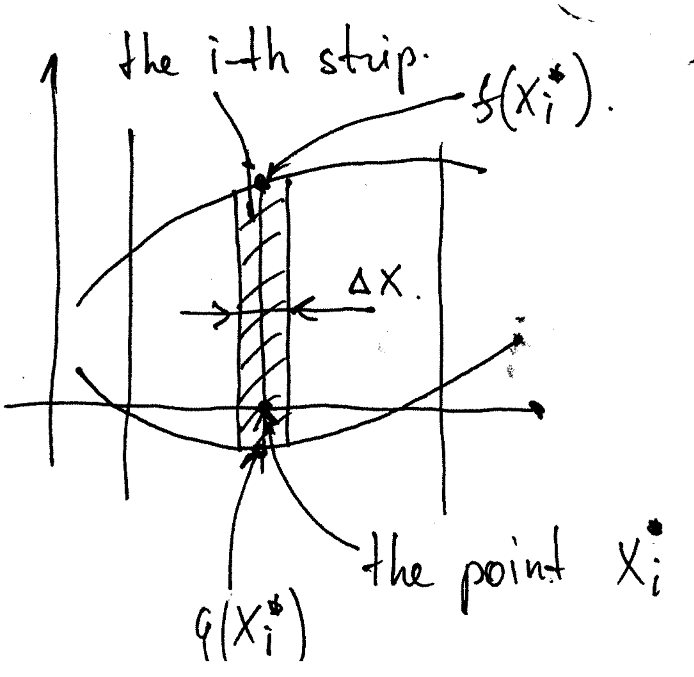
Important Assumption

- $a \le b$
- $f(x) \ge g(x)$ for all x in $[a, b]$

1-2

<u>Idea:</u>

- Subdivide S into n strips of equal width.
- The area of S is then the sum of areas of the respective strips.



The area of the i-strip =

$$= \left(f(x_i^*) - g(x_i^*)\right) \cdot \Delta x.$$

The total area is

$$\sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right) \cdot \Delta x$$

(the Riemann sum).

• We set $n \to \infty$ and obtain. (1-3)

$$\underset{(\text{the area of } S)}{A} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right) \Delta x$$

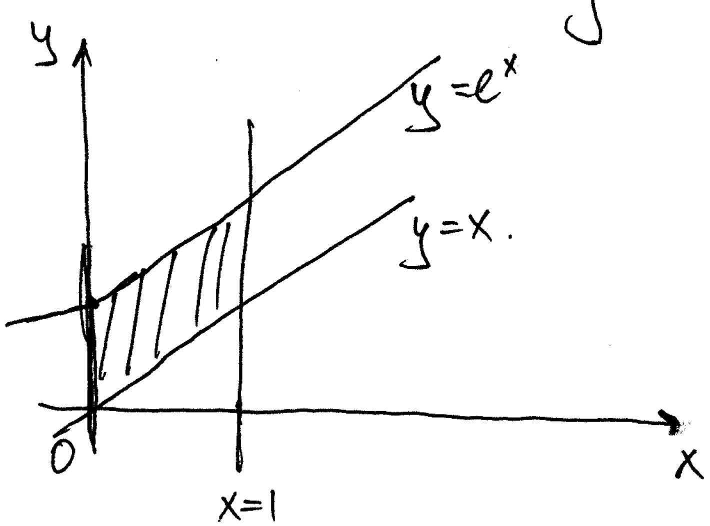
that is the definite integral of $f-g$.

So, if $a \le b$ and $f(x) \ge g(x)$, $\forall x \in [a,b]$, then

$$A = \int_a^b \left(f(x) - g(x)\right) dx.$$

Ex 1. Find the area of the region (1-4)

bounded above by $y = e^x$
below by $y = x$ $\Big] \Rightarrow e^x \ge x$ for all $x \in [0,1]$.
on the sides by $x = 0$ and $x = 1$



$$A = \int_0^1 (e^x - x)\,dx =$$

$$= e^x - \frac{1}{2}x^2 \Big]_0^1 =$$

$$= \left(e - \frac{1}{2}\right) - 1 = e - 1.5$$

Always: 1. Verify the assumption. 2. Apply the formula.

Ex.2. Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$. (1-5)

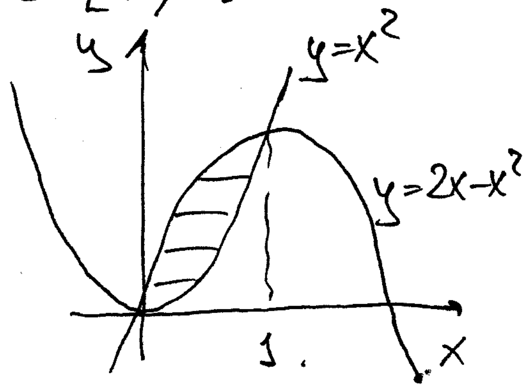
1. find the left and right bounds (intersection points).

$$x^2 = 2x - x^2$$

$$2x(x-1) = 0 \quad \Rightarrow \quad x = 0 \text{ and } x = 1.$$

Check $x^2 \leq 2x - x^2$ for all $x \in [0,1]$.

the second function — the first function



$$A = \int_0^1 (2x - x^2 - x^2)\,dx = \frac{1}{3}.$$

Ex. 3 Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$. (1-6)

1. Find approximate solution to $x/\sqrt{x^2+1} = x^4 - x$

$\Rightarrow x \approx 1.18$ (using e.g. Newton's method).

2. Check the assumption

3. $$A = \int_0^{1.18} \left[\frac{x}{\sqrt{x^2+1}} - (x^4 - x) \right] dx \overset{u=x^2+1 \quad du = 2x. \quad u \approx 2.39}{=\!=}$$

$$= \sqrt{u}\,\Big]_1^{2.39} - \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_0^{1.18} = \sqrt{2.39} - 1 - \frac{(1.18)^5}{5} + \frac{(1.18)^2}{2}$$

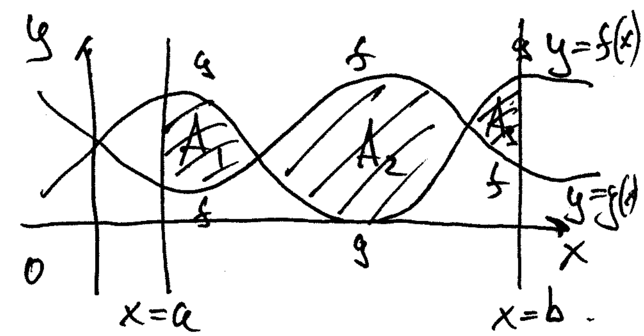
$$\approx 0.785$$

What if the important assumption fails? (1-7)
What if $f(x) \geq g(x)$ for some values of x
..., but $g(x) \geq f(x)$ for other values of x?

<u>Then</u> we split the given region S into several regions $S_1, S_2, \dots$ with areas $A_1, A_2, \dots$ so that the assumption holds for each $S_1, S_2 \dots$

and compute

$$A = A_1 + A_2 + \dots$$



Since $|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ g(x) - f(x) & \text{if } f(x) \leq g(x) \end{cases}$ (1-8)

We obtain the following general formula.

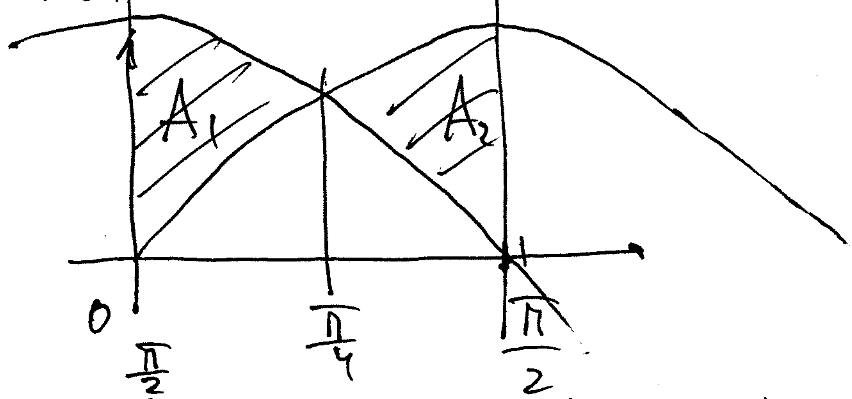
The area between $y = f(x)$, $y = g(x)$ and between $x = a$, $x = b$ $(a \leq b)$

is

$$A = \int_a^b |f(x) - g(x)| \, dx.$$

Ex. 4 $S = \begin{cases} y = \sin x \\ y = \cos x \\ x = 0 \text{ and } x = \frac{\pi}{2}. \end{cases}$ (1-9)

Points of intersection $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$. $\left(0 \le x \le \frac{\pi}{2}\right)$



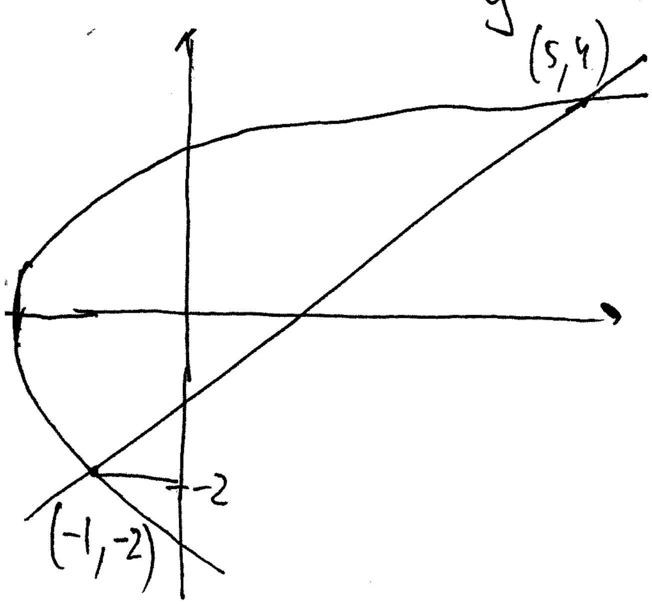
$$A = \int_0^{\frac{\pi}{2}} |\cos x - \sin x|\, dx = A_1 + A_2 \quad \left(\text{but } A_1 = A_2\right).$$

$$= 2A_1 = 2\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx = \sqrt{2} - 1.$$

We can switch $x \longleftrightarrow y$. : (1-10)

Ex. 5 $y = x - 1$ $\quad x_R = y + 1.$

$y^2 = 2x + 6.$ $\quad x_L = \frac{1}{2}y^2 - 3.$



$$A = \int_{-2}^{4} (x_R - x_L)\, dy =$$

$$= \int_{-2}^{4} \left[(y+1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy =$$

$$= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy =$$

$$= 18.$$