MAT135 - LECTURE 1

A function f is a rule that assigns to each element x in a set D to exactly one element denoted by $f(x)$

$$x \longrightarrow \boxed{f} \longrightarrow f(x)$$

Example: $f(x) = \pi x^2$

$f(2) = \pi(2)^2 = 4\pi$

Domain:

The domain of a function denoted by f is the set of points f is defined on.

Range:

The range of a function denoted by f is the set of values that $f(x)$ takes

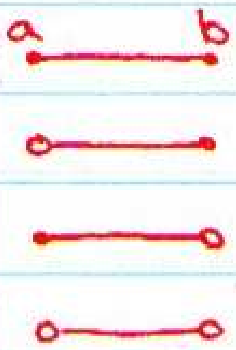
Example: $f(x) = x^2 + 1$ State the domain and range

The domain of f is the set of all real numbers (notation: $\mathbb{R}$)

The range of f is the set $[1, \infty)$ ← don't include this number (∞ never included)

includes 1

Interval vs. Notation



$[a,b]$

$(a,b]$

$[a,b)$

(a,b)

Absolute Value:

The absolute value of a number α, denoted by $|\alpha|$, is the distance from α to zero

Example: Find the absolute value of:

a) $|4| = 4$

b) $|-2| = 2$

also defined as: $|x| = \sqrt{x^2}$

The absolute value function is defined as:

$$\boxed{f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}}$$

Trig Functions:

$f(x) = \sin x$ domain is $\mathbb{R}$ Range is $[-1, 1]$

$$\left.\begin{aligned} \tan x &= \frac{\sin x}{\cos x} \\ \csc x &= \frac{1}{\sin x} \\ \sec x &= \frac{1}{\cos x} \\ \cot &= \frac{1}{\tan x} = \frac{\cos x}{\sin x} \end{aligned}\right\}$$

See Appendix D A28-30

Exponential Functions:

$f(x) = a^x$ where a is a positive constant

case 1: a is a positive fraction less then 1 $(0 < a < 1)$

function increases exponentially in the negative direction range is $(0, \infty)$

case 2: a is 1 $(a = 1)$

function is constant at $y = 1$ range is $[1]$

case 3: a is larger then 1 $(a > 1)$

function increases exponentially in the positive direction. range is $(0, \infty)$

Law of exponents:

(1) $a^{x+y} = a^x a^y$

(2) $a^{x-y} = \dfrac{a^x}{a^y}$

(3) $[a^x]^y = a^{xy}$

(4) $[ab]^x = a^x b^x$

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Translation + Shifting of a graph:

given $y=f(x)$ and $c>0$

$y=f(x)+c$	translating c units upward
$y=f(x)-c$	translating c units downward
$y=f(x+c)$	translating c units to the left
$y=f(x-c)$	translating c units to the right.

In general, to sketch $y=|f(x)|$, the guideline is as follows:

(1) sketch $y=f(x)$

(2) retain the part of the graph that is positive

(3) reflect the negative portions of the graph about the x-axis

In general, to sketch $y=f(|x|)$, the guideline is as follows:

(1) sketch the graph $y=f(x)$ for $x\geq 0$

(2) ~~reflect~~ the remaining portion $(x<0)$ is reflected about the y-axis.

Vertical + Horizontal stretching of a graph:

given $y=f(x)$ and $c>0$

$y=cf(x)$	stretch in the vertical component by a factor of c
$y=-cf(x)$	stretch in the vertical component by a factor of c + reflected x-axis
$y=f(cx)$	stretch in the horizontal component by a factor of $\frac{1}{c}$
$y=f(-cx)$	stretch in the horizontal component by a factor of $\frac{1}{c}$ + reflected y-axis

Inverse functions:

the inverse function, f^{-1}, is defined by $f^{-1}(y)=x$ iff $f(x)=y$ $\exists y$ in the range

A function f is one to one iff no horizontal line intersects the graph more then once

(Horizontal line test)

Example: $f(x)=\dfrac{x+2}{x+1}$ find f^{-1}

$$y=\frac{x+2}{x+1} \qquad f^{-1}(x)=\frac{2-x}{x-1} \quad \text{where } x\neq 1$$

$$x=\frac{y+2}{y+1}$$

$$xy+\cancel{y}^{\,x}=y+2$$

$$xy-y=\cancel{y-2}\ 2-x$$

$$y(x-1)=2-x$$

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Consider the function $f(x) = a^x$ where $a > 0$, $a \neq 1$

This function has an inverse which we call the logarithmic function with base a denoted by $\log_a x$

Log rules:

$\log_a 1 = 0$

$\log_a a = 1$

$\log_a a^x = x$

$a^{\log_a a^x} = x$

$\log_a x + \log_a y = \log_a [xy]$

$\log_a x - \log_a y = \log_a \left[\frac{x}{y}\right]$

$\log_a x^r = r \log_a x$

Section 2.1

MAT135

The slope of the line passing through the points (x_0, y_0) and (x_1, y_1) is given by

$$\boxed{m = \frac{y_1 - y_0}{x_1 - x_0}}$$

Limits:

given a function $f(x)$, we write $\lim_{x \to a} f(x) = L$

Theorem: If f is continuous at $x = a$, and $\lim_{x \to a} g(x) = b$ then

$$\boxed{\lim_{x \to a} f(g(x)) = f(b) = f(g(a))}$$

Example $\lim_{x \to 4} \cos\left[\frac{\sqrt{x} - 1}{4} \pi\right]$

$$\lim_{x \to 4} \frac{\sqrt{x} - 1}{4} \pi = \frac{\sqrt{4} - 1}{4} \pi = \frac{\pi}{4}$$

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

we write $\lim_{x \to a^-} f(x) = L$ /or/ $\lim_{x \to a^+} f(x) = L$

Fact: $\lim_{x \to a} f(x)$ exists iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Infinite Limits:

we write $\lim_{x \to a} f(x) = \pm \infty$ if the value of $f(x)$ can be made arbitrarily large by taking values of x that are sufficiently close to a but not equal

Important! If: $\left.\begin{array}{l} \lim_{x \to a^-} f(x) = -\infty \\ \\ \lim_{x \to a^+} f(x) = \infty \end{array}\right\} \lim_{x \to a} f(x)$ DNE

If: $\left.\begin{array}{l} \lim_{x \to a^-} g(x) = \infty \\ \\ \lim_{x \to a^+} g(x) = \infty \end{array}\right\} \lim_{x \to a} g(x) = \infty$

Vertical Asymptote:

we say the line $x = a$ is a vertical asymptote [VA] of the function $y = f(x)$ if one of the following holds:

CON'T ON BACK

Hilroy

$\lim_{x \to a} f(x) = \infty$ $\lim_{x \to a^-} f(x) = \infty$ $\lim_{x \to a^+} f(x) = \infty$

$\lim_{x \to a} f(x) = -\infty$ $\lim_{x \to a^-} f(x) = -\infty$ $\lim_{x \to a^+} f(x) = -\infty$

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Limit Laws:

Sum Law $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Difference Law $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

Constant Multiplication $\lim_{x\to a}[cf(x)] = c\left[\lim_{x\to a} f(x)\right]$

Product Law $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$

Quotient Law $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\lim_{x\to a} f(x)\right]}{\left[\lim_{x\to a} g(x)\right]}$ where $\lim_{x\to a} g(x) \neq 0$

Power Law $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

Others: $\lim_{x\to a} C = C$ $\lim_{x\to a} x = a$ $\lim_{x\to a} x^n = a^n$

Root Law $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

Examples

(1) $\lim_{x\to 0} \frac{(2x+3)^2-9}{x(x+1)}$

$= \lim_{x\to 0} \frac{4x^2+12x}{x(x+1)}$

$= \lim_{x\to 0} \frac{4x+12}{x+1} = 12$

(2) $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

$= \lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$

$= \lim_{x\to 1} \frac{x-1}{x-1(\sqrt{x+3}+2}$

$= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{4}$

(3) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \frac{\sqrt{x+3}-2}{x-1} & \text{if } x \geq 1 \\ \frac{1}{4}x^2+x-1 & \text{if } x<1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{4}x^2+x-1$

$= \frac{1}{4}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\sqrt{x+3}-2}{x-1}$

$= \frac{1}{4}$

$\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \quad \therefore \lim_{x\to 1} f(x) = \frac{1}{4}$

(4) Show that $\lim_{x\to 0} f(x)$ where $f(x) = \frac{|x|}{x}$ DNE

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-x}{x} = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x} = 1$

$\because \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ DNE

(5) Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ DNE

For $x \neq 0$

$-1 \leq \sin\frac{1}{x} \leq 1$

$-x^2 \leq x^2\sin\frac{1}{x} \leq x^2$

$0 \leq x^2\sin\frac{1}{x} \leq 0$

$\lim_{x\to 0} -x^2 \qquad \lim_{x\to 0} x^2$

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A function is said to be continuous at $x=a$ if the following properties hold

(1) $f(a)$ is defined

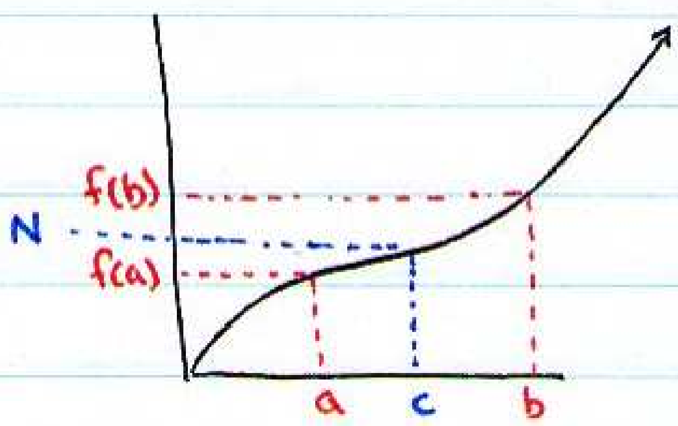
(2) $\lim_{x\to a} f(x)$ is defined

(3) $f(a) = \lim_{x\to a} f(x)$

The fllowing types of functions are continuous at their domai
Polynomials, rational, root, trig, exponential and log functions.

Independent Value Theorm (IVT):

Suppose that f is continuous in the interval $[a,b]$. Let N be any number that is between $f(a)$ and $f(b)$. Then $\exists$ a number c in $[a,b]$ | $f(c)=N$



let f be a function defined on $[a,\infty)$ then $\lim_{x\to\infty} f(x) = L$

means as $x \to \infty$, $f(x) \to L$ and is known as the horizontal asymptote.

Tangent Lines:

The tangent line to the curve $y=f(x)$ at a point p $(a, f(a))$

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$ provided the limit exists.

The derivative of a function $f(x)$ at $x=a$ denoted by $f'(a)$, is defined as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

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MAT135 - Chapter 3

(1) We introduce the derivative of certain basic functions

(2) Differentiation Rules ★ Reference page 5 ★

Section 3.1

Consider $f(x) = x^n$, n is a $\mathbb{R}$

$n = 0$: $f(x) = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$

$n = 3$: $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$

slope of normal line $= \frac{-1}{\text{slope of tangent}}$.

Example 3: Find the eq^n of the tangent and normal line to the curve $f(x) = x\sqrt{x}$ at the point $(1,1)$

$f(x) = x\sqrt{x}$

$f(x) = x^{3/2}$

$f'(x) = \frac{3}{2}x^{1/2}$

so at $(1,1)$: $f'(1) = \frac{3}{2}(1)^{1/2}$

$= \frac{3}{2}$

$\left.\begin{aligned}(y-1) &= \tfrac{3}{2}(x-1)\\ y &= \tfrac{3}{2}x - \tfrac{1}{2}\end{aligned}\right\}$ Both are acceptable

$m_{\perp} = \frac{-2}{3}$

$\left\{\begin{aligned}(y-1) &= \tfrac{-2}{3}(x-1)\\ y &= \tfrac{-2}{3}x + \tfrac{5}{3}\end{aligned}\right.$

Constant Multiple Rule: If c is a constant

$\frac{d}{dx}[cf(x)] = cf'(x)$

Addition + Subtraction Rule:

$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Example 7: The equation of the motion of a partical:

$S = 2t^3 - 5t^2 + 3t + 4$, where s is measured in cm

find acceleration. What is acceleration after 2 seconds?

$S = 2t^3 - 5t^2 + 3t + 4$

$v = 6t^2 - 10t + 3$

$a = 12t - 10$

So at 2 seconds: $a(2) = 12(2) - 10$

$= 24 - 10$

$= 14 \frac{cm}{s^2}$

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Derivative of the exponential function : $f(x) = a^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x [a^h - 1]}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

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Limit Rules

$\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

$\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$

$\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

Differentiation Rules

$f(x) \pm g(x) = f'(x) \pm g'(x)$

$cf(x) = cf'(x)$

$f(x)g(x) \neq f'(x) \cdot g'(x)$

$f(x)g(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Example 1 $f(x) = xe^x$

Find $f'(x)$ and the n^{th} derivative $f^{(n)}(x)$

$f'(x) = e^x + xe^x$
$= (1+x)e^x$

$f''(x) = e^x + (x+1)e^x$
$= (x+2)e^x$

In general, one has:
$f^{(n)}(x) = e^x(x+n)$

Example 3 $f(x) = \sqrt{x}\, g(x)$ where $g(4) = 2$ and $g'(4) = 3$

Find $f'(x)$ at $x=4$

$f'(x) = \sqrt{x}\, g'(x) + \frac{1}{2\sqrt{x}} g(x)$

$f'(4) = \sqrt{4}\, g'(4) + \frac{1}{2\sqrt{4}} g(4)$

$= \frac{13}{2}$

Quotient Rule:

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}}$$

Example 4 $f(x) = \frac{x^2+4x-1}{x^3+1}$ Find $f'(x)$

$$f'(x) = \frac{(x^3+1)(2x+4) - (x^2+4x-1)(3x^2)}{(x^3+1)^2}$$

$$= \frac{-x^4 - 8x^3 + 3x^2 + 2x + 4}{(x^3+1)^2}$$

Example $f(x) = \frac{e^x - 1}{x+1}$ Find $f'(x)$

$$f'(x) = \frac{(x+1)(e^x) - (e^x - 1)}{(x+1)^2}$$

$$= \frac{xe^x + e^x - e^x + 1}{(x+1)^2}$$

$$= \frac{xe^x + 1}{\ldots}$$

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Section 3.4

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MAT135

Chain Rule:

If g is differentiable at x and f is differentiable at $g(x)$ then $f \circ g$ is differentiable at x.

$$\boxed{\frac{d}{dx}[f \circ g(x)] = f'(g(x)) \cdot g'(x)}$$

Leibniz notation:

if $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

evaluated at g(x) — evaluated at g(x)

$$\frac{d}{dx}\Big[\underbrace{f(g(x))}_{\text{outer function}}\Big] = \underbrace{f'}_{\text{derivative of } f}(g(x)) \cdot \underbrace{g'(x)}_{\text{derivative of } g}$$

Example: Differentiate (a) $y = \sin(x^2)$

(b) $y = \sin^2 x$

(c) $y = \sin(\sin x)$

(a) $\frac{dy}{dx} = \cos(x^2) \cdot 2x$

$= 2x\cos(x^2)$

(b) $y = (\sin x)^2$

$\frac{dy}{dx} = [2\sin x] \cdot \cos x$

$= 2\sin x \cos x$

(c) $\frac{dy}{dx} = \cos(\sin x) \cdot \cos x$

Power Rule + Chain Rule:

If n is any real number and $g(x)$ is differentiable, then

$$\boxed{\frac{d}{dx}\left[(g(x))^n\right] = n(g(x))^{n-1} \cdot g'(x)}$$ ☆

Example: Differentiate $y = [e^x + x^2]^{11}[x^3+4]^7$

$$\frac{dy}{dx} = [e^x + x^2]^{11}\left[7(x^3+4)^6(3x^2)\right] + \left[11(e^x+x^2)^{10}(e^x+2x)\right][x^3+4]^7$$

Hilroy

Derivative of Exponential Function:

$a > 0$, $a \neq 1$, $f(x) = a^x$

$$f(x) = a^x = e^{\ln(a^x)}$$
$$= e^{x\ln a}$$
$$f'(x) = e^{x\ln a} \cdot \ln a$$
$$= a^x \ln a$$

$$\boxed{\frac{d}{dx}[a^x] = a^x \ln a}$$

Example Differentiate (a) $y = 2^x$ (b) $y = 2^{\sqrt{x}}$

(a) $\frac{dy}{dx} = 2^x \ln 2$

(b) $\frac{dy}{dx} = \left[2^{\sqrt{x}} \ln 2\right] \cdot \frac{1}{2\sqrt{x}}$

MAT135

Example Differentiate $y = (x+1)^{1000}(x^2+x)^{201}$

$y = (x+1)^{1000}$

$\frac{dy}{dx} = 1000[x+1]^{999} \cdot 1$ $\qquad \frac{dy}{dx} = 1000[x+1]^{999}$

$= 1000[x+1]^{999}[x^2-2]^{201} + [x+1]^{1000}\, 201[x^2-x]^{200}[2x-1]$

Example Differentiate $y = \left[\frac{x+1}{x-1}\right]^{100}$

$\frac{dy}{dx} = 100\left[\frac{x+1}{x-1}\right]^{99} \cdot \frac{d}{dx}\left[\frac{x+1}{x-1}\right]$

$= 100\left[\frac{x+1}{x-1}\right]^{99} \frac{(x-1)-(x+1)}{(x-1)^2}$

$= 100\left[\frac{x+1}{x-1}\right]^{99}\left[\frac{-2}{(x-1)^2}\right]$

Example Differentiate $y = 4^{\sec x}$

$\frac{dy}{dx} = \cancel{4\sec}\left[4^{\sec x}\ln 4\right]\frac{d}{dx}[\sec x]$

$= \left[4^{\sec x}\ln 4\right]\left[\sec x \tan x\right]$

$$\boxed{\frac{dy}{dx}\left[f(g(h(x)))\right] = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)}$$

Example Differentiate $\overset{\sin}{\cancel{\cos}}\left[\sin(\sin(x))\right]$

$f'(x) = \cos\left[\sin(\sin(x))\right] \cdot \cos(\sin(x)) \cdot \cos x$

So far, we have been working with $y=f(x)$. In this section, we will see functions whereby the formula cannot be expressed in the form $y=f(x)$

Hilroy

Example given that $25 = x^2 + y^2$

find the value of $\frac{dy}{dx}$ at the point $(3,4)$ and $(3,-4)$

$$25 = x^2 + y^2 \qquad y = \sqrt{25-x^2}$$
$$y^2 = 25 - x^2 \qquad y = -\sqrt{25-x^2}$$

Consider the case at $(3,4)$

$$y = \sqrt{25-x^2}$$
$$\frac{dy}{dx} = \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$$
$$= -\frac{x}{\sqrt{25-x^2}} = -\frac{3}{4}$$

Consider the case at $(3,-4)$

$$y = -\sqrt{25-x^2}$$
$$\frac{dy}{dx} = -\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(2x)$$
$$= \frac{x}{\sqrt{25-x^2}} = \frac{3}{4}$$

Implicit Differentiation.

$$x^2 + y^2 = 25$$

Differentiate both sides of eq^n with respect to x

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[25]$$

addition law

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 0$$

chain rule.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$= \frac{-x}{y}$$

$\cancel{\frac{dy}{dx} = -\frac{x}{y}}$

Section 3.5

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MATB35

Example: Given $2y^5 - 10x^4y + 5x^5 = 2$ Express $\frac{dy}{dx}$ in terms of x and y

Evaluate at (0,1)

Differentiate both sides with respect to x

$$\frac{d}{dx}(2y^5) - \frac{d}{dy}(10x^4y) + \frac{d}{dx}(5x^5) = \frac{dy}{dx}(2)$$

$$10y^4\frac{dy}{dx} - \left[\frac{10d}{dx}(x^4)y + 10x^4\frac{d}{dx}(y)\right] + 25x^4 = 0$$

$$10y^4\frac{dy}{dx} - \left[40x^3y + 10x^4\frac{dy}{dx}\right] + 25x^4 = 0$$

$$\left[10y^4 - 10x^4\right]\frac{dy}{dx} = 40x^3y - 25x^4$$

$$\frac{dy}{dx} = \frac{40x^3y - 25x^4}{10y^4 - 10x^4}$$

at (0,1) $\frac{dy}{dx} = 0$

Example: Given $y^2 = \sin(x+y) + x$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(\sin(x+y)) + \frac{d}{dx}(x)$$

$$2y\frac{dy}{dx} = \cos(x+y)\frac{d}{dx}(x+y) + 1$$

$$2y\frac{dy}{dx} = \cos(x+y)\left(1 + \frac{dy}{dx}\right) + 1$$

$$\frac{dy}{dx} = \frac{\cos(x+y) + 1}{2y - \cos(x+y)}$$

$$\underbrace{\frac{d}{dx}\left(2y\frac{dy}{dx}\right)} = \underbrace{\frac{d}{dx}\left[\cos(x+y)\left(1+\frac{dy}{dx}\right)+1\right.}$$

$$2\frac{d}{dx}(y)\frac{dy}{dx} + 2y\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

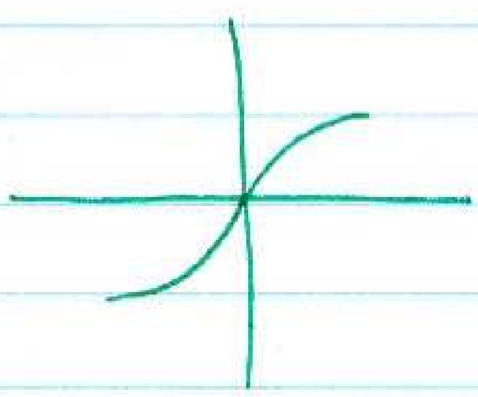
$$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}$$

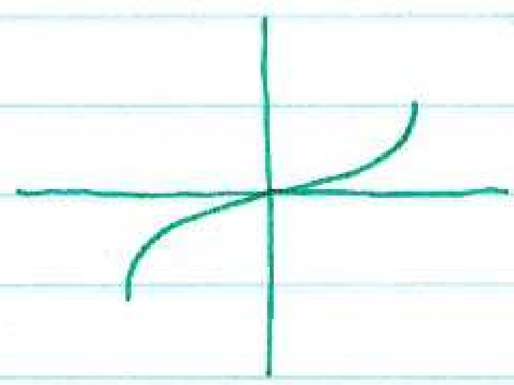
$$-\sin(x+y)\left(1+\frac{dy}{dx}\right)\left(1+\frac{dy}{dx}\right) + \cos(x+y)\left(\frac{d^2y}{dx^2}\right) + 0$$

Solve for $\frac{dy}{dx}$ first. Then substitute the value of $\frac{dy}{dx}$ into second derivative of $\left(\frac{dy}{dx}\right)^2$

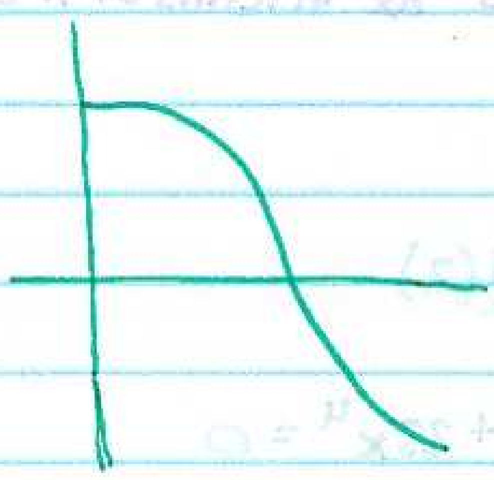
Inverse Trig Functions

$f(x) = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ => inverse => $\arcsin x$

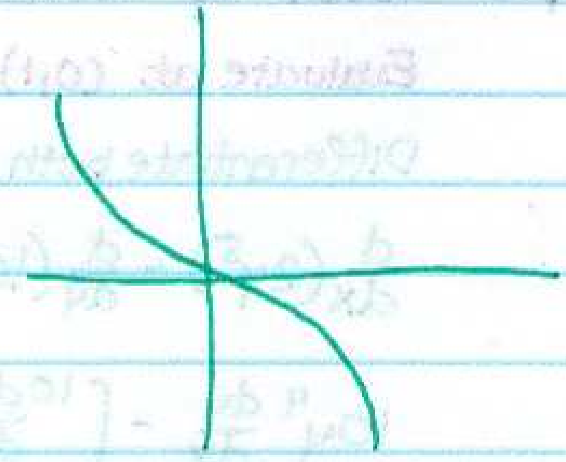




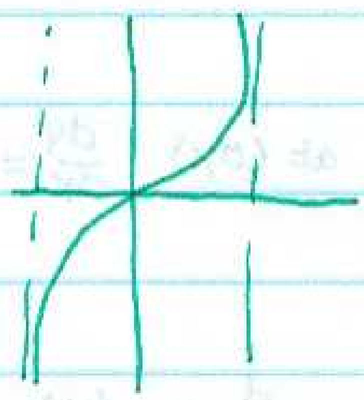
$f(x) = \cos x,\ 0 \le x \le \pi$



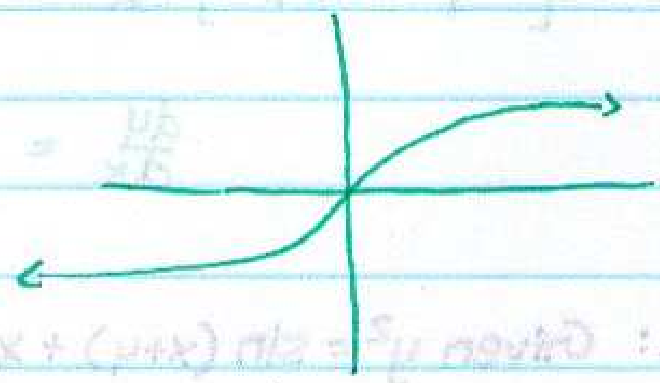
$y = \arccos x$



$f(x) = \tan x$

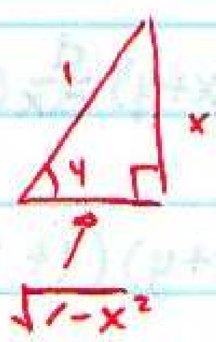


$y = \arctan$



$\frac{d}{dx}(\sin^{-1}(x))$

$\frac{dy}{dx} = \frac{1}{\cos y}$



$\frac{d}{dx}(\sin x) = \cos x$

let $y = \sin^{-1} x$

$\sin y = x$

$\cos y \frac{dy}{dx} = 1$

$\cos y = \sqrt{1-x^2}$

$\therefore$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

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★ $\frac{d}{dx}(\sin^{-1}x) = \arcsin x = \frac{1}{\sqrt{1-x^2}}$

★ $\frac{d}{dx}(\csc^{-1}) = \text{arccsc}x = \frac{-1}{x\sqrt{x^2-1}}$

★ $\frac{d}{dx}(\cos^{-1}x) = \arccos x = \frac{-1}{\sqrt{1-x^2}}$

★ $\frac{d}{dx}(\sec^{-1}x) = \text{arcsec}x = \frac{1}{x\sqrt{x^2-1}}$

★ $\frac{d}{dx}(\tan^{-1}x) = \arctan x = \frac{1}{1+x^2}$

★ $\frac{d}{dx}(\cot^{-1}x) = \text{arccot}\, x = \frac{-1}{1+x^2}$

Example $f(x) = \arctan(x^2-1)$

$$f'(x) = \frac{1}{1+(x^2-1)^2} \cdot 2x$$

$$= \frac{2x}{x^4-2x^2+2}$$

Example $a^y = x$

$$(a^y \ln a)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}\left[\ln(g(x))\right] = \frac{g'(x)}{g(x)}}$$

Example (1) $\ln(x+1)$

$= \frac{1}{x+1}$

(2) $\ln(x-2)$

$= \frac{1}{x-2}$

(3) $\ln\left[\frac{x+1}{\sqrt{x-2}}\right]$ log rules.

$= \ln(x+1) - \ln\sqrt{x-2}$

$= \ln(x+1) - \frac{1}{2}\ln(x-2)$

$= \frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{x-2}$

$= \frac{x-5}{2(x+1)(x-2)}$

Method of log differentiation

Given $y=f(x)$

(1) Take the natural log of both sides of eq$^{\underline{n}}$ and use the law of logs to simplify

(2) Differentiate implicitly w.r.t x

(3) solve in terms of $\frac{dy}{dx} = y'$

Example $y = \frac{e^{7x}\sqrt{\sin^5 x+1}}{(3x^2+7)^{10}\sqrt[3]{e^x+2}}$

$$\ln y = \ln\left[\frac{e^{7x}\sqrt{\sin^5 x+1}}{(3x^2+7)^{10}\sqrt[3]{e^x+2}}\right] = \ln(e^{7x}) + \ln(\sqrt{\sin^5 x+1}) - \ln(3x^2+7)^{10} - \ln(\sqrt[3]{e^x+2})$$

$$= 7x + \frac{1}{2}\ln(\sin^5 x+1) - 10\ln(3x^2+7) - \frac{1}{3}\ln(e^x+2)$$

$$\frac{1}{y}\frac{dy}{dx} = 7 + \frac{1}{2}\cdot\frac{5\sin^4 x\cos x}{\sin^5 x + 1} - 10\,\frac{6x}{3x^2+7} - \frac{1}{3}\cdot\frac{e^x}{e^x+2}$$

$$\frac{dy}{dx} = y\left[7 + \frac{5\sin^4 x\cos x}{2\sin^5 x + 1} - \frac{60x}{3x^2+7} - \frac{e^x}{3(e^x+2)}\right]$$

Section 3.6

October 26 2012

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$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$ $\frac{d}{dx}\ln x = \frac{1}{x}$

Base	Exponent	a, b constants
constant	constant	$\frac{d}{dx}a^b = \varnothing$
variable / function	constant	$\frac{d}{dx}f(x)^b = b f(x)^{b-1}\cdot f'(x)$
constant	variable	$\frac{d}{dx}a^{g(x)} = \left(a^{g(x)}\ln a\right)\cdot g'(x)$
variable	variable	$\frac{d}{dx}f(x)^{g(x)} =$ (1) Take ln of both sides (2) Differentiate w.r.t x (3) Solve in terms of $\frac{dy}{dx}$

/OR/ let $y = x^{\sin x}$

$= (e^{\ln x})^{\sin x}$

$= e^{\sin x \ln x}$

differentiate via chain rule.

Example: Differentiate $y = x^{x^x}$

$y = x^{x^x}$

$\ln y = \ln x^{x^x}$

$\ln y = x^x \ln x$

$\ln\ln y = \ln(x^x \ln x)$

$= \ln x^x + \ln x$

$= x\ln x + \ln\ln x$

$\frac{1}{\ln y}\cdot\frac{1}{y}\frac{dy}{dx} = \ln x + \frac{x}{x} + \frac{1}{\ln x}\cdot\frac{1}{x}$

$\frac{dy}{dx} = y\ln y\left(\ln x + 1 + \frac{1}{x\ln x}\right)$

$= \left[x^{x^x}\ln x^{x^x}\right]\left[\ln x + 1 + \frac{1}{x\ln x}\right]$

Section 3.7

Example: The position of a particle is given by the eq$^{\underline{n}}$

$s(t) = t^3 - 6t^2 + 9t$

(1) when is the particle at rest?

(2) when is the particle moving forward?

(3) when is the particle speeding up?

(1) $v(t) = 3t^2 - 12t + 9$

$0 = 3t^2 - 12t + 9$

$= 3(t-1)(t-3)$

$\therefore$ at $t = 1$

$t = 3$

(2) $0 < 3(t-1)(t-3)$

	$t<1$	$1<t<3$	$t>3$
$v(t)$	+	−	+

$\therefore$ speeding up at $t<1$ and $t>3$

(3) $a(t) = 6t - 12$

$= 6(t-2)$

	$t<1$	$1<t<2$	$2<t<3$	$t>3$
$v(t)$	+	−	−	+
$a(t)$	−	−	+	+

$\therefore$ its speeding up at $1<t<2$ and

Example Suppose a company has estimated that the cost of producing x items is: $C(x) = 10{,}000 + 5x + 0.01x^2$

(1) Find the marginal cost at the production level of 500 items

(2) Find the actual cost of the 501st item

(1) $\frac{dC}{dx} = 5 + 0.02x$

$= 5 + 0.02(500)$

$= 15$/item

(2) $C(501) - C(500)$

$= 15.01$

Section 3.7

Monday October 29

MAT135

- The demand function (or price function)

  $p(x)$ is the price per unit if x units are sold
- If x units are sold and the price per unit is $p(x)$, then the total revenue is $R(x) = x\,p(x)$
- The derivative of the revenue function is called the marginal revenue function and is the rate of change of revenue w.r.t. the number of units sold.
- The profit function $P(x) = R(x) - C(x)$

Example: A company has been selling 100 units of product at \$50 each. A market survey indicates that for each \$2 rebate offered to buyers, the number of units sold will increase by 10 products per week. Find the demand function and the marginal revenue function.

Let x be the number of products sold

$p(100) = 50$

For each increase in 1 unit:

$\frac{1}{10} \times 2 = \frac{1}{5}$

$\therefore\ p(x) = -\frac{1}{5}x + b$

$p(100) = -\frac{1}{5}(100) + b$

$70 = b$

$\therefore\ \underline{p(x) = -\frac{1}{5}x + 70}$

$R(x) = x\,p(x)$

$= x\left(-\frac{1}{5}x + 70\right)$

$= -\frac{1}{5}x^2 + 70x$

$\underline{R'(x) = -\frac{2}{5}x + 70}$

Word problem solving questions:

(1) General type as seen in Sect 3.7

The math involved requires materials covered in and upto 3.7

(2) Exponential Growth (Sect 3.8)

(3) Related Rates (Sect 3.9)

(4) Optimization Problems (Sect. 4.7)

Section 3.8

Given $y = f(t)$

If the rate of change of y w.r.t t is $\propto$ (proportional) to y at any time t

$\frac{dy}{dt} = ky$, k is constant   this is an example of a differential equation

Hilroy

Theorm: The only solution of the derivative $\frac{dy}{dt} = ky$ is

$$\boxed{y(t) = \underbrace{y(0)}_{\text{initial value}} e^{kt}}$$

Example: Given the world population in 1950 (2.56 billion) and in 1960 (3.04 billion). Assume that the population growth rate is proportional to the population size. What is the relative growth rate and Estimate the population in 1993.

$p(t)$ = population size at time t in years ($t = 0$ means 1950)

$\frac{dp}{dt} = kp$ = relative growth rate

$p(0) = 2.56$ $\quad$ $p(10) = 3.04$

$$p(t) = p(0)e^{kt}$$

$$3.04 = 2.56e^{k(10)}$$

$$\frac{3.04}{2.56} = e^{10k}$$

$$\ln\left[\frac{3.04}{2.56}\right] = 10k$$

$$\frac{\ln\left[\frac{3.04}{2.56}\right]}{10} = k \simeq 0.017185$$

$$\simeq 1.7\%$$

$\therefore$ at year 1993 $p(43) \simeq 5.36$