

# MATC01 - Problem Set 2 Solutions

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**2.1** Given  $A \in \mathbb{R}^2$  such that  $A \bullet X = 0$  for all  $X \in \mathbb{R}^2$ . Prove that  $A = (0, 0)$ .

*Solution:* By hypothesis, we have

$$|A|^2 = A \bullet A = 0.$$

This immediately implies that  $A = (0, 0)$  as claimed.

**2.2** Prove that every isometry of the form  $T_h \circ R_\alpha$  with  $\alpha \neq 0$  is a rotation.

*Solution:* We will show this in a more precise form, by proving the following:

**Claim.** Given  $\alpha \in (0, 2\pi)$  and  $h \in \mathbb{C}$ , set

$$C = \frac{h}{1 - e^{i\alpha}}.$$

Then  $T_h \circ R_\alpha = T_C \circ R_\alpha \circ T_{-C}$ . In other words,  $T_h \circ R_\alpha$  is a rotation by  $\alpha$  around  $C$ .

*Proof.* As in lecture, we have

$$T_C \circ R_\alpha \circ T_{-C} = T_C \circ T_{R_\alpha(-C)} \circ R_\alpha = T_{C+R_\alpha(-C)} \circ R_\alpha.$$

Plugging in our choice of  $C$  and simplifying yields

$$T_C \circ R_\alpha \circ T_{-C} = T_h \circ R_\alpha$$

as claimed. □

**2.3** Suppose that  $m$  is an isometry which satisfies  $m(0) = 0$ . Prove that for all  $w, z \in \mathbb{C}$ , we have

$$m(w + z) = m(w) + m(z).$$

*Solution:* Let  $m$  be an isometry that fixes 0. There are many ways to prove that  $m$  is additive. I choose a highly geometric one to present here.

First, we observe that any isometry sends lines to lines. This is simply because a line is defined as the locus of points which are equidistant from a pair of fixed points. For example, the  $x$ -axis can be defined by  $\{x \in \mathbb{R}^2 \mid d(x, (0, 1)) = d(x, (0, -1))\}$ , and any isometry  $f$  will send this

set to the set  $\{x \in \mathbb{R}^2 \mid d(x, f(0, 1)) = d(x, f(0, -1))\}$ , which is also a line. We also observe that since a parallelogram is defined as a quadrilateral with opposite sides having equal lengths, an isometry must send a parallelogram to a parallelogram. Our result follows almost immediately, as it's now clear that since  $m$  fixes the origin, it sends the parallelogram with vertices  $0, w, z,$  and  $w + z$  to the parallelogram with vertices  $0, m(w), m(z)$  and  $m(w) + m(z)$ , which in turn implies that  $m$  must send  $w + z$  to  $m(w) + m(z)$ . (If this explanation isn't clear, draw a few pictures of parallelograms and it will be.)

If you prefer a more explicit algebraic proof, we can use the same idea as in problem 2.1 above. Namely, start by writing

$$|m(w + z) - m(w) - m(z)|^2 = (m(w + z) - m(w) - m(z)) \bullet (m(w + z) - m(w) - m(z))$$

Expanding the right hand side (using the distributive property of dot products) and then simplifying (using the fact that  $m$  preserves dot products) shows that this quantity is 0. But this implies that  $m(w + z) = m(w) + m(z)$  as claimed.

**2.4** Prove that every isometry is either orientation-preserving or orientation-reversing, but not both.

*Solution:* Let  $\phi$  be an isometry, and assume for a contradiction that it is both orientation-preserving and orientation-reversing. That is, assume there exist  $g, h \in \mathbb{C}$  and  $\alpha, \beta \in \mathbb{R}$  such that  $\phi = T_g \circ R_\alpha \circ \rho = T_h \circ R_\beta$ , which implies

$$R_\alpha \circ \rho = T_{h-g} \circ R_\beta.$$

This is only possible if  $h = g$ , since if  $h - g \neq 0$ , the isometry on the left side fixes the origin while the one on the right does not. Then we have  $R_\alpha \circ \rho = R_\beta$ , which in turn implies  $\rho = R_{\beta-\alpha}$ , which is impossible since, for example,  $\rho$  fixes the entire  $x$ -axis while  $R_{\beta-\alpha}$  fixes only the origin (assuming, of course, that  $\alpha \neq \beta$ , which you should verify is impossible).

**2.5** Prove that a composition of (finitely many) primitive isometries is orientation-preserving iff it has an even number of  $\rho$ 's in it.

*Solution:* Let  $\phi$  be a composition of  $n$  primitive isometries. We prove the result by induction on  $n$ .

The base case  $n = 1$  is clear, since a single primitive isometry is orientation-reversing if and only if it is  $\rho$ .

So assume the result is true for compositions of precisely  $n$  primitive isometries, and let  $\phi$  be a composition of  $n + 1$  primitive isometries. We can write  $\phi = \psi \circ f$ , where  $\psi$  is a composition of precisely  $n$  primitive isometries (the first  $n$  from the decomposition of  $\phi$ ), and  $f = T_h, R_\alpha,$  or  $\rho$  (for some  $h \in \mathbb{C}, \alpha \in \mathbb{R}$ , as before).

For the  $(\Rightarrow)$  direction, assume  $\phi$  is orientation preserving but that it contains an odd number of  $\rho$ 's. If  $f = \rho$ , then it follows that  $\psi$  must contain an even number of  $\rho$ 's, which by the inductive hypothesis implies that it must be orientation-preserving. Using the lemma and previous exercise,

there must exist  $g \in \mathbb{C}$  and  $\beta \in \mathbb{R}$  such that  $\psi = T_g \circ R_\beta$ , implying  $\phi = \psi \circ f = T_g \circ R_\beta \circ \rho$ . But then  $\phi$  is orientation-reversing, contradicting our assumption. On the other hand, if  $f \neq \rho$ , it must be that  $\psi$  contains an odd number of  $\rho$ 's, which by the inductive hypothesis implies that it must be orientation-reversing. By our lemma, this means there must exist  $g \in \mathbb{C}$  and  $\beta \in \mathbb{R}$  such that  $\psi = T_g \circ R_\beta \circ \rho$ , and in turn  $\phi = T_g \circ R_\beta \circ \rho \circ f$ . Using the inductive hypothesis again, we see that  $\phi$  is orientation-reversing (remember we're assuming  $f \neq \rho$ ), again contradicting our assumption

Conversely, for the ( $\Leftarrow$ ) direction, assume that  $\phi$  contains an even number of  $\rho$ 's.

If  $\psi$  contains an even number of  $\rho$ 's it is orientation-preserving by the inductive hypothesis, and so we can conclude by the lemma and the previous exercise that  $\psi = T_g \circ R_\beta$  for some  $g, \beta$ . This also forces  $f$  to be a rotation or translation (in order to maintain the parity of  $\rho$ 's), and so we have  $\phi = T_g \circ R_\beta \circ f$ , which is orientation-preserving by the inductive hypothesis. If  $\psi$  contains an odd number of  $\rho$ 's, the inductive hypothesis implies that it is orientation-reversing, whence the lemma says  $\psi = T_g \circ R_\beta \circ \rho$  for some  $g, \beta$ . This also forces  $f = \rho$  (again for parity reasons), and so  $\phi = T_g \circ R_\beta \circ \rho \circ \rho$ , which is orientation-preserving by the inductive hypothesis.

(Note that this proof implicitly makes heavy use of the previous exercise, since we are often using different decompositions of  $\phi$  to conclude whether it preserves orientation.)

**2.6** Suppose  $\mathcal{L}$  and  $\mathcal{L}'$  are two lines in the plane. What can you say about the isometry  $\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}$ ? Try to be as specific as possible. Prove whatever you can.

*Solution:* We'll treat the cases where  $\mathcal{L}$  and  $\mathcal{L}'$  are parallel or not separately.

*Case 1:* They are parallel. We claim that  $\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}$  is a translation by  $2h$ , where  $h$  is the vector normal to  $\mathcal{L}'$ , pointing at  $\mathcal{L}$ , whose length is the distance between  $\mathcal{L}$  and  $\mathcal{L}'$ .

This is easy to visualize, but tricky to write out. Some simplifying assumptions will help. Assume without loss of generality that  $\mathcal{L}$  is the  $y$ -axis of  $\mathbb{R}^2$  and that  $\mathcal{L}'$  is the line  $x = c$ , where  $c$  is some positive real number. Then  $h$  is the vector pointing directly left of length  $c$ , namely  $h = (-c, 0)$ . Now, take an arbitrary  $(x, y) \in \mathbb{R}^2$ . Then we can calculate, entirely from looking at a sketch of the situation:

$$\begin{aligned} \sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}(x, y) &= \sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}(c + (x - c), y) \\ &= \sigma_{\mathcal{L}}(c + (x - c) - 2(x - c), y) \\ &= \sigma_{\mathcal{L}}(-x + 2c, y) \\ &= (x - 2c, y) = (x, y) - 2h \end{aligned}$$

*Case 2:* They are not parallel. Then the two lines intersect at a single point. Let the smaller of the angles between them be  $\theta$ . We claim that  $\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}$  is a rotation about the point at which the lines intersect by an angle of  $2\theta$ , in the direction from  $\mathcal{L}'$  to  $\mathcal{L}$ .

Concluding this intuitively is not difficult, since  $\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}$  is an isometry that fixes precisely the point of intersection of the lines and contains an even number of reflections, and the only such isometries we know are rotations.

Showing this explicitly without relying on a classification of all isometries of the plane, however, is labour intensive.