MATC01 - Problem Set 1 Solutions

Ivan Khatchatourian

18 September 2013

1.1

(a) Prove that for all $z \in \mathbb{C}$, we have $|z|^2 = z\bar{z}$.

Solution: Let $z = x + iy \in \mathbb{C}$. Then

$$z\bar{z} = (x+iy)(x-iy) = x^2 - ixy + ixy - i^2y = x^2 + y^2 = |z|^2.$$

(b) Prove that for any $z, w \in \mathbb{C}$, we have $\overline{zw} = \bar{z}\bar{w}$.

Solution: Let $z = x + iy$, $w = u + iv \in \mathbb{C}$. Then

$$\begin{aligned}\overline{zw} = \overline{(x+iy)(u+iw)} &= \overline{(xu - yw) + i(yu + xw)} \\ &= (xu - yw) - i(yu + xw) \\ &= (x - iy)(u - iw) \\ &= \bar{z}\bar{w}\end{aligned}$$

(c) Simplify $\overline{re^{i\theta}}$. (Give your answer in complex polar coordinates.)

Solution: Let $z = re^{i\theta}$. In rectangular coordinates, we have $z = r(\cos\theta + i\sin\theta)$. Then

$$\bar{z} = \overline{r(\cos\theta + i\sin\theta)} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta)) = re^{-i\theta}.$$

Here, the third equality uses the fact that cosine and sine are even and odd functions, respectively.

(d) Evaluate $(1 + 2i)^2$. (Give your answer in the form $a + bi$.)

Solution:

$$(1+2i)^2 = (1+2i)(1+2i) = 1 + 4i + 4i^2 = 1 + 4i - 4 = -3 + 4i.$$

(e) Determine all compex solutions z to the equation $z^2 = i$.

Solution: Rewriting the equation in polar coordinates, we'd like to find all r and θ such that

$$(re^{i\theta})^2 = e^{i\pi/2}$$

This implies $r^2 = e^{i(\pi/2-2\theta)}$. Taking absolute values of both sides gives $|r^2| = 1$, and so r must equal 1 since we always think of r as the distance from z to the origin. Next, we have $e^{i(\pi/2-2\theta)} = 1$ if and only if $2\theta - \frac{\pi}{2}$ is a multiple of 2π, which is the same as saying $\theta = \frac{\pi}{4} + k\pi$ for some integer k. Thus, all solutions to the equation are of the form

$$z = re^{i\theta} = e^{i(\pi/4+k\pi)} = e^{i\pi/4}e^{k\pi i} = \Big(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\Big)(-1)^k = \pm\Big(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\Big).$$

(f) Rewrite $3e^{7\pi i/6}$ in the form $a + bi$.

Solution:

$$3e^{7\pi i/6} = 3(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}) = 3(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}) = 3(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$

(g) Rewrite $-3\sqrt{2} + 3i\sqrt{2}$ in complex polar coordinates.

Solution: $-3\sqrt{2}+3i\sqrt{2} = 3\sqrt{2}(-1+i)$, and so we need an angle θ whose cosine and sine have the same magnitude but are negative and positive, respectively, so we must have $\theta = \frac{3\pi}{4}$. $e^{i3\pi/4} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, and so we have

$$3\sqrt{2}(-1+i) = 6(\frac{1}{\sqrt{2}}(-1+i)) = 6e^{i3\pi/4}.$$

(h) Rewrite -5 in complex polar coordinates.

Solution: $-5 = 5(-1+0i) = 5(\cos\pi + i\sin\pi) = 5e^{i\pi}$.

(i) Evaluate $R_{2\pi/3}(3-2i)$. Give your answer in the form $a + bi$. No approximations: your answer must be exact!

Solution: The rotation $R_{2\pi/3}$ is simply multiplication by $e^{2i\pi/3}$. So

$$\begin{aligned} R_{2\pi/3}(3-2i) &= e^{2i\pi/3}(3-2i) \\ &= (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})(3-2i) \\ &= (-\frac{1}{2} + i\frac{\sqrt{3}}{2})(3-2i) \\ &= \frac{2\sqrt{3}-3}{2} + \frac{2+3\sqrt{3}}{2}i \end{aligned}$$

1.2 Prove that the composition of two isometries is an isometry.

Solution: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be isometries of the plane, and let $x, y \in \mathbb{R}^2$.

$$\begin{aligned} d\Big(f\circ g(x), f\circ g(y)\Big) &= d\Big(f\big(g(x)\big), f\big(g(y)\big)\Big) \\ &= d\Big(g(x), g(y)\Big) && \text{(since } f \text{ is an isometry)} \\ &= d(x,y) && \text{(since } g \text{ is an isometry)} \end{aligned}$$

This shows that $f \circ g$ is an isometry, as required.

1.3 Prove that an isometry is a bijection.
Solution: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry. We'll show that f is injective and surjective.

To see that f is injective, let $x, y \in \mathbb{R}^2$ be such that $f(x) = f(y)$. Then we have

$$0 = d\big(f(x), f(y)\big) = d(x, y),$$

which implies that $x = y$.

Now for surjectivity. There are many ways to solve this, but all of them are tricky. Here we give two proofs. The first is long but elementary. The second is short, but requires some basic analysis.

Proof 1

The key insight is to think about $\mathbb{R}^2$ as a vector space. What does an isometry do to a basis of $\mathbb{R}^2$? A bit of playing around should convince you that if $\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$, then $\{f(e_1), f(e_2)\}$ is another basis. This quickly leads to a proof of surjectivity, since every point of $\mathbb{R}^2$ can be written in terms of a basis.

Unfortunately, the above idea is quite tricky to formalize. This forces us to be clever. We start by renormalizing the isometry f so that it fixes the origin. More precisely, let $T_{-f(0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the translation given by $T_{-f(0)}(x) = x - f(0)$. This is clearly a bijective isometry, so by Problem 1.2, to show that f is bijective it suffices to show that $g := T_{-f(0)} \circ f$ is bijective. The advantage of working with g is that $g(0) = 0$, which makes calculations easier.

From here, our proof proceeds in two steps. First, we prove that g is a linear map. This means we can study g using linear algebra. Accordingly, we apply the Dimension theorem (aka the Rank-Nullity theorem) to g, which will immediately imply that g is surjective. The second step will turn out to be fairly straightforward, but proving linearity will require a small detour: we first show that g preserves dot products, and then deduce linearity from this.

Having outlined the proof, we get into the details. The fist thing we show is that g preserves dot products, in the sense that $x \bullet y = g(x) \bullet g(y)$. Let $x, y \in \mathbb{R}^2$. Then we have

$$|x - y|^2 = (x - y) \bullet (x - y) = x \bullet x - 2(x \bullet y) + y \bullet y = |x|^2 - 2(x \bullet y) + |y|^2$$

and on the other hand

$$\begin{aligned} |x - y|^2 = |g(x) - g(y)|^2 &= \big(g(x) - g(y)\big) \bullet \big(g(x) - g(y)\big) \\ &= |g(x)|^2 - 2\big(g(x) \bullet g(y)\big) + |g(y)|^2 \\ &= |x|^2 - 2\big(g(x) \bullet g(y)\big) + |y|^2, \end{aligned}$$

where we know $|g(x)| = |x|$ since in general $|x| = d(x, 0)$ and, as we showed, g fixes the origin.

Setting these two final equations equal yields the result about dot products. Next we'll use this along with the linearity of the dot product to see that g is linear. Again letting $x, y, z \in \mathbb{R}^2$, and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}\big(g(\alpha x + y) - \alpha g(x) - g(y)\big) \bullet g(z) &= g(\alpha x + y) \bullet g(z) - \alpha g(x) \bullet g(z) - g(y) \bullet g(z) \\ &= (\alpha x + y) \bullet z - \alpha(x \bullet z) - y \bullet z \\ &= \alpha(x \bullet z) + y \bullet z - \alpha(x \bullet z) - y \bullet z \\ &= 0\end{aligned}$$

Since z (and in turn $g(z)$) was arbitrary, this shows that $g(\alpha x + y) - \alpha g(x) - g(y) = 0$ for all x, y and α, which is to say that $g(\alpha x + y) = \alpha g(x) + g(y)$, as desired.

Now we know that g is linear. We can thus apply the Dimension Theorem (sometimes known as the Rank-Nullity Theorem) from linear algebra:

$$2 = \dim(\mathbb{R}^2) = \dim(\ker(g)) + \dim(\operatorname{im}(g)).$$

Since g is injective (being a composition of injective functions), we see that $\dim(\ker(g)) = 0$. This implies that $\dim(\operatorname{im}(g)) = 2$, whence g is surjective. Finally, we have that g is a bijection. Since $T_{-f(0)}$ is clearly a bijection, this implies f is a bijection, as we originally wanted to prove.

We could of course have given this as a direct proof that f is bijective, but the quick proof that it's injective is useful enough to give on its own.

Proof 2 (*added by Leo; this presupposes some analysis*)

Suppose f is an isometry of the plane. As above, we can renormalize by setting $g := T_{-f(0)} \circ f$. Thus g is an isometry, $g(0) = 0$, and (since translation is bijective) it suffices to prove that g is surjective.

Let S_r denote the circle of radius r centered at the origin. Note that

$$g(S_r) \subseteq S_r.$$

To prove that g is surjective, it therefore suffices to prove that $g(S_r) = S_r$ for every r.

Pick any $x \in S_r$. Given $\epsilon > 0$, there exist positive integers $m \neq n$ such that

$$d\big(g^m(x), g^n(x)\big) < \epsilon.$$

(Otherwise, we'd be packing arbitrarily many well-spaced points into a circumference of finite length!) Since g is an isometry, this implies that there exists an $N > 0$ such that

$$d\big(g^N(x), x\big) < \epsilon.$$

It follows that x is a limit point of $g(S_r)$. Since S_r is closed, $g(S_r)$ must be closed, which means that $g(S_r)$ contains all of its limit points. In particular, $x \in g(S_r)$. This concludes the proof.

1.4 Prove that the inverse of an isometry is an isometry.

Solution: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry and let $x, y \in \mathbb{R}^2$. Then we have

$$\begin{aligned} d\Big(f^{-1}(x), f^{-1}(y)\Big) &= d\Big(f\big(f^{-1}(x)\big), f\big(f^{-1}(y)\big)\Big) \quad \text{(since } f \text{ is an isometry)} \\ &= d(x, y) \end{aligned}$$

This shows that f^{-1} is an isometry, as required.

1.5 Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map iff there exist $a, b, c, d \in \mathbb{R}$ such that f acts like multiplication by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Solution: ($\Rightarrow$) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis vectors of $\mathbb{R}^2$. Let A be the 2×2 matrix whose first and second columns are $f(e_1)$ and $f(e_2)$, respectively. That is, if

$$f(e_1) = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad f(e_2) = \begin{pmatrix} b \\ d \end{pmatrix}, \text{ then } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now let $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2$. Then we have

$$f(x) = f(\alpha e_1 + \beta e_2) = \alpha f(e_1) + \beta f(e_2) = \begin{pmatrix} \alpha a + \beta b \\ \alpha c + \beta d \end{pmatrix} = Ax.$$

where the second equality is by the linearity of f. (This is a technique with which you should be very familiar if you have taken a course on linear algebra.)

($\Leftarrow$) This is immediate from the definition of matrix multiplication.