University of Toronto at Scarborough
Physical Sciences Division, Mathematics

MIDTERM TEST

MATA23H
Linear Algebra I

Examiner: E. Moore

Date: February 9, 2002
Duration: 110 minutes

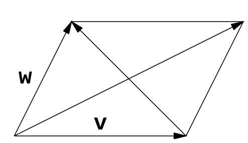
1. **[10 points]** Let $\boldsymbol{v} = [1, -2, 0, 3]$ and $\boldsymbol{w} = [-2, 2, -1, 1]$ be vectors in $\mathbb{R}^4$.
   (a) Find the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.
   (b) i) State the Cauchy-Schwarz inequality.
       ii) Verify the Cauchy-Schwarz inequality for $\boldsymbol{v}$ and $\boldsymbol{w}$.
   (c) i) State the triangle inequality.
       ii) Verify the triangle inequality for $\boldsymbol{v}$ and $\boldsymbol{w}$.

2. **[5 points]** Determine whether or not the triangle in $\mathbb{R}^3$ with vertices $(1, 2, 3)$, $(4, -1, 2)$ and $(2, 4, 0)$ is a right triangle.

3. **[9 points]**
   (a) Give examples to show that the following can happen for $A, B \in M_{2,2}(\mathbb{R})$.
       i) A^2 is the zero matrix, but A is not the zero matrix.
       ii) $AB \neq BA$.
   (b) Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Find all $B \in M_{2,2}(\mathbb{R})$ such that $AB = BA$.

4. **[10 points]**

   Use vector methods to show that the diagonals of a parallelogram bisect each other.



5. **[10 points]** Let $A \in M_{n,n}(\mathbb{R})$.
   (a) Use the properties of matrix transpose to show that
       i) $A + A^T$ is symmetric.
       ii) $A - A^T$ is skew-symmetric.
   (b) Conclude that A can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

6. **[18 points]**

(a) Use the Gauss method with back substitution to find the general solution of the linear system.

$$\begin{aligned} -2x_1 + x_2 - 3x_3 + 2x_4 &= -2 \\ 3x_2 - x_3 - x_4 &= 2 \\ x_1 - 4x_2 + 2x_3 + 3x_4 &= 1 \\ 3x_1 + 2x_3 &= -1 \end{aligned}$$

(b) Use the Gauss-Jordan method to find the general solution of the linear system.

$$\begin{aligned} -2x_1 - 3x_2 + x_3 + 2x_4 &= 0 \\ -x_2 + 3x_3 - x_4 &= 1 \\ x_1 + 2x_2 - 2x_3 + x_4 &= -1 \\ -x_1 + x_2 - 7x_3 + 5x_4 &= -3 \end{aligned}$$

7. **[15 points]**

(a) Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k$ be vectors in $\mathbb{R}^n$

i) Define what is meant by a *linear combination* of $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$.

ii) Define what is meant by the *span* of $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$.

(b) What conditions must be placed on a, b, c so that the vector $[a, b, c]$ is in the span of $[1, 2, 1]$, $[2, 6, -2]$ and $[-3, -11, 7]$?

8. **[13 points]** Let $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$.

(a) Find the inverse of the matrix A.

(b) Solve the linear system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

9. **[10 points]**

(a) Show that, if A is an invertible $n \times n$ matrix, then A^T is also invertible.

(b) Show that an $n \times k$ matrix A is row equivalent to another $n \times k$ matrix B if and only if there exists an invertible matrix C such that $B = CA$.