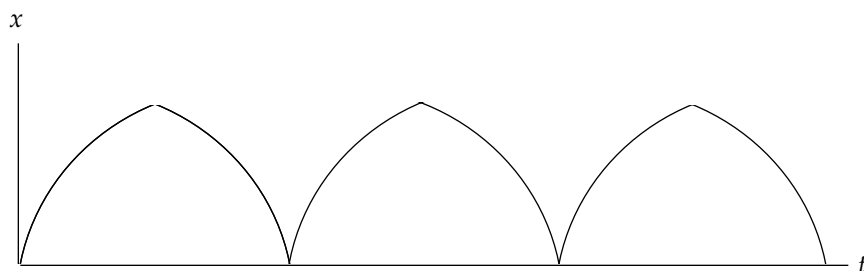


# CHAPTER 13 Oscillatory Motion

## Answers to Understanding the Concepts Questions

1.



2. Suspend a spring vertically and attach a light mass to the end of the spring to stretch it by a certain amount. When an additional, known mass  $m$  is hung from it, the spring is found to stretch further by an amount  $x$ . Then  $mg = kx$ , or  $k = mg/x$ . Make sure that the spring is not overstretched — it should stay within its elastic limit.
3. The maximum kinetic energy, that is, the total energy, is proportional to the square of the amplitude. Therefore a doubling of the amplitude implies a quadrupling of the maximum kinetic energy.
4. At the top of a mountain  $g$  is somewhat less than that at the sea level, so the period of the pendulum,  $T$ , increases as it is inversely proportional to  $\sqrt{g}$ . This means that the pendulum takes longer time to complete each swing, and therefore the pendulum clock would run slower, i.e., it would lose time.
5. At the top of the mountain  $g$  is less than its sea-level value. Since the pendulum period  $T$  is proportional to  $(L/g)^{1/2}$  the period will grow as the value of  $g$  decreases. However, because the air is thinner, the drag force on the pendulum decreases at higher amplitude. There are thus two competing effects, and it is hard to predict which one will win out.
6. No. A force that is not aligned with the spring force (which we assume to be in the  $x$ -direction) must have a component that is perpendicular to the spring force, i.e., in the  $y$ -direction. While the motion is simple harmonic in the  $x$ -direction, it is uniformly accelerated in the  $y$ -direction, in which the only force is the  $y$ -component of the additional constant force.
7. Yes, energy is indeed lost to friction. The system is driven, however, by an external force, which supplies energy to it to compensate the energy loss. As a result the total amount of energy of the oscillator remains constant, and the oscillation goes on with fixed amplitude.
8. The halving of the mass has no effect, so only the doubling of the length is relevant, increasing the period by a factor of  $\sqrt{2}$ .
9. The period of the pendulum is proportional to the square root of the length of the arm of the pendulum. If the clock runs too slow then the period of its pendulum is too long and should be decreased for correction. This means that the length of the pendulum arms needs to be shortened.

10. If the lever arm of the force of gravity about the pivot point of the swing is  $L$  and the rotational inertia about the pivot point is  $I$ , then  $\tau = mgL = I\alpha$  on your way down and  $\tau = -mgL = I\alpha$  on your way up. To gain speed, you want a greater positive  $\alpha$  on your way down and smaller negative  $\alpha$  on your way up. To achieve this, extend the legs and stretch out your body (thereby increasing  $I$ ) during the upswing to get a smaller negative  $\alpha$ , and pull back your legs (to decrease  $I$ ) and get a larger positive  $\alpha$  on your way down. The net result is an increase in amplitude with each complete swing.
11. Without going into details associated with the shape of a sitting versus standing person, we note that a standing person has a center of gravity higher than a sitting one, and effectively the pendulum that is formed by the swinging person is shortened when the person stands. This reduces the period.
12. As sand leaks out the mass  $m$  of the pendulum slowly decreases. As a result the energy of the pendulum,  $E = \frac{1}{2}mv_m^2$ , also slowly decreases, as does the maximum angle  $\theta_0$ . Since  $E/\omega$  is a constant,  $\omega$  must decrease over time, i.e., the period of the pendulum increases.
13. The period of a pendulum is proportional to  $\sqrt{L}$ , and if the length is decreased by a factor of  $3/2$ , the period is decreased by a factor of  $\sqrt{3/2}$ . The energy, in terms of the maximum angle of swing  $\theta_0$  is proportional to  $L\theta_0^2$ , or  $A^2/L$ , where  $A$  is the amplitude of the (small) swing. Thus  $E/\omega$  is proportional to  $ET$ ; that is, to  $A^2/\sqrt{L}$ . This is invariant, so that  $A$  must vary as  $L^{1/4}$ . If  $L$  is decreased by a factor of  $3/2$ , then the amplitude is decreased by a factor of  $(3/2)^{1/4} \approx 1.1$ .
14. Yes it does. The distribution of the mass of the pendulum will affect both its rotational inertia ( $I$ ) and the location of its center of mass. The period of the pendulum depends on both of these factors.
15. Each of the motions described here is oscillatory, but the frequency may vary. (a) Leaves blowing on a tree limb: the oscillation is driven by the wind. (b) Children seasawing: the motion is driven by the force of the ground (as a reaction to the children's feet pushing the ground). (c) Children playing on a swing: it is driven when the child "pumps", and damped if not. (d) A Car bouncing up and down the road: damped oscillation. (e) Water sloshing back and forth in a bathtub: damped oscillation.
16.  $T$  is proportional to the function  $y(r) = (I/r)^{1/2}$ . From the parallel axis theorem  $I = I_{cm} + mr^2$ , so  $y(r) = I_{cm}/r + mr$ . This function reaches its minimum where  $dy/dr = -I_{cm}/r^2 + m = 0$ , or  $r = (I_{cm}/m)^{1/2} = l/(12)^{1/2} \approx 0.289 l$ . As one moves the pivot point from one end of the rod towards its center of mass,  $r$  decreases from  $\frac{1}{2}l$  to 0. So  $T$  first decreases until  $r$  is reduced to  $0.289 l$ , then starts to increase as  $r$  further decreases, reaching infinity as  $r = 0$ .
17. A force diagram shows that if the ropes do not stretch, the suspension is equivalent to a single pendulum, with motion transverse to the line that joins the two points of suspension. This motion is simple harmonic. In fact, any small amplitude motion about a point of stable equilibrium is simple harmonic, as long as the bottom of the potential energy curve can be approximated by a parabola.
18. The centripetal force points from the pendulum bob to the center of its circular motion, so it is horizontal, and is provided by the horizontal component of the tension in the string. Since the motion is in a horizontal plane there is no change in the gravitational potential energy of the bob. The kinetic energy is also fixed as the speed of the bob does not vary, and the total energy is conserved (neglecting all frictions, of course).
19. The motion will still be periodic but no longer simple harmonic. The differential equation governing the motion of the pendulum is now nonlinear, since one cannot approximate  $\theta$  with  $\sin \theta$  when  $\theta$  is large. The restoring force is proportional to  $\sin \theta$ , rather than  $\theta$ , so its magnitude is less than in the case of simple-harmonic approximation. A weaker restoring force results in a slower oscillation, so the resulting period will be longer than that of the small-angle oscillation.

20. It is the harmonic driving force that provides power. This power oscillates with the same frequency as the force, so that energy flows in and out of the system from the external oscillator. The oscillating force may increase the amplitude of the mass on the spring if the phase is right, and that is the phenomenon of resonance. When resonance occurs, the transfer of energy from the external oscillator becomes very efficient!
21. You are adjusting your driving frequency to match the natural frequency of the oscillating diving board so as to get the maximum bounce from it.
22. Assume that the force  $\vec{F}$  of the wind is proportional to the mass  $m$  of the pendulum bob, such that  $\vec{F}/m = \vec{a} = \text{constant}$ . Then we may combine this force with the weight of the bob to yield a new, effective  $\vec{g}$ :  $m\vec{g} + \vec{F} = m(\vec{g} + \vec{a}) = m\vec{g}_{\text{eff}}$ , where  $g_{\text{eff}} = (g^2 + a^2)^{1/2} > g$ . The pendulum will now swing about a new equilibrium position (where the string is parallel to  $\vec{g}_{\text{eff}}$ ), and the new period of the pendulum, being inversely proportional to  $(g_{\text{eff}})^{1/2}$ , will be less than  $T$  without the wind.
23. The spring exerts a harmonic force on the mass, and by Newton's third law, the mass exerts an equal and opposite harmonic force on the spring, and therefore on the point of attachment to the sled. Thus the sled experiences the force of gravity, the normal reaction of the plane and also a harmonic force. The harmonic force on the sled ensures that the sled will not accelerate smoothly down the hill.

## Solutions to Problems

1. We use a trigonometric identity to expand:

$$\sin(\theta + \delta) = \sin \theta \cos \delta + \cos \theta \sin \delta.$$

In order for this to equal  $\cos \theta$ , we must have  $\cos \delta = 0$  and  $\sin \delta = 1$ ; thus  $\delta = \boxed{\pi/2 \text{ rad}}$ .

2. Because the time between maximum and minimum is  $\frac{1}{2}T = 0.3 \text{ s}$ , we have

$$T = \boxed{0.60 \text{ s}} \quad \text{and} \quad \omega = 2\pi/T = 2\pi/0.60 \text{ s} = \boxed{(10\pi/3) \text{ rad/s}}.$$

In the general expression  $x = A \sin(\omega t + \delta)$ , we substitute the known values:

$$0.01 \text{ m} = (0.04 \text{ m}) \sin[(10\pi/3 \text{ rad/s})(0.10 \text{ s}) + \delta], \text{ which gives } \delta = -0.80 \text{ rad}.$$

The general equation of motion is

$$x = \boxed{0.04 \sin[(10\pi/3)t - 0.80]} \text{ with } t \text{ in s, } x \text{ in m}.$$

3. The general expression for  $x$  is

$$x = A \cos(\omega t + \delta), \text{ from which we get}$$

$$v = dx/dt = -A\omega \sin(\omega t + \delta).$$

Comparing this to  $v = 0.4 \sin(\omega t + \pi) \text{ m/s}$ , we see that

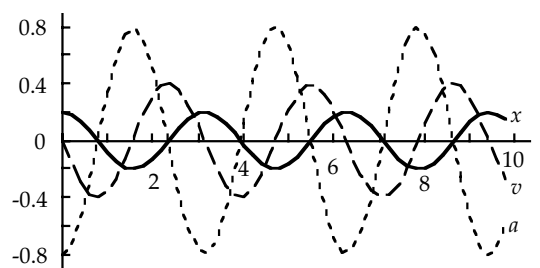
$$A\omega = 0.4 \text{ m/s, so } A = (0.4 \text{ m/s})/(2.00 \text{ rad/s}) = 0.2 \text{ m}.$$

Because  $\sin(\theta + \pi) = -\sin \theta$ , we see that  $\delta = 0$ . Thus

$$x = 0.2 \cos(\omega t) \text{ m}.$$

We obtain  $a$  from

$$a = dv/dt = 0.4\omega \cos(\omega t + \pi) = -0.8 \cos(\omega t) \text{ m/s}^2.$$



4.  $T = 2\pi/\omega = 2\pi/(3.827 \text{ rad/s}) = \boxed{1.642 \text{ s}}$ .

5. From  $v_{\max} = A\omega$ , we get

$$\omega = v_{\max}/A = (4.36 \text{ m/s})/2.84 \text{ m} = 1.54 \text{ rad/s}.$$

The period is  $T = 2\pi/\omega = 2\pi/(1.54 \text{ rad/s}) = \boxed{4.09 \text{ s}}$ .

6. We have  $v_{\max} = A\omega$  and  $a_{\max} = A\omega^2$ , so we get

$$\omega = a_{\max}/v_{\max} = (1.05 \text{ m/s}^2)/(0.371 \text{ m/s}) = 2.83 \text{ rad/s}.$$

Then  $A = v_{\max}/\omega = (0.371 \text{ m/s})/(2.83 \text{ rad/s}) = \boxed{0.131 \text{ m}}$ .

7. From the general expression for  $x$ , we have

$$x = A \cos(\omega t + \delta); \quad 0 = A \cos(0 + \delta), \text{ which gives } \delta = \pm \pi/2. \text{ If we choose } \delta = -\pi/2, \text{ we write}$$

$$x = A \sin(\omega t).$$

We find the position at  $t = 1 \text{ s}$  from

$$x = A \sin(\omega t) = (0.142 \text{ m}) \sin[(3.42 \text{ rad/s})(1 \text{ s})] = \boxed{-0.038 \text{ m}}.$$

8. (a) Using a trigonometric identity, we can write the general expression as

$$x = A \sin(\omega t + \delta) = A[\sin(\omega t) \cos \delta + \cos(\omega t) \sin \delta].$$

Comparing this to  $x = B \cos(\omega t) + C \sin(\omega t)$ , we see that

$$C = A \cos \delta, \quad B = A \sin \delta; \quad \text{the inverse relations are} \\ \tan \delta = B/C \quad \text{and} \quad A^2 = B^2 + C^2.$$

- (b) Because  $v = dx/dt = -B\omega \sin(\omega t) + C\omega \cos(\omega t)$ , we obtain

$$x_0 = B \cos(\omega t_0) + C \sin(\omega t_0) \quad \text{and} \quad v_0 = -B\omega \sin(\omega t_0) + C\omega \cos(\omega t_0).$$

These two equations, after some algebraic manipulation, can be solved for  $B$  and  $C$ :

$$B = \boxed{x_0 \cos(\omega t_0) - (v_0/\omega) \sin(\omega t_0)} \quad \text{and} \quad C = \boxed{(v_0/\omega) \cos(\omega t_0) + x_0 \sin(\omega t_0)}.$$

- (c) Because  $v_{\max} = A\omega$ , from part (a) we have  $v_{\max} = \boxed{\omega(B^2 + C^2)^{1/2}}$ .

9. From  $a_{\max} = A\omega^2$  and  $\omega = 2\pi f$ , we have

$$A = a_{\max} / \omega^2 = (183.25 \text{ m/s}^2) / [2\pi(813.52 \text{ Hz})]^2 = \boxed{7.0137 \times 10^{-6} \text{ m}}.$$

10. The object remains on the platform unless the normal force becomes zero, which occurs when the acceleration becomes  $g$ :

$$a = A\omega^2 = g, \text{ which gives}$$

$$f = \omega / 2\pi = (g/A)^{1/2} / 2\pi = [(9.8 \text{ m/s}^2) / (0.03 \text{ m})]^{1/2} / 2\pi = \boxed{2.9 \text{ Hz}}.$$

11. Using  $\sin \theta = \cos(\pi/2 - \theta)$  and  $\cos \theta = \cos(-\theta)$ , we can write

$$x = A \sin(\omega t + \delta) = A \cos[\pi/2 - (\omega t + \delta)] = A \cos(\omega t + \delta - \pi/2).$$

$$\text{Thus } \boxed{x = A \cos[(2.0 \text{ rad/s})t - 1.17 \text{ rad}], \quad t \text{ in s.}}$$

12. The amplitude is the maximum displacement from the center of the motion:  $A = \boxed{3 \text{ m}}$ .

From the general expression, we have

$$x = A \sin(\omega t + \delta); \quad (-3 \text{ m}) = (3 \text{ m}) \sin(0 + \delta), \text{ which gives the phase } \delta = \boxed{-\pi/2 \text{ rad}}.$$

The period is  $T = 90 \text{ s} / 6 = \boxed{15 \text{ s}}$ .

The frequency is  $f = 1/T = 1/(15 \text{ s}) = \boxed{0.067 \text{ Hz}}$ ; the angular frequency is  $\omega = 2\pi f = \boxed{2\pi/15 \text{ rad/s}}$ .

- 13.** (a) We find the angular frequency from

$$\omega = \sqrt{k/m} = \sqrt{(0.50 \text{ N/m}) / (0.20 \text{ kg})} = 1.58 \text{ rad/s}.$$

$$\text{The period is } T = 2\pi / \omega = 2\pi / (1.58 \text{ rad/s}) = \boxed{4.0 \text{ s}}.$$

- (b) From  $v_{\max} = A\omega$ , we get

$$A = v_{\max} / \omega = (2.0 \text{ m/s}) / (1.58 \text{ rad/s}) = \boxed{1.3 \text{ m}}.$$

14. We get the angular frequency from

$$\omega = 2\pi / T = 2\pi / (5.0 \text{ s}).$$

Because the vertical position is the equilibrium position, the speed is maximum there:

$$v_{\max} = A\omega = (2.5 \text{ m})(2\pi / 5.0 \text{ rad/s}) = \boxed{3.1 \text{ m/s}}.$$

15. The period is the time for one cycle:  $T = 1.71 \text{ s}$ .

Because the total distance traveled is twice that from one extreme to the other, we have

$$A = D/4 = 6.98 \text{ cm} / 4 = 1.75 \text{ cm}.$$

- (a) We define the average speed as total distance traveled/time:

$$\text{average speed} = D/T = 6.98 \text{ cm} / (1.71 \text{ s}) = \boxed{4.08 \text{ cm/s}}.$$

- (b) The angular frequency is  $\omega = 2\pi / T = 2\pi / (1.71 \text{ s}) = 3.67 \text{ rad/s}$ . Thus

$$v_{\max} = A\omega = (1.75 \text{ cm})(3.67 \text{ rad/s}) = \boxed{6.41 \text{ cm/s}} \quad \text{and}$$

$$a_{\max} = A\omega^2 = (1.75 \text{ cm})(3.67 \text{ rad/s})^2 = \boxed{23.6 \text{ cm/s}^2}.$$

16. The angular frequency is  $\omega = (k/m)^{1/2} = [(12 \text{ N/m}) / (0.35 \text{ kg})]^{1/2} = 5.9 \text{ rad/s}$ .

- (a) Because the mass is released from its maximum displacement,  $A = 0.040 \text{ m}$ .

From  $x = A \sin(\omega t + \delta)$ , we have  $A = A \sin(0 + \delta)$ , which gives  $\delta = \pi/2$ . Thus

$$x(t) = (0.040 \text{ m}) \sin[(5.9 \text{ rad/s})t + \pi/2] = \boxed{(0.040 \text{ m}) \cos[(5.9 \text{ rad/s})t]}.$$

- (b) From  $a = d^2x/dt^2$ , we get  $a = -A\omega^2 \cos(\omega t) = -\omega^2 x$ ; therefore the maximum positive acceleration occurs at the minimum value of  $x$ , which is  $-A = \boxed{-0.040 \text{ m}}$ .

- (c)  $a_{\max} = A\omega^2 = (0.040 \text{ m})(5.86 \text{ rad/s})^2 = \boxed{1.4 \text{ m/s}^2}$ .

- (d)  $v_{\max} = A\omega = (0.040 \text{ m})(5.86 \text{ rad/s}) = \boxed{0.23 \text{ m/s}}$ .

17. (a) If we write  $x = A \sin(\omega t + \delta)$ , then  $v = dx/dt = A\omega \cos(\omega t + \delta)$ .

From the initial conditions at  $t = 0$ , for these two equations we have

$$3.0 \text{ cm} = A \sin \delta, \text{ and } -5.0 \text{ cm/s} = A(3.0 \text{ rad/s}) \cos \delta.$$

Dividing these two equations, we have

$\tan \delta = -1.8$ , which gives  $\delta = -1.06 \text{ rad}$  and  $+2.08 \text{ rad}$ . We choose the one that gives a positive value for the sine and a negative value for the cosine:  $\delta = 2.1 \text{ rad}$ .

From this value, we obtain  $A = 3.4 \text{ cm}$ . The displacement is

$$x = (3.4 \text{ cm}) \sin[(3.0 \text{ rad/s})t + 2.1 \text{ rad}].$$

- (b) We find the times when the position is 3 cm from

$$3.0 \text{ cm} = (3.44 \text{ cm}) \sin[(3.0 \text{ rad/s})t + 2.08 \text{ rad}], \text{ which gives}$$

$$(3.0 \text{ rad/s})t + 2.08 \text{ rad} = 1.06 \text{ rad} + n2\pi, \quad n = 0, \pm 1, \pm 2, \dots, \text{ and}$$

$$(3.0 \text{ rad/s})t + 2.08 \text{ rad} = 2.08 \text{ rad} + n2\pi, \quad n = 0, \pm 1, \pm 2, \dots.$$

Because 2.08 rad corresponds to  $t = 0$ , the next time the position is 3 cm occurs when

$$(3.0 \text{ rad/s})t + 2.08 \text{ rad} = 1.06 \text{ rad} + 2\pi, \text{ which gives } t = 1.75 \text{ s} \approx \boxed{1.8 \text{ s}}.$$

- (c) We find the times when the speed is 5 cm/s from

$$\pm 5.0 \text{ cm/s} = (3.44 \text{ cm})(3.0 \text{ rad/s}) \cos[(3.0 \text{ rad/s})t + 2.08 \text{ rad}].$$

These two equations have four solutions:

$$(3.0 \text{ rad/s})t + 2.08 \text{ rad} = 1.06 \text{ rad} + n2\pi, \quad n = 0, \pm 1, \pm 2, \dots,$$

$$(3.0 \text{ rad/s})t + 2.08 \text{ rad} = 2.08 \text{ rad} + n2\pi, \quad n = 0, \pm 1, \pm 2, \dots,$$

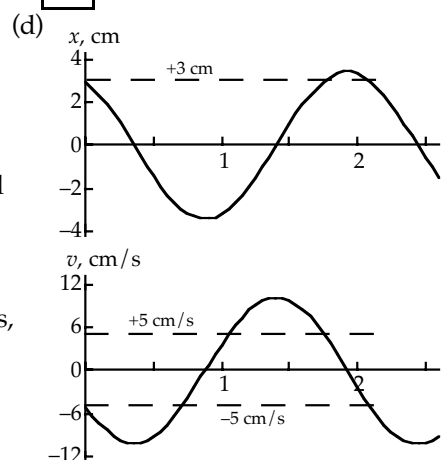
$$(3.0 \text{ rad/s})t + 2.08 \text{ rad} = 4.20 \text{ rad} + n2\pi, \quad n = 0, \pm 1, \pm 2, \dots, \text{ and}$$

$$(3.0 \text{ rad/s})t + 2.08 \text{ rad} = 5.22 \text{ rad} + n2\pi, \quad n = 0, \pm 1, \pm 2, \dots.$$

Beginning with  $t = 0$ , the four times are

$$\boxed{0, 0.71 \text{ s}, 1.1 \text{ s}, 1.8 \text{ s}}.$$

Additional times occur when the period of the motion,  $T = 2.1 \text{ s}$ , is added to these times.



18. For uniform circular motion, we have  $x = R \cos(\omega t + \delta)$ , and  $y = R \sin(\omega t + \delta)$ .

From the initial conditions, for these two equations we have

$$-R = R \cos \delta, \text{ and } 0 = R \sin \delta, \text{ or } \sin \delta = 0, \text{ and } \cos \delta = -1, \text{ which gives } \delta = \boxed{\pi}.$$

19. The distance from the center is

$$D = (x^2 + y^2)^{1/2} = [R^2 \cos^2(\omega t + \delta) + R^2 \sin^2(\omega t + \delta)]^{1/2}$$

$$= R[\cos^2(\omega t + \delta) + \sin^2(\omega t + \delta)]^{1/2}.$$

Because  $\cos^2 \phi + \sin^2 \phi = 1$ , we have  $D = \boxed{R}$ .

20. (a) We are given  $x = R \sin(\omega t + \delta - \pi/2)$ ;  $y = R \sin(\omega t + \delta)$ .

At a time when  $\omega t + \delta = \pi/2$ ,  $\Delta t = T/4$ , the position is  $x = 0$ ,  $y = +R$ .

When  $t$  increases so  $\omega t + \delta = \pi$ , the position is  $x = +R$ ,  $y = 0$ .

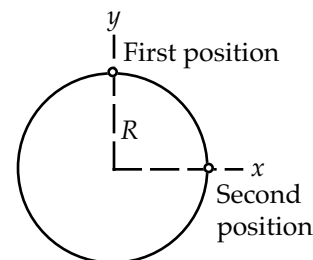
Thus the motion is clockwise.

- (b) We are given  $x = R \sin(\omega t + \delta)$ ;  $y = R \sin(\omega t + \delta + \pi/2)$ .

At a time when  $\omega t + \delta = 0$ ,  $\Delta t = T/4$ , the position is  $x = 0$ ,  $y = +R$ .

When  $t$  increases so  $\omega t + \delta = \pi/2$ , the position is  $x = +R$ ,  $y = 0$ .

Thus the motion is clockwise.



21. Because the observer is in the plane of the orbit, the motion will be seen as the projection of the circular motion. Choosing the  $x$ -axis along the motion, we have

$$\boxed{x = R \cos(\omega t + \delta)},$$

with  $R = 1.5 \times 10^{11} \text{ m}$  and  $\omega = 2\pi/(3.16 \times 10^7 \text{ s}) = 1.99 \times 10^{-7} \text{ rad/s}$ .

22. The amplitude of the motion is the radius of the record:

$$A = (4 \text{ in.})(2.54 \text{ cm/in}) = 10 \text{ cm}.$$

The angular frequency is

$$\omega = (45 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s}) = \frac{3}{2}\pi \text{ rad/s}.$$

- (a) Because  $x = 0$  at  $t = 0$ , we assume that the object starts at the top of the circle. The angle turned in time  $t$  is given by

$\theta = \omega t$ , so we find the projection from the figure:

$$x = A \sin(\omega t) = (10 \text{ cm}) \sin[(\frac{3}{2}\pi \text{ rad/s})t], \quad t \text{ in s}.$$

- (b)  $A = 10 \text{ cm}$ ,  $\omega = \frac{3}{2}\pi \text{ rad/s}$ ,

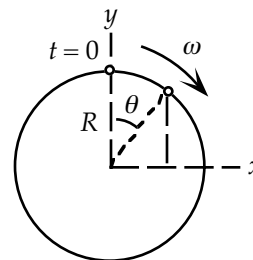
$$v_{\max} = A\omega = (10 \text{ cm})(\frac{3}{2}\pi \text{ rad/s}) = 47 \text{ cm/s}.$$

- (c)  $v = dx/dt = d[A \sin(\omega t)]/dt = \omega A \cos(\omega t)$ , so

$$a = dv/dt = d[\omega A \cos(\omega t)]/dt = -\omega^2 A \sin(\omega t)$$

$$= -(3\pi/2 \text{ rad/s})^2 (10 \text{ cm}) \sin[(\frac{3}{2}\pi \text{ rad/s})t]$$

$$= -(2.2 \text{ m/s}^2) \sin[(\frac{3}{2}\pi \text{ rad/s})t], \quad t \text{ in s}.$$



23. From  $x(t) = R_1 \cos(\omega t + \delta)$  and  $y(t) = R_2 \sin(\omega t + \delta)$ , we can write

$$x/R_1 = \cos(\omega t + \delta) \quad \text{and} \quad y/R_2 = \sin(\omega t + \delta).$$

Using  $\cos^2 \phi + \sin^2 \phi = 1$ , we can write

$$\cos^2(\omega t + \delta) + \sin^2(\omega t + \delta) = (x/R_1)^2 + (y/R_2)^2 = 1.$$

Thus the curve is an ellipse with axis lengths of  $R_1$  and  $R_2$ .

24. The force exerted by the spring is always along the spring and is proportional to the amount by which the spring is stretched or compressed. To make it move in a circle, the length of the spring, and therefore the magnitude of the restoring force it exerts on the mass, cannot change. Note that this restoring force is radial and therefore acts as the centripetal force for the circular motion. So we can pull on the mass to make the spring stretch by a certain amount, measure the resulting restoring force  $F$ , and then give the object (of mass  $m$ ) a tangential push (i.e., in the direction perpendicular to the spring) such that it starts off with a speed  $v$  that satisfies  $F = mv^2/r$ , with  $r$  the distance between the peg and the mass. The object will then continue on its circular motion with that radius and speed. To make it move in a straight line, we must make sure that the object cannot have a velocity component that is perpendicular to the force exerted on it, i.e., a component that is perpendicular to the orientation of the spring. This is easily achieved if the spring is either compressed or stretched and the object is then let go from rest, or if the object is given an initial push along the spring either towards or away from the peg. A one-dimensional simple harmonic oscillation along the orientation of the spring will ensue.

25. The magnitude of the net force exerted on the mass when it is just released is  $F = kA$ , where  $k$  is the spring constant and  $A = 6.0 \text{ cm}$ . According to Newton's second law this should be equal to  $ma$ :

$$F = kA = ma, \text{ which gives}$$

$$T = 2\pi (m/k)^{1/2} = 2\pi (A/a)^{1/2} = 2\pi [6.0 \text{ cm}/(40 \text{ cm/s}^2)]^{1/2} = 24 \text{ s}.$$

The speed of the mass at its equilibrium point is

$$v_{\max} = \omega A = (k/m)^{1/2} A = [(40 \text{ cm/s}^2)/6.0 \text{ cm}]^{1/2} (6.0 \text{ cm}) = 15 \text{ cm/s}.$$

26. (a) From the equilibrium condition that the restoring force and the weight must balance, we get

$$k \Delta y - mg = 0;$$

$$\Delta y = mg/k = (4.0 \text{ kg})(9.8 \text{ m/s}^2)/(300 \text{ N/m}) = 0.13 \text{ m}.$$

- (b) On the Moon the only change is in  $g$ :  $g_M = 1.62 \text{ m/s}^2$ .

$$\text{Thus } \Delta y = (4.0 \text{ kg})(1.62 \text{ m/s}^2)/(300 \text{ N/m}) = 0.022 \text{ m}.$$

27. The period depends only on the mass and force constant:  $T = 2\pi\sqrt{m/k}$  :

$$1.0 \text{ s} = 2\pi\sqrt{m/(200 \text{ N/m})}, \text{ which gives } m = 5.1 \text{ kg}.$$

28. At equilibrium, the spring force balances the weight, so we have

$$k = mg/x = (60 \text{ kg})(9.8 \text{ m/s}^2)/(0.0245 \text{ m}) = \boxed{2.4 \times 10^4 \text{ N/m}}.$$

29. The period depends only on the mass and force constant:  $T = 2\pi\sqrt{m/k}$  :

$$3.1 \text{ s} = 2\pi\sqrt{(0.045 \text{ kg})/k}, \text{ which gives } k = 0.18 \text{ N/m}.$$

30. The period depends only on the mass and force constant:  $T = 2\pi\sqrt{m/k}$  .

- (a) Because the spring constant does not change, we can write the ratio as  $T_2/T_1 = \sqrt{m_2/m_1}$ .

$$\text{For } m_2 = 2m_1, T_2 = \sqrt{2} T_1 = \sqrt{2} (1.3 \text{ s}) = \boxed{1.8 \text{ s}}.$$

$$\text{For } m_2 = \frac{1}{2}m_1, T_2 = (1/\sqrt{2})T_1 = (1/\sqrt{2})(1.3 \text{ s}) = \boxed{0.92 \text{ s}}.$$

- (b) Because  $T$  is independent of  $A$ , the period is  $\boxed{1.3 \text{ s}}$  for any amplitude.

31. The angular frequency is  $\omega = 2\pi f = 2\pi(20 \text{ s}^{-1}) = 40\pi \text{ rad/s}$ .

The maximum force provides the maximum acceleration:

$$F_{\max} = mA\omega^2 = (1.20 \text{ kg})(0.80 \times 10^{-2} \text{ m})(40\pi \text{ rad/s})^2 = \boxed{1.5 \times 10^2 \text{ N}}.$$

32. We use a coordinate system with up positive.

- (a) At the equilibrium position we have  $\Sigma F = k \Delta y - mg = 0$ .

$$\text{Thus } k = mg/\Delta y = (180 \times 10^{-3} \text{ kg})(9.8 \text{ m/s}^2)/[(45 - 20) \times 10^{-2} \text{ m}] = \boxed{7.1 \text{ N/m}}.$$

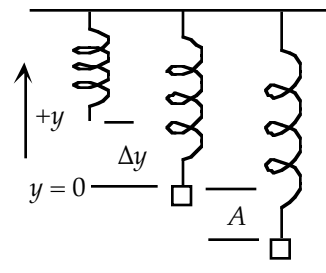
- (b) The angular frequency is

$$\omega = \sqrt{k/m} = \sqrt{(7.1 \text{ N/m})/(180 \times 10^{-3} \text{ kg})} = 6.3 \text{ rad/s}.$$

- (c) We put the origin at the equilibrium position and set  $t = 0$  at release, when  $y = -A = A \sin(-\pi/2)$ . Then

$$y = A \sin(\omega t - \pi/2) = (15 \text{ cm})\sin[(6.3 \text{ rad/s})(5 \text{ s}) - \pi/2] = -14.9 \text{ cm}.$$

$$\text{The position is } y_f = h + y = 20 \text{ cm} + (-14.9 \text{ cm}) = \boxed{5.1 \text{ cm above the floor}}.$$



33. We use a coordinate system with up positive.

At the equilibrium position of the 25-g mass, we have  $\Sigma F = k \Delta y - mg = 0$ . Thus

$$k = mg/\Delta y = (25 \times 10^{-3} \text{ kg})(9.8 \text{ m/s}^2)/(12 \times 10^{-2} \text{ m}) = 2.0 \text{ N/m}.$$

For the 75-g mass, the angular frequency is

$$\omega = \sqrt{k/m} = \sqrt{(2.0 \text{ N/m})/(75 \times 10^{-3} \text{ kg})} = 5.2 \text{ rad/s}.$$

Thus the period is  $T = 2\pi/\omega = 2\pi/(5.2 \text{ rad/s}) = \boxed{1.2 \text{ s}}.$

34. If we apply a force  $F$  to stretch the springs, the total displacement  $\Delta y$  is the sum of the displacements of the two springs:  $\Delta y = \Delta y_1 + \Delta y_2$ .

The effective spring constant is defined from  $F = -k_{\text{eff}} \Delta y$ .

Because they are in series, the force must be the same in the each spring:

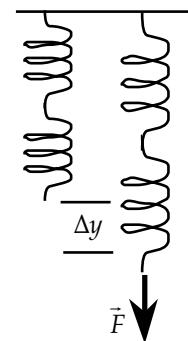
$$F_1 = F_2 = F = -k_1 \Delta y_1 = -k_2 \Delta y_2;$$

Then  $\Delta y = \Delta y_1 + \Delta y_2$  becomes

$$-F/k_{\text{eff}} = -F/k_1 - F/k_2, \text{ or}$$

$$1/k_{\text{eff}} = 1/k_1 + 1/k_2 = 2/k; \text{ so}$$

$$k_{\text{eff}} = k/2.$$





35. In the equilibrium position, we have

$$F_{\text{net}} = F_{20} - F_{10} = 0, \text{ or } F_{10} = F_{20}.$$

When the object is moved to the right a distance  $x$ , we have

$$F_{\text{net}} = F_{20} - k_2x - (F_{10} + k_1x) = -(k_1 + k_2)x.$$

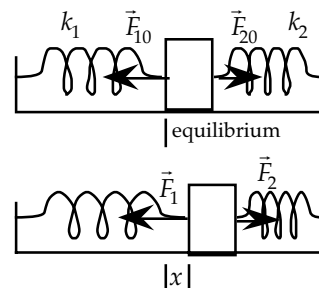
The effective spring constant is  $k_{\text{eff}} = k_1 + k_2$ , so the angular frequency is

$$\omega = \sqrt{k_{\text{eff}} / m} = \sqrt{(100 \text{ N/m} + 200 \text{ N/m}) / (0.060 \text{ kg})} = 71 \text{ rad/s}.$$

If we write the displacement as  $x = A \sin(\omega t + \delta)$ , at  $t = 0$  we have

$$A = 1.0 \text{ cm} = A \sin(0 + \delta), \text{ which gives } \delta = \pi/2, \text{ so}$$

$$x = (1.0 \text{ cm}) \sin[(71 \text{ rad/s})t + \pi/2].$$



36. The energy is the maximum potential energy:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(375 \text{ N/m})(14 \times 10^{-2} \text{ m})^2 = \boxed{3.7 \text{ J}}.$$

37. At the equilibrium position we have  $\Sigma F = k \Delta y - mg = 0$ , thus  $m = k \Delta y / g$ .

$$\text{The period is } T = 2\pi\sqrt{m/k} = 2\pi\sqrt{\Delta y/g} = 2\pi\sqrt{(0.30 \text{ m})/(9.8 \text{ m/s}^2)} = 1.1 \text{ s}.$$

38. The angular frequency is  $\omega = 2\pi/T = 2\pi/(2.5 \text{ s}) = 2.5 \text{ rad/s}$ .

Because the total energy is the maximum kinetic energy, we have

$$E = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}mA^2\omega^2;$$

$$2.7 \text{ J} = \frac{1}{2}(1.2 \text{ kg})A^2(2.5 \text{ rad/s})^2, \text{ which gives } A = \boxed{0.85 \text{ m}}.$$

39. The total energy is the maximum potential energy and the maximum kinetic energy:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2;$$

$$\frac{1}{2}(2 \text{ N/m})(3 \times 10^{-2} \text{ m})^2 = \frac{1}{2}(1 \text{ kg})v_{\text{max}}^2, \text{ which gives } v_{\text{max}} = \boxed{0.042 \text{ m/s}}.$$

40. At maximum compression, the kinetic energy has been changed to potential energy:

$$E = \frac{1}{2}mv^2 = \frac{1}{2}kA^2;$$

$$\frac{1}{2}(0.350 \text{ kg})(0.88 \text{ m/s})^2 = \frac{1}{2}(140 \text{ N/m})A^2, \text{ which gives } A = 0.044 \text{ m} = \boxed{4.4 \text{ cm}}.$$

$$\text{If } m_2 = 2m_1, \text{ then the amplitude would increase by } \sqrt{2} \text{ to } \sqrt{2}(4.4 \text{ cm}) = \boxed{6.2 \text{ cm}}.$$

41. Because the tension is central, it does no work. The total energy, which is conserved, is

$$E = \frac{1}{2}mv_{\text{top}}^2 + U_{\text{top}} = \frac{1}{2}mv_{\text{bottom}}^2 + U_{\text{bottom}}.$$

Each speed is  $R\omega$ . We choose the reference level for  $U$  at the bottom, so we have

$$\frac{1}{2}mR^2\omega_{\text{top}}^2 + mg(2R) = \frac{1}{2}mR^2\omega_{\text{bottom}}^2 + 0.$$

After canceling common factors, we have

$$R\omega_{\text{top}}^2 + 4g = R\omega_{\text{bottom}}^2;$$

$$(1.2 \text{ m})(2.2 \text{ rad/s})^2 + 4(9.8 \text{ m/s}^2) = (1.2 \text{ m})\omega_{\text{bottom}}^2, \text{ which gives } \omega_{\text{bottom}} = \boxed{6.1 \text{ rad/s}}.$$

42. The total energy, which is conserved, is

$$E = \frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2x^2.$$

Then  $dE/dt = mv(dv/dt) + m\omega^2x(dx/dt) = 0$ , which can be written  $(a + \omega^2x)mv = 0$ .

Thus  $a = -\omega^2x$ , which is the equation of motion for simple harmonic motion.

43. Because the speed is tangential, the energy is  $E = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega d)^2$ . For circular motion, the distance from the center can be expressed as  $d^2 = x^2 + y^2$ , so

$$E = \frac{1}{2}m\omega^2(x^2 + y^2) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2y^2.$$

44. Because the mass is released from rest at maximum amplitude, the next time it comes to rest is half a period; so

$$T = 2(0.45 \text{ s}) = 0.90 \text{ s}, \text{ and the angular frequency is}$$

$$\omega = 2\pi/T = 2\pi/(0.90 \text{ s}).$$

The maximum speed is

$$v_{\max} = A\omega = (0.13 \text{ m})[2\pi/(0.90 \text{ s})] = \boxed{0.90 \text{ m/s}}.$$

45. The angular frequency is  $\omega = \sqrt{k/m} = \sqrt{(1 \text{ N/m})/(0.2 \text{ kg})} = 2.24 \text{ rad/s}$ .

Because  $x = x_{\max}$  at  $t = 0$ , we can write

$$x = x_{\max} \sin(\omega t + \pi/2), \text{ from which we get}$$

$$v = x_{\max} \omega \cos(\omega t + \pi/2) = v_{\max} \cos(\omega t + \pi/2).$$

Using the data at  $t = 0.5 \text{ s}$ , we get

$$1.5 \text{ m/s} = v_{\max} \cos[(2.24 \text{ rad/s})(0.5 \text{ s}) + \pi/2], \text{ which gives}$$

$$v_{\max} = \boxed{2 \text{ m/s}}.$$

Then  $x_{\max} = v_{\max}/\omega = (1.67 \text{ m/s})/(2.24 \text{ rad/s}) = \boxed{0.7 \text{ m}}.$

The total energy is  $E = \frac{1}{2}kx_{\max}^2 = \frac{1}{2}(1 \text{ N/m})(0.74 \text{ m})^2 = \boxed{0.3 \text{ J}}.$

46. (a) Because the force is given by  $F = -dU/dx$ , the equilibrium positions, where  $F = 0$ , are found from the maxima and minima of  $U$ :

$$dU/dx = d[U_0(x^2 - a^2)^2]/dx = 2U_0(x^2 - a^2)(2x) = 0, \text{ which gives}$$

$$x_{\text{eq}} = \boxed{0, \pm a}.$$

To test for stability, we let  $x = x_{\text{eq}} + z$ , where  $z$  is small.

For  $x_{\text{eq}} = 0$ :

$$F = -dU/dx = -2U_0(z^2 - a^2)(2z) \approx -2U_0(-a^2)(2z) = 4U_0a^2z.$$

Thus for  $z > 0$ ,  $F > 0$ ; while for  $z < 0$ ,  $F < 0$ .

The force is away from the equilibrium position, therefore the position is **unstable**.

For  $x_{\text{eq}} = +a$ :

$$F = -dU/dx = -2U_0[(a+z)^2 - a^2]2(a+z) \approx -2U_0(2az)(2a) = -8U_0a^2z.$$

Thus for  $z > 0$ ,  $F < 0$ ; while for  $z < 0$ ,  $F > 0$ .

The force is toward the equilibrium position, therefore the position is **stable**.

For  $x_{\text{eq}} = -a$ :

$$F = -dU/dx = -2U_0[(-a+z)^2 - a^2]2(-a+z)$$

$$\approx -2U_0(-2az)(-2a) = -8U_0a^2z.$$

Thus for  $z > 0$ ,  $F < 0$ ; while for  $z < 0$ ,  $F > 0$ .

The force is toward the equilibrium position, therefore the position is **stable**.

The stable equilibrium positions are  $x = \pm a$ .

- (b) Near the stable equilibrium positions at  $x = \pm a$ , we have

$$U[(a+z)^2 - a^2]^2 = U_0(a^2 + 2az + z^2 - a^2)^2 \approx U_0(4a^2z^2).$$

If we write this as  $U = \frac{1}{2}kz^2$ , the effective force constant is  $k = 8U_0a^2$ , from which we get

$$\omega = \sqrt{k/m} = \sqrt{8U_0a^2/m}.$$

47. (a) Assuming uniform density, we have  $\rho = M_E / \frac{4}{3}\pi R_E^3$ .

The mass within a sphere of radius  $\ell$  is

$$M = \rho \frac{4}{3}\pi \ell^3 = [M_E / \frac{4}{3}\pi R_E^3] (\frac{4}{3}\pi \ell^3) = \boxed{(\ell^3 / R_E^3) M_E}.$$

- (b) The gravitational force is

$$F = GMm / \ell^2 = \boxed{GM_E m \ell / R_E^3 \text{ toward the center}}.$$

- (c) Because the sides of the tunnel provide sufficient normal forces to make the mass move along the tunnel, the net force must be the component of the force in part (b) toward the deepest point:

$$F_{\text{net}} = -(GM_E m \ell / R_E^3) \sin \theta \\ = \boxed{-GM_E m x / R_E^3 \text{ toward the center of the tunnel}},$$

where  $\theta$  is the angle subtended at the center by  $x$ :  $x = \ell \sin \theta$ .

There is no net force perpendicular to the tunnel because the normal forces balance the perpendicular component of the gravitational force.

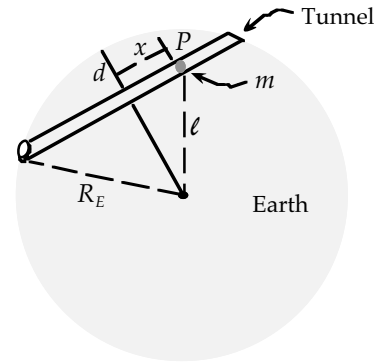
- (d) From part (c), the effective force constant is  $k = GM_E m / R_E^3$ , so the period is

$$T = 2\pi \sqrt{m/k} = 2\pi \sqrt{R_E^3 / GM_E} \\ = 2\pi \sqrt{(6.37 \times 10^6 \text{ m})^3 / [(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2)(5.98 \times 10^{24} \text{ kg})]} = 5.05 \times 10^3 \text{ s} = 1.4 \text{ h}.$$

- (e) Because the satellite has a radial acceleration provided by the gravitational attraction, from  $\Sigma F = ma$  we get

$$GM_E m / R_E^2 = mv^2 / R_E, \text{ which gives } v^2 = GM_E / R_E.$$

The period is  $T = 2\pi R_E / v = 2\pi (R_E^3 / GM_E)^{1/2}$ , which is the same as in part (d):  $\boxed{1.4 \text{ h}}$ .



48. (a) With  $h = 0$  at the bottom of the bowl, the potential energy is

$$U = mgh = mg(R - R \cos \theta) = mgR(1 - \cos \theta).$$

For small angles, we expand the cosine function:

$$U \approx mgR[1 - (1 - \frac{1}{2}\theta^2 + \dots)] = \boxed{\frac{1}{2}mgR\theta^2}.$$

- (b) Because  $U$  is proportional to  $\theta^2$ , the angular motion is simple harmonic.

With the bead released from rest at  $\theta_0$ , we have

$$\theta(t) = \theta_0 \cos(\omega t).$$

The angular frequency is

$$\omega = (k/I)^{1/2} = (mgR/mR^2)^{1/2} = (g/R)^{1/2} = [(9.8 \text{ m/s}^2)/(0.15 \text{ m})]^{1/2} = 8.1 \text{ rad/s}.$$

We have  $\theta(t) = \boxed{0.1 \cos(8.1t) \text{ rad}}$ , with  $t$  in s; the frequency is  $f = \omega/2\pi = \boxed{1.3 \text{ Hz}}$ .

- (c) The total energy is the maximum potential energy:

$$E = \frac{1}{2}(mgR)\theta_0^2 = \frac{1}{2}(0.040 \text{ kg})(9.8 \text{ m/s}^2)(0.15 \text{ m})(0.1 \text{ rad})^2 = \boxed{2.9 \times 10^{-4} \text{ J}}.$$

- (d) The angular velocity of the bead is

$$d\theta/dt = -\theta_0 \omega \sin(\omega t).$$

The velocity is tangential:

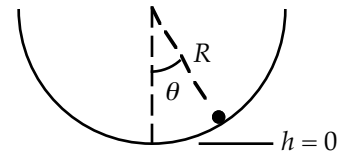
$$v = R d\theta/dt = -(0.15 \text{ m})(0.1 \text{ rad})(8.08 \text{ rad/s}) \sin[(8.08 \text{ rad/s})(0.1 \text{ s})] \\ = \boxed{-0.088 \text{ m/s (to the left)}}.$$

- (e) The angular acceleration of the bead is

$$d^2\theta/dt^2 = -\theta_0 \omega^2 \cos(\omega t).$$

The acceleration is tangential:

$$a = R d^2\theta/dt^2 = -(0.15 \text{ m})(0.1 \text{ rad})(8.08 \text{ rad/s})^2 \cos[(8.08 \text{ rad/s})(0.2 \text{ s})] \\ = \boxed{0.044 \text{ m/s}^2 \text{ (to the right)}}.$$

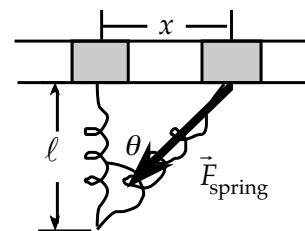


49. The force exerted by the spring is  $F_{\text{spring}} = k[(x^2 + \ell^2)^{1/2} - \ell]$  directed toward the pin. Because no friction is present, the restoring force along the groove is the component of  $F_{\text{spring}}$  along the groove:

$$F_{\text{net}} = -k[(x^2 + \ell^2)^{1/2} - \ell] \sin \theta \\ = -k[(x^2 + \ell^2)^{1/2} - \ell] [x/(x^2 + \ell^2)^{1/2}].$$

For  $x \ll \ell$ ,  $(x^2 + \ell^2)^{1/2} \simeq \ell + \frac{1}{2}(x^2/\ell)$ , so we have

$$F_{\text{net}} \simeq -kx[\ell + \frac{1}{2}(x^2/\ell) - \ell]/\ell = -\frac{1}{2}kx^3/\ell^2.$$



50.  $T = 2\pi\sqrt{L/g} = 2\pi\sqrt{(1.20 \text{ m})/(9.8 \text{ m/s}^2)} = 2.20 \text{ s}.$

51.  $g = (2\pi/T)^2 L = 4\pi^2 f^2 L = 4\pi^2 (0.342 \text{ Hz})^2 (2.12 \text{ m}) = \boxed{9.79 \text{ m/s}^2}.$

52. The maximum displacement is  $\theta_0 = 0.075 \text{ rad} = 4.3^\circ$ , small enough that we can treat the motion as simple harmonic.

The period is  $T = 2\pi\sqrt{L/g}$ , which means  $\omega = (g/L)^{1/2} = [(9.8 \text{ m/s}^2)/(3.88 \text{ m})]^{1/2} = 1.6 \text{ rad/s}.$

Because the pendulum is released at maximum displacement, we can write

$$\theta = \theta_0 \cos(\omega t) = \boxed{0.055 \cos(1.6t) \text{ rad}}, \text{ with } t \text{ in s}.$$

53. (a) From  $T = 2\pi\sqrt{L/g}$ , we get  $L = gT^2/4\pi^2 = (9.80 \text{ m/s}^2)(3.0000 \text{ s})^2/4\pi^2 = \boxed{2.23 \text{ m}}.$

(b) Because  $g$  depends on the distance from the center of the Earth,  $g = GM/r^2$ , we can write

$$g_2/g_1 = (r_1/r_2)^2 = [R_E/(R_E + h)]^2, \text{ or } g_1/g_2 = (1 + h/R_E)^2,$$

where  $R_E$  is the radius of the Earth and the height of the skyscraper is  $h \ll R_E$ .

With a constant length, we get

$$T_2/T_1 = (g_1/g_2)^{1/2} = 1 + h/R_E. \text{ Thus}$$

$$3.0002 \text{ s}/3.0000 \text{ s} = 1 + h/R_E, \text{ which gives}$$

$$h/R_E = 6.7 \times 10^{-5}, \text{ or } h = (6.7 \times 10^{-5})(6.37 \times 10^6 \text{ m}) = \boxed{430 \text{ m}}.$$

54. Because the fractional change is so small, we will treat it as a differential.

From  $T = 2\pi\sqrt{L/g}$ , we get  $dT = \frac{1}{2}2\pi\sqrt{1/Lg} dL.$

Dividing by  $T = 2\pi\sqrt{L/g}$ , we get  $dT/T = \frac{1}{2}(dL/L) = \frac{1}{2}(1/30,000) = 1/60,000.$

Thus  $dT = (1/60,000)T = (1/60,000)(7 \text{ d})(24 \text{ h/d})(3600 \text{ s/h}) = \boxed{10.1 \text{ s}}.$

55. (a) Without the peg, we have

$$T_1 = 2\pi\sqrt{L/g} = 2\pi\sqrt{(1.0 \text{ m})/(9.8 \text{ m/s}^2)} = 2.0 \text{ s}.$$

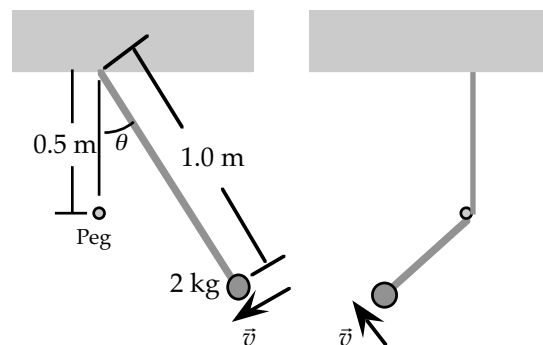
When the string is caught by the peg, we have

$$T_2 = 2\pi\sqrt{L/g} = 2\pi\sqrt{(0.5 \text{ m})/(9.8 \text{ m/s}^2)} = 1.4 \text{ s}.$$

Because there is half of each of these swings in each basic period, we have

$$T = \frac{1}{2}T_1 + \frac{1}{2}T_2 = \frac{1}{2}(2.0 \text{ s}) + \frac{1}{2}(1.4 \text{ s}) = \boxed{1.7 \text{ s}}.$$

- (b) Because total energy,  $E = \frac{1}{2}mv^2 + mgh$ , is conserved and  $v = 0$  at the maximum height on each side, the ball must rise to the same height:  $\boxed{0.05 \text{ m}}.$



56. Let the length of the pendulum string be  $L$ . Without the additional impulse the energy of the pendulum is  $E = U_{\max} = mgL(1 - \cos \theta_0) = mgL[1 - (1 - \frac{1}{2}\theta_0^2 + \dots)] \approx \frac{1}{2}mgL\theta_0^2$ , where the approximation  $\cos \theta_0 \approx 1 - \frac{1}{2}\theta_0^2$  has been used. With the impulse the pendulum bob acquires a velocity  $v_1$  at the top of its path, with  $mv_1 = -J$ . (Here the negative sign indicates that the impulse delivered is to the left.) The additional kinetic energy the pendulum acquires is  $K' = \frac{1}{2}mv_1^2 = \frac{1}{2}m(-J/m)^2 = J^2/2m$ , so after the impulse the total mechanical energy of the pendulum is

$$E' = U_{\max} + K' \approx \frac{1}{2}mgL\theta_0^2 + J^2/2m.$$

Equate this to the expression of the new maximum potential energy, when the bob swing to the new turning point which is at an angle  $\theta_0'$  from the vertical:

$$E' = U'_{\max}; \text{ or}$$

$$\frac{1}{2}mgL\theta_0^2 + J^2/2m = \frac{1}{2}mgL(\theta_0')^2, \text{ which gives}$$

$$\theta_0' = [\theta_0^2 + J^2/m^2gL]^{1/2} = [\theta_0^2 + (J/m\omega L)^2]^{1/2}, \text{ where we noted that } \omega = (g/L)^{1/2}.$$

To find the phase angle  $\delta$  of the oscillation, write  $\theta(t) = \theta_0' \sin(\omega t + \delta)$ . At  $t = T/4$  we have

$$v(T/4) = L d\theta/dt = L\omega\theta_0' \cos(\omega T/4 + \delta) = L\omega\theta_0' \cos(\pi/2 + \delta) = -L\omega\theta_0' \sin \delta$$

$$= v_1 = -J/m; \text{ so}$$

$$\sin \delta = J/m\omega L\theta_0' = (J/m\omega L)/[\theta_0^2 + (J/m\omega L)^2]^{1/2}, \text{ or } \tan \delta = J/m\omega L\theta_0. \text{ Thus}$$

$$\delta = \tan^{-1}(J/m\omega L\theta_0).$$

57. Set the zero level of gravitational potential energy at the lowest point of the pendulum bob's swing.

Then as the pendulum swings to the maximum angle  $\alpha$  the bob's vertical displacement is  $h = L(1 - \cos \alpha)$ , whereupon the potential energy of the bob is

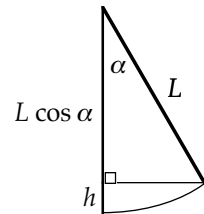
$$U = mgh = mgL(1 - \cos \alpha). \text{ Note that, at that point, the bob is not moving}$$

and so  $K = K_{\min} = 0$  and  $U = U_{\max} = E$ , i.e.,

$$E = mgL(1 - \cos \alpha).$$

If the angle  $\alpha$  is small, then the approximation  $\cos \alpha \approx 1 - \frac{1}{2}\alpha^2$  may be applied, which yields

$$E \approx mgL[1 - (1 - \frac{1}{2}\alpha^2)] = \frac{1}{2}mgL\alpha^2 \quad (|\alpha| \ll 1).$$



58. The rotational inertia of the door about the hinge is  $I = \frac{1}{3}ML^2$ , where  $L = 35$  cm.

For a uniform door,  $r = \frac{1}{2}L$  is the distance of the center of mass from the hinge, then

$$T = 2\pi (I/Mgr)^{1/2} = 2\pi [(\frac{1}{3}ML^2)/(Mg\frac{1}{2}L)]^{1/2} = 2\pi [\frac{2}{3}(0.40 \text{ m}) / (9.8 \text{ m/s}^2)]^{1/2} = 1.03 \text{ s}.$$

The frequency is  $f = 1/T = 0.96 \text{ Hz}$ .

59. We use the parallel-axis theorem to get the rotational inertia about the pivot:

$$I = I_{\text{CM}} + Md^2 = \frac{1}{2}MR^2 + Mr^2 = \frac{1}{2}(0.200 \text{ kg})(0.10)^2 + (0.200 \text{ kg})(0.08)^2 = 2.28 \times 10^{-3} \text{ kg}\cdot\text{m}^2. \text{ Then}$$

$$T = 2\pi \sqrt{\frac{I}{Mgr}} = 2\pi \sqrt{\frac{(2.28 \times 10^{-3} \text{ kg}\cdot\text{m}^2)}{(0.200 \text{ kg})(9.8 \text{ m/s}^2)(0.08 \text{ m})}} = 0.76 \text{ s}.$$

60. We need the rotational inertia about the corner of the book. We use the parallel-axis theorem to get

$$I_A = I_{\text{CM}} + Md^2 = (1/12)M(a^2 + b^2) + Md^2,$$

where  $d = [(\frac{1}{2}a)^2 + (\frac{1}{2}b)^2]^{1/2}$  is the distance from the center to the corner. Thus  $I_A = \frac{1}{3}M(a^2 + b^2)$ .

For the angular frequency we get

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Mgr}{I_A}} = \sqrt{\frac{Mg\sqrt{a^2/4 + b^2/4}}{M(a^2 + b^2)/3}} = \sqrt{\frac{3g}{2\sqrt{a^2 + b^2}}}$$

$$= \sqrt{\frac{3(9.8 \text{ m/s}^2)}{2\sqrt{(0.15 \text{ m})^2 + (0.21 \text{ m})^2}}} = 7.5 \text{ rad/s}.$$

61. We use the parallel-axis theorem to get the rotational inertia about the pin:

$$I = I_{\text{CM}} + Md^2 = (1/12)ML^2 + M(\frac{1}{2}L - y)^2,$$

where  $d = \frac{1}{2}L - y$  is the distance from the center to the pin. Then

$$T = 2\pi\sqrt{\frac{I}{Mg}} = 2\pi\sqrt{\frac{M[\frac{1}{12}L^2 + (\frac{1}{2}L - y)^2]}{Mg(\frac{1}{2}L - y)}} = 2\pi\sqrt{\frac{2(L^2 - 3Ly + 3y^2)}{3(L - 2y)g}}.$$

62. We can treat the wire as a physical pendulum. If we call the length of each side  $L$ , the rotational inertia about the pivot is  $I = \frac{1}{3}(\frac{1}{2}M)L^2 + \frac{1}{3}(\frac{1}{2}M)L^2 = \frac{1}{3}ML^2$ . In the equilibrium position the center of mass is directly below the pivot and midway between the center of mass of each side, so its distance from the pivot is  $r = \frac{1}{2}L \cos 45^\circ$ . Thus

$$\begin{aligned} f = \frac{1}{T} &= \frac{1}{2\pi}\sqrt{\frac{Mg r}{I}} = \frac{1}{2\pi}\sqrt{\frac{Mg(\frac{L}{2} \cos \theta)}{\frac{1}{3}ML^2}} = \frac{1}{2\pi}\sqrt{\frac{3g \cos \theta}{2L}} \\ &= \frac{1}{2\pi}\sqrt{\frac{3(9.8 \text{ m/s}^2) \cos 45^\circ}{2(0.40 \text{ m})}} = 0.81 \text{ Hz}. \end{aligned}$$

63. (a) We find the rotational inertia about the nail from the parallel-axis theorem:

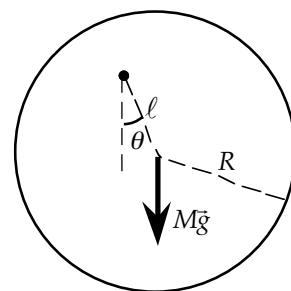
$$I = I_{\text{CM}} + Md^2 = \frac{1}{2}MR^2 + M\ell^2.$$

- (b) From Newton's second law for rotation,  $\tau = I\alpha$ , we have  
 $-Mg\ell \sin \theta = I d^2\theta/dt^2$ , which for small angles becomes  
 $-Mg\ell \theta = M(\frac{1}{2}R^2 + \ell^2) d^2\theta/dt^2$ , or  
 $-g\ell \theta = (\frac{1}{2}R^2 + \ell^2) d^2\theta/dt^2$ .

- (c) The result for part (b) shows that the motion is simple harmonic with  
 $\omega^2 = 2g\ell/(R^2 + 2\ell^2)$ , so

$$T = 2\pi/\omega = 2\pi[(R^2 + 2\ell^2)/2g\ell]^{1/2}.$$

- (d) For small  $\ell$ , we have  $T = 2\pi R/(2g\ell)^{1/2}$ , so as  $\ell \rightarrow 0$ ,  $T \rightarrow \infty$ ; no torque, which means uniform rotation.



64. We find the rotational inertia about the suspension point from the parallel-axis theorem:

$$I = I_{\text{CM}} + Md^2 = \frac{1}{2}MR^2 + ML^2.$$

$$\text{The period is } T = 2\pi\sqrt{\frac{I}{MgL}} = 2\pi\sqrt{\frac{R^2 + 2L^2}{2gL}}.$$

65. (a) The rotational inertia of the dumbbell about the center is

$$I = 2[m(\frac{1}{2}\ell)^2] = \frac{1}{2}m\ell^2.$$

Because  $\gamma$  corresponds to the rotational force constant, we find the period from

$$T = 2\pi\sqrt{\frac{I}{\gamma}} = 2\pi\sqrt{\frac{m\ell^2}{2\gamma}} = 2\pi\sqrt{\frac{(80 \text{ g})(30 \text{ cm})^2}{2(2 \times 10^5 \text{ g} \cdot \text{cm}^2/\text{s}^2)}} = 2.7 \text{ s}.$$

- (b) The total energy is the maximum potential energy:

$$E = \frac{1}{2}\gamma\theta^2 = \frac{1}{2}(2 \times 10^5 \text{ g} \cdot \text{cm}^2/\text{s}^2)(10^{-3} \text{ kg/g})(10^{-2} \text{ m/cm})^2(0.1 \text{ rad})^2 = \boxed{1.0 \times 10^{-4} \text{ J}}.$$

- (c) The total energy is the maximum kinetic energy:

$$E = \frac{1}{2}I(d\theta/dt)_{\text{max}}^2 = \frac{1}{2}(\frac{1}{2}m\ell^2)(d\theta/dt)_{\text{max}}^2;$$

$$1.0 \times 10^{-4} \text{ J} = \frac{1}{2}[\frac{1}{2}(0.080 \text{ kg})(0.30 \text{ m})^2](d\theta/dt)_{\text{max}}^2, \text{ which gives } (d\theta/dt)_{\text{max}} = 0.24 \text{ rad/s}.$$

Because the speed is tangential, we have

$$v_{\text{max}} = \frac{1}{2}\ell(d\theta/dt)_{\text{max}} = \frac{1}{2}(0.30 \text{ m})(0.24 \text{ rad/s}) = \boxed{3.6 \times 10^{-2} \text{ m/s}}.$$

66. The rotational inertia about the suspension point is found from the parallel-axis theorem:

$$I = I_{\text{CM}} + Md^2 = (1/12)m\ell^2 + m(\frac{1}{2}\ell - z\ell)^2 = m\ell^2(\frac{1}{3}\frac{1}{4} - z + z^2).$$

From Newton's second law for rotation,  $\tau = I\alpha$ , we have

$$-mg(\frac{1}{2}\ell - z\ell) \sin \theta = I d^2\theta/dt^2.$$

For small angles this becomes

$$-mg\ell(\frac{1}{2} - z)\theta = I d^2\theta/dt^2, \text{ or } d^2\theta/dt^2 = -g(\frac{1}{2} - z)\theta/\ell(\frac{1}{3} - z + z^2) = -\omega^2\theta.$$

The period is

$$T = 2\pi/\omega = \boxed{2\pi[2\ell(1 - 3z + 3z^2)/3g(1 - 2z)]^{1/2}}.$$

As  $z \rightarrow 0.5$ ,  $t \rightarrow \infty$  (uniform rotation).



67. The loudness ratio is  $L(t)/L(0) = e^{-t/\tau}$ .

At  $t = 2$  min, we get

$$L(t)/L(0) = e^{-(2 \text{ min})(60 \text{ s/min})/(55 \text{ s})} = \boxed{0.113}.$$

For  $L(t)/L(0) = 1/10,000$ , we have

$$1/10,000 = e^{-t/(55 \text{ s})}.$$

When we take the natural logarithm, we get

$$-\ln(10,000) = -9.2 = -t/(55 \text{ s}), \text{ which gives } t = 507 \text{ s} = \boxed{8.4 \text{ min}}.$$

68. From  $A/A_0 = e^{-bt/2m}$ , we have

$$0.35 = e^{-b(12.9 \text{ s})/2m}, \text{ which gives } b/2m = 0.0814 \text{ /s}.$$

The angular frequency is  $\omega' = \sqrt{\omega_0^2 - (b/2m)^2}$ . With  $b/2m \ll \omega_0$ , we have

$$\omega' \approx \omega_0[1 - \frac{1}{2}(b/2m\omega_0)^2] = (3.6 \text{ rad/s})[1 - \frac{1}{2}[(0.0814 \text{ /s})/(3.6 \text{ rad/s})]^2] = \boxed{(3.6 - 0.011) \text{ rad/s}}.$$

69. For the undamped angular frequency we have  $\omega_0^2 = k/m$ , or  $m = k/\omega_0^2$ .

For critical damping we have

$$b_c = (4mk)^{1/2} = 2k/\omega_0 = 2(184 \text{ N/m})/(3880 \text{ rad/s}) = \boxed{0.095 \text{ kg/s}}.$$

70. As the mass oscillates back and forth the sliding friction opposes its motion and does negative work on it, reducing its total mechanical energy and hence its amplitude. Eventually the motion dies out. This is the case of underdamping, when friction is not sufficient to completely prevent the mass from completing at least one cycle of oscillation.

If friction is large enough that the negative work it does on the object reduces the mechanical energy of the mass-spring system to zero as the mass is brought back to its equilibrium position, i.e.,

$$W_f = -fA = -\mu mgA = \Delta E = -\frac{1}{2}kA^2, \text{ or } \mu = kA/2mg.$$

In such a case the mass would simply stop once it is back at the equilibrium position.

In the case of overdamping  $\mu > \boxed{kA/2mg}$ , and the mass will not pass through the equilibrium position.

71. The weight of a typical sedan is about 3000 lbs, which corresponds to a mass of about 1500 kg. The period of oscillation is around 1 s. So from  $T \approx 2\pi(m/k)^{1/2}$  we get

$$k \approx 4\pi^2 m/T^2 \approx 4\pi^2 (1500 \text{ kg})/(1 \text{ s})^2 \approx \boxed{6 \times 10^4 \text{ kg/s}^2}.$$

The life time  $\tau$  of the oscillation is  $\tau \approx m/b$ , which gives the damping constant  $b$ :

$$b \approx m/\tau \approx 10^4 \text{ kg}/10^0 \text{ s} \approx \boxed{10^3 \text{ kg/s}}.$$

72. (a) We use the initial position to determine the phase constant:

$$x = A \sin(\omega t + \delta); \quad 0 = A \sin(0 + \delta), \text{ which gives } \delta = 0.$$

We use the initial velocity to determine the amplitude:

$$v = dx/dt = A\omega \cos(\omega t); \quad -v_0 = A\omega \cos(0), \text{ which gives}$$

$$A = -v_0/\omega.$$

Thus we have  $x = (-v_0/\omega) \sin(\omega t)$ , where  $\omega = \sqrt{k/m}$ .

- (b) With
- $f_D = -bv$
- , we have

$$x = -(v_0/\omega') e^{-bt/2m} \sin(\omega' t), \text{ where } \omega' = \sqrt{(k/m)^2 - (b/2m)^2}.$$

- (c) With a force
- $f$
- , constant in magnitude, opposing
- $v$
- , the equation of motion is

$$m d^2x/dt^2 = -kx \pm f = -k(x - f/k).$$

This is similar to the vertical motion of a spring in Sec. 13-3. The motion is sinusoidal with  $\omega^2 = k/m$ . The new equilibrium position is shifted by  $-f/k$ , with the sign dependent on the direction of motion. If we pull the mass to  $x = A$  and release, the equilibrium position (where  $F_{\text{net}} = 0$ ) is  $x_{\text{eq}} = f/k$ . The mass starts at  $A - f/k$  from equilibrium, so it will stop at  $x = -(A - f/k) + f/k = -(A - 2f/k)$ . Thus each half-cycle, the amplitude will decrease by  $2f/k$ . Eventually the mass will come to rest at a point where the friction force is large enough to keep it stopped.

73. From the natural period,
- $T_0 = 2.0$
- s, we find
- $\omega_0 = 2\pi/T_0 = 3.14$
- rad/s.

From  $x = A_0 e^{-bt/2m} \sin(\omega' t)$ , we find the ratio of amplitudes:

$$A_1/A_0 = e^{-bt/2m}/e^0; \quad 0.70 = e^{-b(5.0 \text{ s})/2m}, \text{ which gives } b/2m = 0.071 \text{ s}^{-1}.$$

The angular frequency is  $\omega' = \sqrt{\omega_0^2 - (b/2m)^2}$ . With  $b/2m \ll \omega_0$ , we have

$$\omega' \approx \omega_0 [1 - \frac{1}{2}(b/2m\omega_0)^2]. \text{ In terms of the period, we have}$$

$$\begin{aligned} T &= T_0 / [1 - \frac{1}{2}(bT_0/4\pi m)^2] \approx T_0 [1 + \frac{1}{2}(bT_0/4\pi m)^2] \\ &= (2.0 \text{ s}) \{1 + \frac{1}{2}[(0.071 \text{ s}^{-1})(2.0 \text{ s})/2\pi]^2\} \\ &= \boxed{(2.0 + 0.00051) \text{ s}}. \end{aligned}$$

74. For each cycle,
- $E = \frac{1}{2}kA^2$
- , so we have

$$E/E_0 = (A/A_0)^2 = e^{-bt/m}, \text{ or}$$

$$E = \boxed{E_0 e^{-bt/m}}.$$

75. If we write
- $x = Ae^{-bt/2m} \sin(\omega' t + \delta)$
- , the maxima will occur approximately when
- $\sin(\omega' t + \delta) = 1$
- .

From the given data we have

$$6.0 \text{ cm} = Ae^{-b(1.5 \text{ s})/2m} \quad \text{and} \quad 5.6 \text{ cm} = Ae^{-b(2.5 \text{ s})/2m}.$$

We have two equations for  $b/2m$  and  $A$ , which give  $b/2m = 0.0690 \text{ s}^{-1}$  and  $A = 6.65 \text{ cm}$ .

We find the angular frequency from the time between adjacent maxima:

$$\omega' = 2\pi/T' = 2\pi/(2.5 \text{ s} - 1.5 \text{ s}) = 2\pi \text{ rad/s}.$$

Because the first maximum occurs at 1.5 s, we can find  $\delta$  from

$$\sin(\omega' t + \delta) = \sin[(2\pi \text{ rad/s})(1.5 \text{ s}) + \delta] = 1, \text{ or } 3\pi + \delta = \pi/2, \text{ which gives } \delta = -5\pi/2 \text{ rad, or } -\pi/2.$$

The positions are found from  $x = Ae^{-bt/2m} \sin(\omega' t + \delta)$ :

$$x_{3.0} = (6.65 \text{ cm})e^{-(0.0690/\text{s})(3.0 \text{ s})} \sin[(2\pi \text{ rad/s})(3.0 \text{ s}) - \pi/2] = \boxed{-5.40 \text{ cm}}.$$

$$x_{4.8} = (6.65 \text{ cm})e^{-(0.0690/\text{s})(4.8 \text{ s})} \sin[(2\pi \text{ rad/s})(4.8 \text{ s}) - \pi/2] = \boxed{-1.48 \text{ cm}}.$$

$$x_0 = (6.65 \text{ cm})e^{-(0.0690/\text{s})(0 \text{ s})} \sin[(2\pi \text{ rad/s})(0 \text{ s}) - \pi/2] = \boxed{-6.65 \text{ cm}}.$$



76. We need the value of  $b$  from Problem 75. First we find  $\omega_0$ :

$$\omega_0^2 = \omega'^2 + (b/2m)^2 = (2\pi \text{ rad/s})^2 + (0.0690 \text{ s}^{-1})^2, \text{ which gives } \omega_0 \approx 2\pi \text{ rad/s}.$$

From  $\omega_0^2 = k/m$ , we have

$$b/2m = b\omega_0^2/2k; \quad 0.0690 \text{ s}^{-1} = b(2\pi \text{ rad/s})^2/2(12.0 \text{ N/m}), \text{ which gives } b = 0.0419 \text{ kg/s}.$$

We find the critical mass from  $b^2 = 4m_c k$ :

$$m_c = (0.0419 \text{ kg/s})^2/[4(12.0 \text{ N/m})] = \boxed{3.67 \times 10^{-5} \text{ kg}}.$$

$$\text{The lifetime is } \tau = m_c/b = (3.67 \times 10^{-5} \text{ kg})/(0.0419 \text{ kg/s}) = \boxed{8.75 \times 10^{-4} \text{ s}}.$$

$$\text{The } Q \text{ factor is } Q = \sqrt{\frac{k}{m_c}} \tau = \sqrt{\frac{120 \text{ N/m}}{3.67 \times 10^{-5} \text{ kg}}} (8.75 \times 10^{-4} \text{ s}) = 0.50.$$

77. We write the equation of motion, Eq. (13-45), as

$$m(d^2x/dt^2) + b(dx/dt) + kx = 0.$$

We need the derivatives of our trial solution:

$$x = Ae^{-\alpha t} \sin(\omega' t);$$

$$dx/dt = -\alpha Ae^{-\alpha t} \sin(\omega' t) + Ae^{-\alpha t} \omega' \cos(\omega' t)$$

$$d^2x/dt^2 = \alpha^2 Ae^{-\alpha t} \sin(\omega' t) - 2\alpha Ae^{-\alpha t} \omega' \cos(\omega' t) - Ae^{-\alpha t} \omega'^2 \sin(\omega' t).$$

When we substitute these in the equation of motion and collect the sine and cosine terms, we have

$$A[m(\alpha^2 - \omega'^2) - b\alpha + k]e^{-\alpha t} \sin(\omega' t) + A(-m2\alpha\omega' + b\omega')e^{-\alpha t} \cos(\omega' t) = 0.$$

The only way this can be true for all values of  $t$  is for the coefficients of the sine and cosine terms to separately be 0, so we have

$$-m2\alpha\omega' + b\omega' = 0, \text{ or } \alpha = b/2m, \text{ which is Eq. (13-47); and}$$

$$m(\alpha^2 - \omega'^2) - b\alpha + k = 0, \text{ or } \omega'^2 = \alpha^2 - b\alpha/m + k/m.$$

When we use Eq. (13-47), this becomes

$$\omega'^2 = (b/2m)^2 - b^2/2m^2 + k/m, \text{ or } \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}, \text{ which is Eq. (13-48).}$$

78. We find the effective force constant from the equilibrium condition,  $k \Delta y = mg$ :  $k = mg/\Delta y$ .

The maximum amplitude will occur when the tapping is done with the natural frequency. Thus

$$\omega_0 = (k/m)^{1/2} = (g/\Delta y)^{1/2} = [(9.8 \text{ m/s}^2)/(0.006 \text{ m})]^{1/2} = 40.4 \text{ rad/s, so}$$

$$f = \omega_0/2\pi = (40.4 \text{ rad/s})/2\pi = \boxed{6.4 \text{ Hz}}.$$

**79.** We find the effective force constant from the equilibrium condition,  $k \Delta y = mg$ :  $k = mg/\Delta y$ .

Resonance will occur if the driving frequency is the natural frequency:

$$\omega_0 = \sqrt{k/m} = \sqrt{g/\Delta y} = \sqrt{(9.80 \text{ m/s}^2)/(8 \times 10^{-2} \text{ m})} = 11 \text{ rad/s, so}$$

$$f = \omega_0/2\pi = (11 \text{ rad/s})/2\pi = \boxed{1.8 \text{ Hz}}.$$

80. For a system driven with a force of amplitude  $F_0$ , the displacement amplitude is

$$A = F_0/[m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2]^{1/2}.$$

To find the maxima, we set  $dA/d\omega = 0$ :

$$\begin{aligned} dA/d\omega &= (-\frac{1}{2}F_0)[2m^2(\omega^2 - \omega_0^2)(2\omega) + 2b^2\omega]/[m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2]^{3/2} \\ &= -F_0\omega[2m^2(\omega^2 - \omega_0^2) + b^2]/[m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2]^{3/2} = 0. \end{aligned}$$

We set the numerator equal to zero:

$$2m^2(\omega^2 - \omega_0^2) + b^2 = 0, \text{ which gives } \omega = \sqrt{\omega_0^2 - (b^2/2m^2)}.$$

(The other solutions of  $\omega = 0$  and  $\omega = \infty$  are the minima.)

81. (a) The natural frequency of the system is  $\omega_0 = \sqrt{k/m} = \sqrt{(86 \text{ N/m})/(0.548 \text{ kg})} = 12.5 \text{ rad/s}$ .  
 From Problem 80 we have  
 $\omega^2 = \omega_0^2 - (b^2/2m^2) = k/m - b^2/2m^2$ ;  
 $(12.2 \text{ rad/s})^2 = (86 \text{ N/m})/(0.548 \text{ kg}) - b^2/[2(0.548 \text{ kg})^2]$ , which gives  $b = \boxed{2.2 \text{ N}\cdot\text{s/m}}$ .
- (b) The lifetime is  $\tau = m/b = (0.548 \text{ kg})/(2.2 \text{ N}\cdot\text{s/m}) = \boxed{0.25 \text{ s}}$ .
- (c) The sharpness can be expressed as  $\Delta\omega = 2b/m = 2(2.2 \text{ N}\cdot\text{s/m})/(0.548 \text{ kg}) = \boxed{8.0 \text{ rad/s}}$ ,  
 or with the  $Q$  factor:  $Q = \omega_0\tau = (12.5 \text{ rad/s})(0.25 \text{ s}) = \boxed{3.1}$ .
82. (a) The equation of motion for the driven oscillator, after sufficient time, is  
 $x = A \sin(\omega t + \delta)$ , so the velocity is  $v = A\omega \cos(\omega t + \delta)$ .  
 The power generated by a force is  $P = Fv$ . For the damping force, which opposes the velocity,  
 we have  $P = -f_D v = -bv^2 = \boxed{-b(A\omega)^2 \cos^2(\omega t + \delta)}$ .
- (b) The average power loss over a cycle is  $P_{\text{average}} = bA^2\omega^2[\cos^2(\omega t + \delta)]_{\text{average}}$ .  
 Because the cosine function is a sine function shifted by  $90^\circ$ , over a cycle  
 $[\cos^2(\omega t + \delta)]_{\text{average}} = [\sin^2(\omega t + \delta)]_{\text{average}} = 1/2$ .  
 Thus we have  $P_{\text{average}} = \boxed{bA^2\omega^2/2}$ .
83. For small damping the resonant frequency is  $\omega_0 = 2\pi/T = (k/m)^{1/2}$ . From the equilibrium condition,  $k\Delta y = mg$ , we have  
 $\Delta y = mg/k = g/\omega_0^2 = g(T/2\pi)^2 = (9.8 \text{ m/s}^2)[(1.2 \text{ s})/2\pi]^2 = \boxed{0.36 \text{ m}}$ .
84. (a) If we evaluate  $x = (0.35 \text{ m}) \sin(\omega t + \delta)$  at  $t = 0 \text{ s}$ , we get  
 $-0.080 \text{ m} = (0.35 \text{ m}) \sin(0 + \delta)$ , which gives  $\delta = -0.23 \text{ rad}$ , or  $\pi - 0.23 \text{ rad}$ .  
 To choose between these two, we must use  $v = dx/dt = (0.35 \text{ m})\omega \cos(\omega t + \delta)$ . At  $t = 0 \text{ s}$ , we get  
 $-2.1 \text{ m/s} = (0.35 \text{ m})\omega \cos(-0.23)$ , or  $(0.35 \text{ m})\omega \cos(\pi - 0.23)$ .  
 Because  $\omega$  must be positive, we select the cosine that is negative:  $\delta = \pi - 0.23 \text{ rad} = \boxed{2.91 \text{ rad}}$ .
- (b) From  $-2.1 \text{ m/s} = (0.35 \text{ m})\omega \cos(\pi - 0.23)$ , we get  $\omega = 6.16 \text{ rad/s}$ , which gives  $f = \omega/2\pi = \boxed{0.98 \text{ Hz}}$ .
- (c) We find the acceleration from  $a = dv/dt = -(0.35 \text{ m})\omega^2 \sin(\omega t + \delta)$ . At  $t = 0 \text{ s}$ , we get  
 $a = -(0.35 \text{ m})(6.16 \text{ rad/s})^2 \sin(0 + \pi - 0.23 \text{ rad}) = \boxed{-3.0 \text{ m/s}^2}$  ( $-x$ -direction).
- (d) The total energy is  $E = \frac{1}{2}kA^2$ ;  
 $6 \text{ J} = \frac{1}{2}k(0.35 \text{ m})^2$ , which gives  $k = \boxed{98 \text{ N/m}}$ .
- (e) We find the mass  $m$  from  $\omega^2 = k/m$ ;  
 $m = k/\omega^2 = (98 \text{ N/m})/(6.16 \text{ rad/s})^2 = \boxed{2.6 \text{ kg}}$ .
85. If  $\Delta y$  is the stretch of the spring at equilibrium, from the initial condition  $k\Delta y_1 = mg$ , or  $k = mg/\Delta y_1$ .  
 The force must have the same magnitude at any position in the spring, while the stretch increases linearly from the top. For the same hanging mass, when the spring is cut in two, the stretch at equilibrium will be  $\Delta y_2 = \frac{1}{2}\Delta y_1$ . Thus  
 $k_2 = mg/\Delta y_2 = 2(mg/\Delta y_1) = 2k$  and  $\omega_2 = \sqrt{k_2/m} = \sqrt{2k/m} = \sqrt{2} \omega$ .
86. The effective spring constant  $k$  of the foam rubber can be found from  $m_1 g = kx_1$ , where  $m_1 = 100 \text{ g}$  and  $x_1 = 4.0 \text{ cm}$ . So  $k = m_1 g/x_1 = (0.100 \text{ kg})(9.8 \text{ m/s}^2)/(0.040 \text{ m}) = 24.5 \text{ kg/s}^2$ . Suppose the second mass is dropped from a height  $h$  and compresses the mattress by an amount  $x_2$  before it stops. Then the change in its gravitational potential energy is  $\Delta U_g = -m_2 g(h + x_2)$ , while the change in the elastic potential energy of the mattress is  $\Delta U_e = \frac{1}{2}kx_2^2$ . There is no net change in the kinetic energy of the object. Conservation of energy requires that  
 $\Delta E = \Delta U_g + \Delta U_e + \Delta K = -m_2 g(h + x_2) + \frac{1}{2}kx_2^2 = 0$ , which gives  
 $x_2 = m_2 g/k + [(m_2 g)^2 + 2m_2 gkh]^{1/2}/k$  as its positive solution.  
 Plug in  $m_2 = 200 \text{ g} = 0.200 \text{ kg}$ ,  $k = 24.5 \text{ kg/s}^2$ ,  $g = 9.8 \text{ m/s}^2$ , and  $h = 30.0 \text{ cm} = 0.300 \text{ m}$  to obtain  
 $x_2 = 0.31 \text{ m} = \boxed{31 \text{ cm}}$ .

87. The acceleration of gravity on the surface of the Moon is given by

$$g_M = GM_M/R_M^2 = (6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(7.35 \times 10^{22} \text{ kg})/(1738 \times 10^3 \text{ m})^2 = 1.62 \text{ m/s}^2.$$

The period of the pendulum on Earth is  $T_E = 2\pi(L/g_E)^{1/2}$ , while that on the Moon is  $T_M = 2\pi(L/g_M)^{1/2}$ .

Thus  $T_M/T_E = (g_E/g_M)^{1/2}$ , or

$$T_M = T_E (g_E/g_M)^{1/2} = (1 \text{ s})[(9.8 \text{ m/s}^2)/(1.62 \text{ m/s}^2)]^{1/2} = \boxed{2.5 \text{ s}}.$$

88. (a) The natural angular frequency of the spring is  $\omega_0 = \sqrt{k/m}$ . We call the unstretched length  $L$  and the stretched length  $r$ , so the extension is  $\Delta r = r - L$ .

For uniform circular motion, the radial acceleration is  $r\omega^2$  and is provided by the elastic force of the spring. Thus

$$k \Delta r = mr\omega^2, \text{ which gives } \omega = \sqrt{k \Delta r / mr}.$$

Because  $\omega = \frac{2}{3}\omega_0$ , we have

$$\sqrt{k \Delta r / mr} = \frac{2}{3}\sqrt{k/m}, \text{ which gives } \Delta r / r = 4/9.$$

Thus we have  $r = (9/4)\Delta r = L + \Delta r$ , which gives  $\Delta r = 4L/5 = 4(30 \text{ cm})/5 = \boxed{24 \text{ cm}}$ .

- (b) If  $\omega = \alpha\omega_0$ ,  $\Delta r/r = \alpha^2$ , so we have

$$r = \Delta r / \alpha^2 = L + \Delta r, \text{ which gives } \Delta r = \boxed{\alpha^2 L / (1 - \alpha^2)}.$$

Because there must be a tension in the spring to provide the radial force,  $\Delta r$  cannot be negative, which means that  $\alpha < 1$ .

89. We call the length of the unstretched cord  $L$ .

- (a) If we consider the energies, at the bridge there is gravitational potential energy only, with  $h = 0$  at the river level. At the maximum extension at the water surface, there is elastic potential energy only. Calling the maximum extension of the cord  $\Delta L$ , we use energy conservation:

$$K_i + U_{gi} + U_{cordi} = K_f + U_{gf} + U_{cordf};$$

$$0 + mgH + 0 = 0 + 0 + \frac{1}{2}k(\Delta L)^2, \text{ or } mgH = \frac{1}{2}(10 \text{ mg}/H)(\Delta L)^2, \text{ which gives } \Delta L = H/\sqrt{5}.$$

From  $L + \Delta L = H$ , we have  $L = H - \Delta L = H(1 - 1/\sqrt{5}) = \boxed{0.553 H}$ .

- (b) At the final equilibrium position, the stretch is  $\Delta L_0 = mg/k = mg/(10 \text{ mg}/H) = H/10$ .

The final height above the water is  $h = H - L - \Delta L_0 = H(1 - 0.553 - 0.1) = \boxed{0.347 H}$ .

90. (a) We use energy conservation to find the maximum compression, with  $h = 0$  at the initial position:

$$\frac{1}{2}mv_0^2 + 0 + 0 = 0 + mgh + \frac{1}{2}kh^2;$$

$$\frac{1}{2}(0.80 \text{ kg})(1.7 \text{ m/s})^2 = 0 + (0.80 \text{ kg})(9.8 \text{ m/s}^2)h + \frac{1}{2}(3.5 \text{ N/m})h^2.$$

(Note that we let the sign of  $h$  determine the final position.

A negative value represents a downward displacement.)

When we solve this quadratic equation, we get

$$h_1 = -4.6 \text{ m}, \text{ and } h_2 = +0.14 \text{ m}.$$

The negative value corresponds to the compression of the spring:  $\boxed{4.6 \text{ m}}$ .

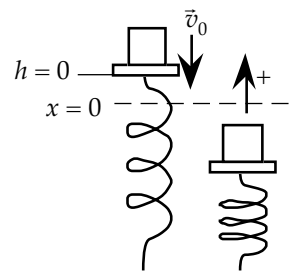
- (b) Consider the simple harmonic motion of the object, described as  $x = A \sin(\omega t + \delta)$ , where  $\omega = (k/m)^{1/2} = [(3.5 \text{ N/m})/0.80 \text{ kg}]^{1/2} = 2.1 \text{ rad/s}$ . Choose up as the positive direction, and place the origin at the equilibrium position. At  $t = 0$  the position of the object is  $x_0 = mg/k = (0.80 \text{ kg})(9.8 \text{ m/s}^2)/(3.5 \text{ N/m}) = 2.24 \text{ m}$ , while its velocity is  $v_0 = -1.7 \text{ m/s}$  (negative since it moves downward). But  $x_0 = A \sin(\omega t + \delta) = A \sin \delta$  and  $v_0 = \omega A \cos(\omega t + \delta) = \omega A \cos \delta$ ; so

$$x_0/v_0 = A \sin \delta / (\omega A \cos \delta) = \tan \delta / \omega; \text{ which gives}$$

$$\delta = \tan^{-1}(\omega x_0/v_0) = \tan^{-1}[(4.1 \text{ rad/s})(2.24 \text{ m})/(-1.7 \text{ m/s})] = 1.754 \text{ rad}.$$

Upon reaching the lowest position the phase of the motion is  $\omega t + \delta = \pi$ , so the time it takes to get there is

$$t = (\pi - \delta)/\omega = (\pi - 1.754 \text{ rad})/(2.1 \text{ rad/s}) = \boxed{0.66 \text{ s}}.$$



91. (a) The rotational inertia of each weight about the wire is

$$mL^2, \text{ where } L^2 = (12.5 \text{ cm})^2 + (5 \text{ cm})^2 = 181 \text{ cm}^2.$$

The center of mass is midway between the weights at  $r = 5 \text{ cm}$ .

We find the period of the physical pendulum from

$$T = 2\pi \sqrt{\frac{I}{2mgr}} = 2\pi \sqrt{\frac{2mL^2}{2mgr}} = 2\pi \sqrt{\frac{L^2}{gr}} = 2\pi \sqrt{\frac{181 \times 10^{-4} \text{ m}^2}{(9.8 \text{ m/s}^2)(5 \times 10^{-2} \text{ m})}} = 1.2 \text{ s}.$$

- (b) Because each spring has the same elongation
- $\Delta y$
- , the equilibrium condition is
- $2k \Delta y = mg$
- .

The effective spring constant is  $k_{\text{eff}} = 2k$ , so we have  $T = 2\pi \sqrt{\frac{m}{2k}} = \pi \sqrt{\frac{2m}{k}}$ .

- (c) Each mass oscillates as if the center of the spring is fixed. If
- $x$
- is the total extension, the extension for each spring is
- $\frac{1}{2}x$
- . The elastic force on each mass is
- $-kx$
- , but the motion of each mass is equivalent to the force creating an extension of
- $\frac{1}{2}x$
- , so we have

$$-kx = -k_{\text{eff}}(\frac{1}{2}x), \text{ which gives } k_{\text{eff}} = 2k.$$

$$\text{Thus } T = 2\pi \sqrt{\frac{m}{2k}} = \pi \sqrt{\frac{2m}{k}}.$$

- (d) This is the common arrangement, where the net force is
- $-ky$
- , with
- $y$
- measured from the equilibrium position. Thus
- $T = 2\pi \sqrt{\frac{m}{k}}$
- .

92. With respect to a coordinate system on the surface, we find the center of mass of the system of cart and pendulum from

$$(M + m)x_{\text{CM}} = Mx_M + mx_m,$$

where  $x_M$  locates the center of mass of the cart. Because the cart rolls freely, the position of the center of mass does not change:

$$(M + m) \Delta x_{\text{CM}} = M \Delta x_M + m \Delta x_m = 0, \text{ so } \Delta x_m = -(M/m) \Delta x_M.$$

The negative sign is an indication that the cart and pendulum move in opposite directions. The amplitude of the pendulum, relative to the cart, is

$$A = L \sin \theta \approx L\theta = (0.65 \text{ m})(10^\circ)(\pi \text{ rad}/180^\circ) = 0.113 \text{ m}.$$

The displacement of the pendulum relative to the cart is

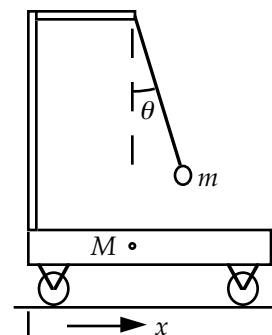
$$x = x_m - x_M, \text{ and changes are}$$

$$\Delta x = \Delta x_m - \Delta x_M = -(M/m) \Delta x_M - \Delta x_M = -(1.0 \text{ kg}/0.28 \text{ kg} + 1) \Delta x_M = 4.57 \Delta x_M.$$

When the pendulum moves through an amplitude, the cart will also, so for the amplitude of the cart's motion we have

$$A_M = A/4.57 = 0.113 \text{ m}/4.57 = 0.025 \text{ m} = \boxed{2.5 \text{ cm}}.$$

The two objects move in opposite directions. The pendulum executes simple harmonic motion and, in order to preserve the position of the center of mass of the system, so must the cart.



93. We find the rotational inertia about the pivot, with the twins as point masses and the bar as two rods:

$$I = 2[\frac{1}{3}(\frac{1}{2}M_{\text{Bar}})(\frac{1}{2}L)^2] + 2M_T(\frac{1}{2}L)^2 = (\frac{1}{3}M_{\text{Bar}} + 2M_T)(\frac{1}{2}L)^2 \\ = [\frac{1}{3}(12 \text{ kg}) + 2(32 \text{ kg})][\frac{1}{2}(5.0 \text{ m})]^2 = 425 \text{ kg} \cdot \text{m}^2.$$

Before oscillating, the center of mass is directly below the pivot at a distance of

$$r = [2M_T(\frac{1}{2}L \sin 7^\circ) + 2(\frac{1}{2}M_{\text{Bar}})(\frac{1}{2})(\frac{1}{2}L \sin 7^\circ)] / (2M_T + M_{\text{Bar}}) \\ = \{2(32 \text{ kg})[\frac{1}{2}(5.0 \text{ m}) \sin 7^\circ] + 2[\frac{1}{2}(12 \text{ kg})][\frac{1}{2}][\frac{1}{2}(5.0 \text{ m}) \sin 7^\circ]\} / [2(32 \text{ kg}) + 12 \text{ kg}] \\ = 0.28 \text{ m}.$$

We find the period of the physical pendulum from

$$T = 2\pi \sqrt{\frac{I}{mgr}} = 2\pi \sqrt{\frac{425 \text{ kg} \cdot \text{m}^2}{[2(32 \text{ kg}) + 12 \text{ kg}](9.8 \text{ m/s}^2)(0.28 \text{ m})}} = 9.0 \text{ s}.$$

94. (a) Because the bucket is filled slowly, we consider the bucket to be in equilibrium, so we have

$$mg = f_1(x/L) + f_2(x/L)^2.$$

Initially, when  $x$  is small, the rope stretches linearly. As the  $x^2$  term becomes more significant for larger  $x$ , the rope will stretch more slowly.

- (b) We find the potential energy from

$$U = -\int F dx = \int (f_1/L)x dx + \int (f_2/L)x^2 dx$$

$$U = \boxed{\frac{1}{2}(f_1/L)x^2 + \frac{1}{3}(f_2/L^2)x^3, \text{ with } U = 0 \text{ at } x = 0}.$$

When set in motion, the initial kinetic energy goes into potential energy.

For small  $v_0$  (small kinetic energy), the amplitude of oscillation will be small. The  $x^2$  term will dominate and the motion will be simple harmonic.

For large  $v_0$  (large kinetic energy), the  $x^3$  term of  $U$  becomes significant. For the same  $|x|$  from the equilibrium position,  $\Delta U$  will be greater below the equilibrium point. Because  $\Delta U = -\Delta K$ , the turning points will not be equidistant from the equilibrium point. The amplitude of the motion will be greater above the equilibrium position than below. Thus the motion will be periodic but not simple harmonic.

95. (a) The equilibrium separation occurs where the potential energy is a minimum, which we find from the derivative of  $U = A/r^2 - e^2/r$ :

$$dU/dr = -2A/r^3 + e^2/r^2 = (-2A + e^2r)/r^3 = 0, \text{ which gives}$$

$$r_0 = \boxed{2A/e^2}.$$

- (b) We express  $U$  in terms of the displacement from equilibrium,  $r_0x$ :

$$U = A/[r_0^2(1+x)^2] - e^2/[r_0(1+x)]. \text{ When we use the expansions for small } x, \text{ we get}$$

$$U = (A/r_0^2)(1 - 2x + 3x^2 - \dots) - (e^2/r_0)(1 - x + x^2 - \dots). \text{ We collect the powers of } x \text{ up to } x^2:$$

$$U = A/r_0^2 - e^2/r_0 + (-2A/r_0^2 + e^2/r_0)x + (3A/r_0^2 - e^2/r_0)x^2.$$

If we use the result of part (a), the coefficient of the  $x$  term is 0 and we get

$$U = -e^4/4A + (e^4/4A)x^2.$$

To find the effective force constant, we express this in terms of the displacement  $r_0x$ :

$$U = -e^4/4A + (e^4/4A)(e^2/2A)^2(r_0x)^2 = -e^4/4A + \frac{1}{2}(e^8/8A^3)(r_0x)^2.$$

This has the form of the elastic potential energy  $\frac{1}{2}k(\Delta r)^2$ , so the motion will be simple harmonic, with  $k = e^8/8A^3$ . The kinetic energy is  $K = \frac{1}{2}\mu v^2$ , with the reduced mass  $\mu = mm/(m+m) = \frac{1}{2}m$ .

The angular frequency is

$$\omega = \sqrt{k/\mu} = \sqrt{(e^8/8A^3)/(m/2)} = e^4/\sqrt{4A^3m}.$$

96. (a) Because the mass  $m$  is attracted to both  $M$ 's, the net force is

$$F = GMm/(\frac{1}{2}L)^2 - GMm/(\frac{1}{2}L)^2 = \boxed{0}.$$

- (b) For the two attractive forces the components parallel to the line between the two  $M$ 's cancel, but the components perpendicular to this line will add.

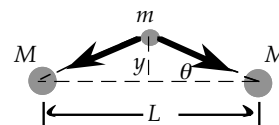
$F_{\text{net}}$  will be toward the original position (opposite to  $y$ ).

- (c) If  $\theta$  is the angle of  $m$  away from the line joining the  $M$ 's, so  $\sin \theta = y/[(\frac{1}{2}L)^2 + y^2]^{1/2}$ , we have

$$F_{\text{net}} = -2GMm/[(\frac{1}{2}L)^2 + y^2](\sin \theta) = -2GMmy/[(\frac{1}{2}L)^2 + y^2]^{3/2}, \text{ which, for } y \ll L, \text{ becomes}$$

$$F_{\text{net}} \approx \boxed{-16GMmy/L^3}.$$

- (d) From the expression for the force, we see that the net force is a restoring force, with  $k = 16GMm/L^3$ . The motion of  $m$  will be simple harmonic, with  $\omega = (k/m)^{1/2} = 4(GM/L^3)^{1/2}$ .



97. We find the rotational inertia about the suspension point from the parallel-axis theorem:

$$I = I_{\text{CM}} + Md^2 = mR^2 + mR^2 = 2mR^2.$$

For the physical pendulum, the period is  $T = 2\pi\sqrt{\frac{I}{MgR}} = 2\pi\sqrt{\frac{2MR^2}{MgR}} = 2\pi\sqrt{\frac{2R}{g}}$ ,

and the frequency is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{2R}}.$$

98. We choose a coordinate system with the origin at the center of the machine gun's motion, with  $x$  toward the target and  $y$  parallel to the target and the motion of the gun.

Because the  $x$ -motion of a bullet has constant speed, the time to reach the target is

$$\Delta t = L/v_{\text{muzzle}} = (180 \text{ m})/(370 \text{ m/s}) = 0.486 \text{ s}.$$

The frequency of firing is much higher than the vibration frequency, so the maximum speed of a bullet in the  $y$ -direction occurs for a bullet that leaves from the center of the vibration:

$$v_{y\text{max}} = A\omega = A2\pi f = (8.50 \times 10^{-2} \text{ m})2\pi(4.0 \text{ Hz}) = 0.68 \pi \text{ m/s}.$$

The distance from the center of the target where this bullet strikes the target is

$$\Delta y = v_{y\text{max}} \Delta t = (0.68 \pi \text{ m/s})(0.486 \text{ s}) = 1.04 \text{ m}.$$

Because this is larger than the amplitude of the vibration, this will be the maximum distance from the center; the impact points will be spread over a range 1.04 m on each side of the target's center. A more detailed analysis would show that they are bunched more at the extremes than they are at the center.

99. Because the cylinder is rolling, the contact point  $A$  is instantaneously at rest, so we can use Newton's law for rotation:  $\sum \tau_A = I_A \alpha$ .

We use the parallel-axis theorem to find the rotational inertia about  $A$ :

$$I_A = I_{\text{CM}} + Md^2 = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2.$$

The only torque is from the restoring spring force with a moment arm of  $R$ . From Newton's second law, we have

$$-kxR = (\frac{3}{2}mR^2)\alpha.$$

The stretch of the spring  $x$  is also the distance rolled,  $x = R\theta$ ,

so we have

$$-kR^2 \theta = \frac{3}{2}mR^2 d^2\theta/dt^2,$$

which corresponds to angular simple harmonic motion.

We see that the  $k_{\text{effective}} = kR^2$ , so

$$\omega = \sqrt{k_{\text{effective}}/I} = \sqrt{kR^2/\frac{3}{2}mR^2} = \sqrt{2k/3m}.$$

The frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{2k/3m}.$$

