UNIVERSITY OF TORONTO
Joseph L. Rotman School of Management

Oct. 25, 2011 **Kan/Simutin**
RSM332 **MID-TERM EXAMINATION** **Yang**

SOLUTIONS

1. Let one period be six months and D be the amount of semi-annual deposits. Kathy will deposit D from $t = 1$ when Larry is five an half years old, till $t = 26$ when Larry is 18 years old. This forms a 26-period annuity with an effective semi-annual rate of $r_s = 0.091/2 = 0.0455$. The future value of this annuity on date $t = 26$ is

$$\begin{aligned} & (1+r_s)^{26} DA_{r_s}^{26} \\ = \; & (1+0.0455)^{26} \frac{D}{0.0455}\left[1-\frac{1}{(1+0.0455)^{26}}\right] \\ = \; & 47.912D. \end{aligned}$$

The education cost forms a 4-year annuity with payment of \$24,000 per year. But now the discount rate is the effective annual interest rate:

$$r_a = \left(1+\frac{0.091}{2}\right)^2 - 1 = 0.09307.$$

The value of this annuity at $t = 26$ is

$$24000A_{r_a}^4 = \frac{24000}{r_a}\left[1-\frac{1}{(1+r_a)^4}\right] = 77232.17.$$

Thus,

$$47.912D = 77232.17 \Rightarrow D = 1611.97.$$

(b) Let P be the amount of annual installment. At the beginning of the 11th year, we still owe 10 installments. Therefore, the principal outstanding is $P \times A_r^{10}$ and the interest proportion of the 11th installment is:

$$rPA_r^{10} = P\left[1-\frac{1}{(1+r)^{10}}\right] = 150.$$

Similarly, at the beginning of 16th year, we still owe 5 installments, and hence the interest portion is

$$rPA_r^5 = P\left[1-\frac{1}{(1+r)^5}\right] = 125.$$

Let $x = 1/(1+r)^5$. Then, we have the following equations:

$$\begin{aligned} P(1-x^2) &= 150, & (1) \\ P(1-x) &= 125. & (2) \end{aligned}$$

Dividing the first equation by the second equation, we obtain

$$1 + x = \frac{150}{125} \Rightarrow x = 0.2 \Rightarrow \frac{1}{1+r} = (0.2)^{\frac{1}{5}}.$$

It follows that $P = 125/(1-x) = 125/(1-0.2) = 156.25$.

At the beginning of 19th year, we will owe 2 installments, and the interest portion of the 19th installment is

$$rPA_r^2 = P\left[1 - \frac{1}{(1+r)^2}\right] = 156.25 \times \left[1 - (0.2)^{\frac{2}{5}}\right] = 74.17.$$

2. (a) Since the yield-to-maturity of a pure discount bond is equal to the spot rate, we have $r_1 = 0.05$ from Bond 1. For bond 2, we have

$$1000 = \frac{80}{1+r_1} + \frac{1080}{(1+r_2)^2} = \frac{80}{1.05} + \frac{1080}{(1+r_2)^2}.$$

Solving this equation, we have $r_2 = 0.0812$. Finally for bond 3, we have

$$750 = \frac{1000}{(1+r_3)^3} \Rightarrow r_3 = \left(\frac{1000}{750}\right)^{\frac{1}{3}} - 1 = 0.1006.$$

(b) Since bond 2 trades at par, we know its yield-to-maturity is the same as its coupon rate, i.e., 8%. For bond 3, it is a pure discount bond, so its yield-to-maturity is the same as the 3-year spot rate, i.e., 10.06%.

(c) The forward rate for the third year is

$$f_3 = \frac{(1+r_3)^3}{(1+r_2)^2} - 1 = 0.1405.$$

(d) Based on the spot rates obtained from bonds 1 to 3, the theoretical price of bond 4 should be

$$P_4 = \frac{100}{1+r_1} + \frac{100}{(1+r_2)^2} + \frac{1100}{(1+r_3)^3} = 1005.78,$$

so the government underprices bond 4. In order to obtain an arbitrage opportunity, we need to buy bond 4 at $900 and sell a replicating portfolio of bonds 1 to 3 that has the same cashflows as bond 4. Let x_1, x_2 and x_3 be the number of units of bonds 1 to

3 in the replicating portfolio. In order for the replicating portfolio to have the same cashflows as bond 4, we need

$$
\begin{aligned}
100 &= 1000x_1 + 80x_2, && (3)\\
100 &= 1080x_2, && (4)\\
1100 &= 1000x_3. && (5)
\end{aligned}
$$

Solving these equations, we obtain $x_1 = 5/54 = 0.0926$, $x_2 = 5/54 = 0.0926$ and $x_3 = 1.1$. The cost of this portfolio is

$$x_1P_1 + x_2P_2 + x_3P_3 = 0.0926 \times \frac{1000}{1.05} + 0.0926 \times 1000 + 1.1 \times 750 = 1005.78.$$

Therefore, by buying bond 4 and selling the replicating portfolio, we obtain an arbitrage profit of \$1005.78 − \$900 = \$105.78.

3. (a) If Jack does not invest, his utility is zero. If he makes investment A, his utility is

$$U_A = (500 - 244)^{1/4}400^{1/2} = 80.$$

If he decides to invest an amount of I in production opportunity B, his consumption at time 0 and 1 will be $C_0 = 500 - I$ and $C_1 = 50\sqrt{I}$ and his utility is

$$U_B(I) = (500 - I)^{\frac{1}{4}}(50I^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{50}[(500 - I)I]^{\frac{1}{4}}.$$

Differentiating U_B with respect to I, we have

$$\frac{\mathrm{d}U_B(I)}{\mathrm{d}I} = \sqrt{50}\frac{1}{4}[(500 - I)I]^{-\frac{3}{4}}(500 - 2I).$$

Setting it to zero, we have $I^* = 250$. Therefore, $U_B(I^*) = \sqrt{50}[(500 - 250)250]^{\frac{1}{4}} = 111.80$. Clearly, he is better off going with investment B.

Alternatively, we can set MRS = MRT to obtain I^*. The MRS of Jack's utility function is given by

$$\text{MRS} = -\frac{\partial U/\partial C_0}{\partial U/\partial C_1} = -\frac{C_1}{2C_0}$$

The MRT of production opportunity B is

$$\text{MRT} = -\frac{\partial F}{\partial I} = -\frac{25}{\sqrt{I}}$$

By setting MRS = MRT and substituting $C_0 = 500 - I$ and $C_1 = 50\sqrt{I}$:

$$-\frac{50\sqrt{I}}{2\,(500 - I)} = -\frac{25}{\sqrt{I}} \Rightarrow I = 500 - I \Rightarrow I^* = 250.$$

(b) Jack will choose the investment with the highest NPV. For investment A, its NPV is

$$\text{NPV}_A = -244 + \frac{400}{1.2} = 89.33.$$

To solve for NPV of investment B, we first calculate optimal investment by setting MRT $= -(1 + r) = -1.2$:

$$-\frac{25}{\sqrt{I}} = -1.2,$$

which gives $I^* = 434.03$, $F = 50\sqrt{I^*} = 1041.67$, and the NPV of investment B is therefore

$$\text{NPV}_B = -434.03 + \frac{1041.67}{1.2} = 434.03.$$

Jack should choose investment B. After he makes this optimal investment, he has a wealth of $500 + 434.03 = 934.03$ and his budget constraint is

$$C_0 + \frac{C_1}{1.2} = 934.03 \Rightarrow C_1 = 1120.84 - 1.2C_0.$$

Plugging this expression C_1 in the utility function, we have

$$U(C_0, C_1) = C_0^{\frac{1}{4}}(1120.84 - 1.2C_0)^{\frac{1}{2}}.$$

Differentiating this with respect to C_0, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}C_0} = \frac{1}{4}C_0^{-\frac{3}{4}}(1120.84 - 1.2C_0)^{\frac{1}{2}} - 1.2 \times \frac{1}{2}C_0^{\frac{1}{4}}(1120.84 - 1.2C_0)^{-\frac{1}{2}}.$$

Setting it equal to zero, we have

$$1120.84 - 1.2C_0 = 2.4C_0 \Rightarrow C_0^* = 311.34$$

and hence $C_1^* = 1120.84 - 1.2C_0^* = 747.22$.

Alternatively, we can obtain C_0^* and C_1^* by setting MRS $= -(1 + r)$, which leads to

$$-\frac{C_1}{2C_0} = -1.2 \Rightarrow C_1 = 2.4C_0.$$

Together with the budget constraint $C_1 = 1120.84 - 1.2C_0$, we obtain $C_0^* = 311.34$ and $C_1^* = 747.22$.

(c) Jack can now pay wages to take on investment A he could not take on in part (b). The present value of the maximum total wages he will be willing to pay is equal to the NPV of investment A. Denoting the maximum wage per period by W, we have:

$$W + \frac{W}{1.20} = 89.33,$$

which gives us the maximum wage per period of $W = 48.73$.

(d) With a perfect capital market, the Fisher Separation Theorem applies, so the maximum amount that Jill is willing to pay is the net present value of investment opportunity B, which is 434.03.

4. (a) The growth rate of dividends is

$$g = b \times \text{ROE} = (1 - 0.4) \times 0.15 = 0.09,$$

where b is the retention ratio. From the Gordon growth model, the discount rate for a telecom company is

$$r = \frac{D_1}{P_0} + g = 0.02 + 0.09 = 0.11.$$

(b) The dividend for year 1 is $D_1 = 1.6 \times 1.15 = 1.84$. The dividend for year 2 is $D_2 = D_1 \times 1.15 = 2.116$. The dividend for year 3 is $D_3 = D_2 \times 1.15 = 2.4334$. Since the dividend will grow at $g_2 = 0.07$ from year 3 onward, the ex-dividend stock price at year 2 is

$$P_2^{ex} = \frac{D_3}{r - g_2} = \frac{2.4334}{0.11 - 0.07} = 60.835.$$

Therefore, the current share price of ABC is

$$P_0 = \frac{D_1}{1+r} + \frac{D_2 + P_2^{ex}}{(1+r)^2} = \frac{1.84}{1.11} + \frac{2.116 + 60.835}{(1.11)^2} = 52.75.$$

(c) The per share net present value of growth opportunities of ABC is given by

$$\text{NPVGO} = P_0 - \frac{E_1}{r} = 52.75 - \frac{2.2}{0.11} = 32.75.$$

(d) If the growth rate of dividend is constant, then we have

$$\begin{aligned}
P_0 &= \frac{D_0(1+g)}{r-g} \\
\Rightarrow \; r - g &= \frac{D_0}{P_0}(1+g) \\
\Rightarrow \; g &= \frac{P_0 r - D_0}{P_0 + D_0} \\
\Rightarrow \; g &= \frac{52.75 \times 0.11 - 1.6}{52.75 + 1.6} = 0.07732.
\end{aligned}$$

5. (a) The price of the 30-year 5% annual coupon bond is

$$P_0 = 50 A_{0.1}^{30} + \frac{1000}{(1.1)^{30}} = 528.65.$$

(b) Let S be the price of the security, we have

$$S = \frac{1}{(1+r)} + \frac{2}{(1+r)^2} + \cdots + \frac{30}{(1+r)^{30}}.$$

Multiplying this equation by $(1+r)$, we have

$$(1+r)S = 1 + \frac{2}{(1+r)} + \frac{3}{(1+r)^2} \cdots + \frac{30}{(1+r)^{29}}.$$

Subtracting the second equation from the first equation, we obtain

$$rS = 1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^{29}} - \frac{30}{(1+r)^{30}} = 1 + A_r^{29} - \frac{30}{(1+r)^{30}}.$$

It follows that

$$S = \frac{1}{r}\left[1 + A_r^{29} - \frac{30}{(1+r)^{30}}\right] = 86.50.$$

(c) The duration of the 30-year 5% annual coupon bond is

$$D = \frac{1}{P_0}\left[\sum_{t=1}^{30} t\frac{50}{(1+r)^t} + 30 \times \frac{1000}{(1+r)^{30}}\right] = \frac{1}{P_0}\left[50S + \frac{30000}{(1+r)^{30}}\right] = 11.43 \text{ years.}$$

6. (a) Since the interest rate is 10% per year compounded semi-annually, the interest rate for a 6-month period is 5%. Imagine the length of one period is 18 months, the interest rate for one period is therefore $(1.05)^3 - 1 = 0.157625$. Standing at six months before today, the cashflows are just a perpetuity, and its present value discounted to six months before today is given by $\$100/0.157625 = \$634.42$. Compounding this amount for six months to today, the present value of the cashflows is $\$634.42(1.05) = \$666.14$.

(b) The cashflows of the growing annuity discounted to year 5 is

$$\frac{\$100}{0.05 - 0.04}\left[1 - \left(\frac{1.04}{1.05}\right)^{25}\right] = \$2{,}127.71.$$

The present value of the deferred growing annuity is therefore $\$2{,}127.71/(1.05)^5 = \$1,667.12$.

(c) The outstanding loan balance at the end of the 12th quarter is given by

$$1{,}000A_{0.03}^{28} = \frac{1{,}000}{0.03}\left[1 - \frac{1}{(1.03)^{28}}\right] = 18{,}764.11.$$

Let n be the number of additional quarters it takes to finish paying off this balance at the interest rate of 3.5%/quarter, we have

$$\begin{aligned}
1000A_{0.035}^{n} &= 18{,}764.11 \\
\Rightarrow \frac{1{,}000}{0.035}\left[1 - \frac{1}{(1.035)^n}\right] &= 18{,}764.11 \\
\Rightarrow 1 - \frac{1}{(1.035)^n} &= 0.6567 \\
\Rightarrow (1.035)^n &= 2.9133 \\
\Rightarrow n &= \ln(2.9133)/\ln(1.035) \\
\Rightarrow n &= 31.08.
\end{aligned}$$

Therefore, it takes altogether 43 quarters for you to payoff the mortgage.

(d) Let T be the number of years it takes to quadruple the money when the continuously compounded interest rate is 10% per year, we have

$$\begin{aligned} & e^{0.1T} = 4 \\ \Rightarrow \quad & 0.1T = \ln(4) \\ \Rightarrow \quad & T = 1.3863/0.1 \\ \Rightarrow \quad & T = 13.863. \end{aligned}$$

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Oct. 23, 2012		**Brean/Kadar/**
RSM332	**MID-TERM EXAMINATION**	**Kan/Yang/Yung**

SOLUTIONS

1. (a) If the individual invests in Project A, then his consumption at $t = 0$ will be $C_0 = Y_0 - I_A = 300 - 50 = 250$ and his consumption at $t = 1$ will be $C_1 = Y_1 + P_A = 200 + 100 = 300$. Therefore, his utility for investing in project A is $U(250, 350) = 250^{\frac{1}{3}} 300^{\frac{2}{3}} = 282.31$.

   If the individual invests in Project B, then his consumption at $t = 0$ will be $C_0 = Y_0 - I_B = 300 - 100 = 200$ and his consumption at $t = 1$ will be $C_1 = Y_1 + P_B = 200 + 180 = 380$. Therefore, his utility for investing in project B is $U(200, 380) = 200^{\frac{1}{3}} 380^{\frac{2}{3}} = 306.81$.

   If the individual invests in Project C, then his consumption at $t = 0$ will be $C_0 = Y_0 - I_C = 300 - 200 = 100$ and his consumption at $t = 1$ will be $C_1 = Y_1 + P_C = 200 + 300 = 500$. Therefore, his utility for investing in project C is $U(100, 500) = 100^{\frac{1}{3}} 500^{\frac{2}{3}} = 292.40$.

   If the individual chooses not to invest, then his consumption at $t = 0$ will be $C_0 = Y_0 = 300$ and his consumption at $t = 1$ will be $C_1 = Y_1 = 200$. Therefore, his utility for investing in project C is $U(300, 200) = 300^{\frac{1}{3}} 200^{\frac{2}{3}} = 228.94$.

   Out of these four possible choices, investing in Project B is the best one, and his optimal consumptions are $C_0^* = 200$ and $C_1^* = 380$.

   (b) Since the individual has only \$40 at $t = 0$, he cannot invest in any of the three projects, so his optimal consumptions are simply $C_0^* = Y_0 = 40$ and $C_1^* = Y_1 = 400$.

   (c) In order to figure out which project is the best, we need to compute their NPVs. The NPVs of the three projects are given by

$$
\begin{aligned}
\text{NPV}_A &= -50 + \frac{100}{1.1} = 40.91, \\
\text{NPV}_B &= -100 + \frac{180}{1.1} = 63.63, \\
\text{NPV}_C &= -200 + \frac{300}{1.1} = 72.73.
\end{aligned}
$$

The optimal investment decision is to pick the project with the highest NPV, i.e., Project C. After we pick the project, we need to solve the optimal consumption problem:

$$\max_{C_0,C_1} U(C_0, C_1)$$
$$\text{s.t.} \quad C_0 + \frac{C_1}{1.1} = Y_0 + \frac{Y_1}{1.1} + \text{NPV}_C = 300 + \frac{200}{1.1} + 72.73 = 554.55.$$

From the budget constraint, we have

$$C_1 = 610 - 1.1C_0.$$

Substituting this into the utility function, we have

$$U(C_0, C_1) = C_0^{\frac{1}{3}} C_1^{\frac{2}{3}} = C_0^{\frac{1}{3}} (610 - 1.1C_0)^{\frac{2}{3}}.$$

Taking derivative with respect to C_0 and setting it equal to zero, we obtain

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}C_0} C_0^{\frac{1}{3}} (610 - 1.1C_0)^{\frac{2}{3}} \\ \Rightarrow \quad 0 &= \frac{1}{3} C_0^{-\frac{2}{3}} (610 - 1.1C_0)^{\frac{2}{3}} + \frac{2}{3} C_0^{\frac{1}{3}} (610 - 1.1C_0)^{-\frac{1}{3}} (-1.1) \\ \Rightarrow \quad 2.2C_0 &= 610 - 1.1C_0 \\ \Rightarrow \quad C_0 &= 184.85. \end{aligned}$$

It follows that the optimal consumption decisions are given by $C_0^* = 184.85$ and $C_1^* = 610 - 1.1C_0^* = 406.67$.

(d) With a capital market, the optimal investment decision is independent of the preference and endowments, so the optimal decision is still to invest in project C. The optimal consumption decisions are obtained by solving

$$\max_{C_0,C_1} U(C_0, C_1)$$
$$\text{s.t.} \quad C_0 + \frac{C_1}{1.1} = Y_0 + \frac{Y_1}{1.1} + \text{NPV}_C = 40 + \frac{400}{1.1} + 72.73 = 476.36.$$

From the budget constraint, we have

$$C_1 = 524 - 1.1C_0.$$

Substituting this into the utility function, we have

$$U(C_0, C_1) = C_0^{\frac{1}{3}} C_1^{\frac{2}{3}} = C_0^{\frac{1}{3}} (524 - 1.1C_0)^{\frac{2}{3}}.$$

Taking derivative with respect to C_0 and setting it equal to zero, we obtain

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}C_0} C_0^{\frac{1}{3}}(524-1.1C_0)^{\frac{2}{3}} \\
\Rightarrow\ 0 &= \frac{1}{3}C_0^{-\frac{2}{3}}(524-1.1C_0)^{\frac{2}{3}} + \frac{2}{3}C_0^{\frac{1}{3}}(524-1.1C_0)^{-\frac{1}{3}}(-1.1) \\
\Rightarrow\ 2.2C_0 &= 524-1.1C_0 \\
\Rightarrow\ C_0 &= 158.79.
\end{aligned}$$

It follows that the optimal consumption decisions are given by $C_0^* = 158.79$ and $C_1^* = 524 - 1.1C_0^* = 349.33$.

2. (a) You need to buy 1-year and 2-year pure discount bonds, each with face value of \$1,000. The price of this bond portfolio is

$$\frac{1000}{1+r_1} + \frac{1000}{(1+r_2)^2} = \frac{1000}{1.08} + \frac{1000}{(1.085)^2} = \$1175.38.$$

(b) Suppose the bond's face value is \$100. For the coupon bond to be selling at par, we must have

$$\begin{aligned}
100 &= \sum_{t=1}^{3} \frac{100c}{(1+r_t)^t} + \frac{100}{(1+r_3)^3} \\
\Rightarrow 100 &= 100c\left[\frac{1}{1.08} + \frac{1}{(1.085)^2} + \frac{1}{(1.09)^3}\right] + \frac{100}{(1.09)^3} \\
\Rightarrow 100 &= 254.76c + 77.22 \\
\Rightarrow c &= 0.0894.
\end{aligned}$$

Therefore, the par bond has a coupon rate of 8.94%.

(c) The forward rate for the fourth year is given by

$$f_4 = \frac{(1+r_4)^4}{(1+r_3)^3} - 1 = \frac{(1.095)^4}{(1.09)^3} - 1 = 0.11.$$

(d) The price of the bond (with a face value of \$100) today is

$$P_0 = \frac{8}{1.08} + \frac{8}{(1.085)^2} + \frac{108}{(1.09)^3} = 97.5989.$$

At the end of the first year, the one-year spot rate is 10% and the 2-year spot rate is 10.5%. Therefore, the price of the 8% coupon bond at the end of the first year is

$$P_1 = \frac{8}{1.1} + \frac{108}{(1.105)^2} = 95.723.$$

The 1-year holding period return is then given by

$$\frac{P_1 + 8}{P_0} - 1 = \frac{95.723 + 8}{97.5989} - 1 = 0.0627.$$

(e) With an increasing term structure of interest rates, we expect the longer term bond (6-year) to have a higher yield-to-maturity than the shorter term bond (2-year). This is because for the 6-year bond, most of its cash flows will come in at the end of the 6th year, so its yield-to-maturity will be close to the 6-year spot rate. For the 2-year bond, most of its cash flows will come in at the end of the 2-year, so its yield-to-maturity will be close to the 2-year spot rate. As the 6-year spot rate is higher than the 2-year spot rate, we expect the yield-to-maturity of the 6-year bond to be higher than the yield-to-maturity of the 2-year bond.

3. (a) The stated annual interest rate is 12%/year, compounded quarterly, so the interest rate for one quarter is 12%/4 = 3% and the interest rate for a six-month period is $r^* = (1 + 0.03)^2 - 1 = 6.09\%$. We first compute the present value of the 5 payments, which is equal to

$$PV = 100 + 100A^4_{r^*} = 100 + \frac{100}{0.0609}\left[1 - \frac{1}{(1.0609)^4}\right] = \$445.80.$$

The future value of these five payments are then given by

$$FV = 445.80(1 + r^*)^{20} = 445.80(1.0609)^{20} = \$1454.21.$$

(b) The effective annual rate of bank A is

$$EAR_A = \left(1 + \frac{0.08}{12}\right)^{12},$$

and the effective annual rate for bank B is

$$EAR_B = \left(1 + \frac{q}{2}\right)^2,$$

For bank B to be competitive with bank A, we need

$$\begin{aligned} EAR_B &\geq EAR_A \\ \Rightarrow \left(1 + \frac{q}{2}\right)^2 &\geq \left(1 + \frac{0.08}{12}\right)^{12} \\ \Rightarrow q &\geq 2\left[\left(1 + \frac{0.08}{12}\right)^{\frac{12}{2}} - 1\right] = 8.13\%. \end{aligned}$$

So, bank B should set a rate at least q as high as 8.13%.

(c) The monthly interest rate is given by

$$r_m = \left(1 + \frac{0.06}{2}\right)^{\frac{1}{6}} - 1 = 0.49386\%.$$

(i) You need to repay the mortgage using $30 \times 12 = 360$ monthly payments. Thus, the monthly mortgage payment is

$$C = \frac{100000}{A_{r_m}^{360}} = \frac{100000}{\frac{1}{r_m}\left[1 - \frac{1}{(1+r_m)^{360}}\right]} = \$594.82.$$

(ii) After the 10th payments, you still owe the bank $360 - 10 = 350$ payments, so the outstanding balance is

$$CA_{r_m}^{350} = \frac{C}{r_m}\left[1 - \frac{1}{(1+r_m)^{350}}\right] = \$98967.65.$$

(iii) For the 11th payment, you need to pay an interest of

$$98967.65 r_m = \$488.76$$

and the remaining is the principal repayment:

$$594.82 - 488.76 = \$106.06.$$

4. (a) From the prices of the three zero-coupon bonds, we can figure out the spot rates as follows:

$$\begin{aligned}
P_A &= 95.24 = \frac{100}{1+r_1} \Rightarrow r_1 = \frac{100}{95.24} - 1 = 5\%, \\
P_B &= 85.73 = \frac{100}{(1+r_2)^2} \Rightarrow r_2 = \left(\frac{100}{85.73}\right)^{\frac{1}{2}} - 1 = 8\%, \\
P_C &= 75.13 = \frac{100}{(1+r_3)^3} \Rightarrow r_3 = \left(\frac{100}{75.13}\right)^{\frac{1}{3}} - 1 = 10\%.
\end{aligned}$$

(b) The theoretical price of Bond D is:

$$P_D = \frac{5}{1+r_1} + \frac{5}{(1+r_2)^2} + \frac{105}{(1+r_3)^3} = 87.935.$$

(c) A year from now, Bond B and Bond C become 1-year and 2-year zeros respectively. Hence, the new spot rates are:

$$\begin{aligned}
P_B &= 94.34 = \frac{100}{1+r_1} \Rightarrow r_1 = 6\%; \\
P_C &= 87.34 = \frac{100}{(1+r_2)^2} \Rightarrow r_2 = 7\%.
\end{aligned}$$

Bond D also becomes a 2-year level coupon, and hence its price is:

$$P_D = \frac{5}{1+r_1} + \frac{105}{(1+r_2)^2} = 96.424.$$

Alternatively, we can easily figure out that a year from now, the discount factors for year 1 and 2 are $DF_1 = 0.9434$ and $DF_2 = 0.8734$, so the price of bond D should be

$$P_D = 5DF_1 + 105DF_2 = 5 \times 0.9434 + 105 \times 0.8734 = 96.424.$$

Let y be the yield to maturity of bond D and $x = 1/(1+y)$. We have:

$$\begin{aligned} P_D &= \frac{5}{1+y} + \frac{105}{(1+y)^2} \\ \Rightarrow \ 0 &= 105x^2 + 5x - 96.424 \\ \Rightarrow \ x &= \frac{-5 + \sqrt{5^2 + 4 \times 105 \times 96.424}}{2 \times 105} \quad \text{(negative root is dropped)} \\ \Rightarrow \ y &= \frac{2 \times 105}{-5 + \sqrt{5^2 + 4 \times 105 \times 96.424}} - 1 = 6.977\%. \end{aligned}$$

(d) The theoretical forward rate for 2014 is given by

$$f_3 = \frac{(1+r_3)^2}{(1+r_2)^2} - 1 = \frac{(1.10)^3}{(1.08)^2} - 1 = 14.11\%.$$

Since the actual forward rate for 2014 is 15% and it is higher than the theoretical one, you should lend using the actual forward rate. In order to eliminate the risk, you need to borrow using a synthetic forward contract, which is constructed by buying a 2-year bond (i.e., Bond B) and selling a 3-year bond (i.e., Bond C). The following table shows the positions and the cashflows of this arbitrage trade:

	Cashflows		
Position	$t = 0$	$t = 2$	$t = 3$
Lend \$10,000 using the forward rate of 15%	0	−10000	11500
Buy a 2-year bond with FV = 10000	−8573	10000	0
Sell a 3-year bond with FV = 11500	8639.95	0	−11500
Total	66.95	0	0

5. (a) Let x, y, and z be the number of units of Bonds A to C in your synthetic portfolio. In order for the synthetic portfolio to replicate a 3-year, zero-coupon bond which pays

off \$2,200 at maturity, we need

$$
\begin{aligned}
0 &= 1000x + 10z, && (1)\\
0 &= 100y + 10z, && (2)\\
2200 &= 110z. && (3)
\end{aligned}
$$

Solving the three equations, we obtain $x = -0.2$, $y = -2$, and $z = 20$. Therefore, we need to short sell 0.2 units of Bond A, short sell 2 units of Bond B, and buy 20 units of Bond C to replicate the 3-year zero coupon bond.

(b) In order to avoid arbitrage opportunities, the price of the 3-year zero-coupon bond should be the same as the cost of the synthetic bond, which is

$$P = -0.2 \times 985.2 - 2 \times 95 + 20 \times 118.21 = \$1977.16.$$

(c) The 3-year spot rate is given by

$$1977.16 = \frac{2200}{(1+r_3)^3} \Rightarrow r_3 = \left(\frac{2200}{1977.16}\right)^{\frac{1}{3}} - 1 = 3.62\%.$$

(d) The theoretical price of Bond F is

$$P_F = \frac{1000}{(1+r_3)^3} = \$898.71.$$

Since the actual price is \$901.45 is higher than the theoretical price, there is an arbitrage opportunity. All we need to do is short sell Bond F, and buy the synthetic bond as we constructed in part (a). The following table provides the cashflows of this trading strategy:

	Cashflows			
Position	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Sell 0.2 units of Bond A	197.04	−200	0	0
Sell 2 units of Bond B	190	0	−200	0
Buy 20 units of Bond C	−2364.20	200	200	2200
Sell 2.2 units of Bond F	1983.19	0	0	−2200
Total	6.03	0	0	0

This gives us an arbitrage profit.

6. (a) The earnings for year 3 is $E_3 = 2(1.08)^2(1.04) = \$2.426$. As the dividend payout rate is 30% of earnings, so dividend for year 3 is $D_3 = 0.3E_3 = \$0.728$ and it will be growing at a rate of 4%/year from year 3 onward. Using the Gordon's formula, the expected price of the stock at year 2 is

$$P_2 = \frac{D_3}{r - g_2} = \frac{0.728}{0.1 - 0.04} = \$12.13,$$

and the price of the stock today is

$$P_0 = \frac{P_2}{(1+r)^2} = \frac{12.13}{(1.1)^2} = \$10.03.$$

(b) Jill's revised estimate of the value of a share of XL uses a cost of equity of 12%/year as opposed to 10%/year. The revised estimate on that basis reduces the price per share from $10.03 to $7.25 because under the new cost of equity, we have

$$P_0 = \frac{D_3}{(r-g_2)(1+r)^2} = \frac{0.728}{(0.12-0.04)(1.12)^2} = \$7.25.$$

Jack keeps the original cost of equity (10%/year). For the estimate of the share price of XL to fall in Jack's analysis, the steady-state (year 3 onward) earnings growth rate, g_2, (estimated to be 4 percent per year in the original analysis) must be lower than 4 percent. Under the new steady state growth rate, the dividend at year 3 will be

$$D_3 = 0.3E_3 = 0.3 \times 2(1.08)^2(1+g_2) = 0.7(1+g_2)$$

For the current price of the stock to be $7.25, the new g_2 must satisfy

$$7.25 = \frac{D_3}{(r-g_2)(1+r)^2} = \frac{0.7(1+g_2)}{(0.1-g_2)(1.1)^2}.$$

Solving for g_2, we obtain $g_2 = 0.01873$ or 1.873%/year. Jack had adjusted XL's steady state growth rate to 1.873%/year.

(c) The effective monthly interest rate is

$$r_m = (1.06)^{\frac{1}{12}} - 1 = 0.487\%.$$

We label June 2013 as $t = 0$ and let s to be the saving rate. The present value of Justin's 60 months of savings (which is a growing annuity) is

$$PV = \frac{4000s}{r_m - g}\left[1 - \left(\frac{1+g}{1+r_m}\right)^{60}\right] = \frac{4000s}{0.00487-0.002}\left[1 - \left(\frac{1.002}{1.00487}\right)^{60}\right] = 219797s.$$

The future value of the saving at the end of five years must be equal to the amount of the down payment, so we have

$$219797s(1+r_m)^{60} = 50000 \Rightarrow s = \frac{50000}{219797s(1.00487)^{60}} = 0.17.$$

Therefore, Justin should save 17% of his salary every month.

(d) The effective quarterly interest rate is

$$r_q = (1.06)^{\frac{1}{4}} - 1 = 1.47\%.$$

Let C be the recurring cash flow of the perpetuity. The present value of the perpetuity is

$$PV = C + \frac{C}{r_q} = 69.15C.$$

The first term in the above equation accounts for the fact that the first cashflow of the perpetuity will arrive immediately at $t = 0$. The future value of the perpetuity at the end of five years must be equal to the amount of the down payment, so we have

$$69.15C(1 + r_q)^{20} = 50000 \Rightarrow C = \frac{50000}{69.15(1.0147)^{20}} = \$540.33.$$

UNIVERSITY OF TORONTO
Joseph L. Rotman School of Management

Oct. 22, 2013 **Brean/Kan/**
RSM332 **MID-TERM EXAMINATION** **Yang/Yung**

SOLUTIONS

1. (a) Since the mortgage interest rate in Canada is a semi-annually compounded rate, the monthly interest rate is given by

$$r_m = \left(1 + \frac{0.04}{2}\right)^{\frac{1}{6}} - 1 = 0.0033059.$$

We plan to repay the mortgage using $25 \times 12 = 300$ monthly payments. Therefore, the monthly mortgage payment is

$$C = \frac{500000}{A_{r_m}^{300}} = \frac{500000 r_m}{1 - \frac{1}{(1+r_m)^{300}}} = 2630.10.$$

(b) At the beginning of month t, you still owe the bank $301 - t$ payments, so the outstanding balance is

$$CA_{r_m}^{301-t} = \frac{C}{r_m}\left[1 - \frac{1}{(1+r_m)^{301-t}}\right].$$

It follows that the interest payment for month t is

$$CA_{r_m}^{301-t} r_m = C\left[1 - \frac{1}{(1+r_m)^{301-t}}\right],$$

and the principal payment for month t is

$$C - CA_{r_m}^{301-t} r_m = \frac{C}{(1+r_m)^{301-t}}.$$

Therefore, the interest payment for the 24th month is

$$C\left[1 - \frac{1}{(1+r_m)^{301-24}}\right] = 1575.88$$

and the principal repayment for the 24th month is

$$\frac{C}{(1+r_m)^{301-24}} = 1054.22.$$

(c) After the 120th monthly payment, you still owe the bank 180 monthly payments, so the outstanding balance is

$$CA_{r_m}^{180} = \frac{C}{r_m}\left[1 - \frac{1}{(1+r_m)^{180}}\right] = 356363.55.$$

(d) From part (c), we know the outstanding balance at the end of 120 months is 356363.55. Therefore, the total amount of principal repayment for the first 120 months is $500000 - 356363.55 = 143636.45$. It follows that the total amount of interest in the first 120 payments is

$$120C - 143636.45 = 171975.67.$$

2. (a) Without a capital market, we have $C_0 = Y_0 - I_0 = 12 - I_0$ and $C_1 = Y_1 + f(I_0) = 15 + 30\sqrt{I_0}$. It follows that

$$U(C_0, C_1) = C_0 C_1 = (12 - I_0)(15 + 30\sqrt{I_0}).$$

Differentiating U with respect to I_0, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}I_0} = -(15 + 30\sqrt{I_0}) + (12 - I_0)\frac{15}{\sqrt{I_0}}.$$

In order to maximize utility, we set

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}I_0} &= 0 \\
\Rightarrow\ (12 - I_0)\frac{15}{\sqrt{I_0}} &= 15 + 30\sqrt{I_0} \\
\Rightarrow\ 45I_0 + 15\sqrt{I_0} - 180 &= 0 \\
\Rightarrow\ \sqrt{I_0} &= \frac{-15 + \sqrt{(15)^2 - 4(45)(-180)}}{2(45)} \\
\Rightarrow\ \sqrt{I_0} &= 1.8403 \\
\Rightarrow\ I_0 &= 3.3866.
\end{aligned}$$

Therefore, the optimal investment in the production is $I_0^* = 3.3866$, and the optimal consumption at $t = 0$ and $t = 1$ are $C_0^* = 12 - I_0^* = 8.6134$ and $C_1^* = 15 + 30\sqrt{I_0^*} = 70.208$, respectively.

(b) With a capital market, we should invest up to the point such that the marginal rate of return from the production function is equal to the interest rate, i.e.,

$$\frac{\mathrm{d}f(I_0)}{\mathrm{d}I_0} = 1.25 \Rightarrow \frac{15}{\sqrt{I_0}} = 1.25 \Rightarrow I_0 = 144.$$

Therefore, the optimal investment in the production is $I_0^* = 144$.

The budget constraint is given by

$$C_0 + \frac{C_1}{1.25} = Y_0 + \frac{Y_1}{1.25} - I_0^* + \frac{30\sqrt{I_0^*}}{1.25} = 12 + \frac{15}{1.25} - 144 + \frac{30(12)}{1.25} = 168.$$

It follows that

$$U(C_0, C_1) = \left(168 - \frac{C_1}{1.25}\right) C_1.$$

Taking derivative with respect to C_1 and setting it equal to zero, we obtain the optimal consumption for $t = 1$ as

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}C_1} &= 0 \\ \Rightarrow \quad 168 - \frac{2C_1}{1.25} &= 0 \\ \Rightarrow \quad C_1^* &= 105. \end{aligned}$$

Therefore, the optimal consumption for $t = 0$ is $C_0^* = 168 - C_1^*/1.25 = 84$. In order to support the consumption and investment at $t = 0$, we need to borrow $C_0^* + I_0^* - Y_0 = 84 + 144 - 12 = 216$ from the capital market today.

(c) Fisher's separation theorem suggests that in a perfect capital market, the investment and consumption decisions can be separated. When proportional taxes are imposed on the output of production, investors will only receive the after-tax payoffs from the production function. This will alter the optimal amount of investment but since the capital market is still perfect, Fisher's separation theorem continues to hold.

3. Let DF_1 and DF_2 be the 1-year and 2-year discount factors. From the prices of the two coupon bonds, we have

$$\begin{aligned} 1039.20 &= 100DF_1 + 1100DF_2, \\ 893.72 &= 20DF_1 + 1020DF_2. \end{aligned}$$

Solving these two equations, we obtain $DF_1 = 0.9612$ and $DF_2 = 0.8573$. It follows that

$$\begin{aligned} \frac{1}{1 + r_1} &= 0.9612 \quad \Rightarrow \quad r_1 = 4.04\%, \\ \frac{1}{(1 + r_2)^2} &= 0.8573 \quad \Rightarrow \quad r_2 = 8\%. \end{aligned}$$

(b) The forward rate for second year is

$$f_2 = \frac{DF_1}{DF_2} - 1 = \frac{0.9612}{0.8573} - 1 = 12.11\%.$$

(c) Let $x = 1/(1+y)$, where y is the yield to maturity of Bond A. We have

$$\begin{aligned}
1039.20 &= 100x + 1100x^2 \\
\Rightarrow \quad x &= \frac{-100 + \sqrt{(100)^2 - 4(1100)(-1039.2)}}{2(1100)} \\
\Rightarrow \quad x &= 0.9276 \\
\Rightarrow \quad y &= 7.81\%.
\end{aligned}$$

(d) The fair price of the annuity is

$$P_0 = 500DF_1 + 500DF_2 = 500 \times 0.9612 + 500 \times 0.8573 = \$909.25.$$

(e) The market price of the annuity is \$800 and it is is less than its theoretical price of \$909.25. Therefore, we can construct a trading strategy to make an arbitrage profit. Specifically, we will buy the annuity and sell a portfolio that replicates the payoffs of the annuity.

Let x_A and x_B denote the units of Bonds A and B in the portfolio. In order to replicate the payoffs of the annuity, we need

$$\begin{aligned}
500 &= 100x_A + 20x_B, \\
500 &= 1100x_A + 1020x_B.
\end{aligned}$$

Solving these two equations, we obtain $x_A = 6.25$ and $x_B = -6.25$. The price of this portfolio is $6.25 \times 1039.20 - 6.25 \times 893.72 = 909.25$ (the theoretical price of the annuity). Therefore, by buying the annuity and selling this portfolio, we make an instant profit of $909.25 - 800 = 109.25$ today and we do not have any future obligations.

4. (a) The spot rates are given by

$$\begin{aligned}
952.38 &= \frac{1000}{1+r_1} \quad \Rightarrow \quad r_1 = 5\%, \\
857.34 &= \frac{1000}{(1+r_2)^2} \quad \Rightarrow \quad r_2 = 8\%, \\
751.31 &= \frac{1000}{(1+r_3)^2} \quad \Rightarrow \quad r_3 = 10\%.
\end{aligned}$$

(b) For zero-coupon bonds, their yields to maturities are just the spot rates for their maturities. Therefore, the yields to maturities of bonds A, B, C are simply 5%, 8%, and 10%, respectively.

(c) The forward rates for years 2 and 3 are

$$
\begin{aligned}
f_2 &= \frac{(1+r_2)^2}{1+r_1} - 1 = \frac{(1.08)^2}{1.05} - 1 = 11.09\%, \\
f_3 &= \frac{(1+r_3)^3}{(1+r_2)^2} - 1 = \frac{(1.10)^3}{(1.08)^2} - 1 = 14.11\%.
\end{aligned}
$$

(d) Since the actual forward rate for year 3 (15%) is different from the theoretical forward rate for year 3 (14.11%), there is an arbitrage opportunity. In order to take advantage of this arbitrage opportunity, you should lend \$1000 using the forward rate of 15%. At the same time, you should use the 2-year and 3-year bond to sell a synthetic forward contract which has a forward rate of 14.11%. The exact transactions and their cashflows are given in the following table:

	$t=0$	$t=2$	$t=3$
Lend \$1000 using the forward contract	0	−1000	1150
Buy 1 unit of 2-year bond	−857.34	1000	0
Sell 857.34/751.31 units of 3-year bond	857.34	0	−1141.11
Total	0	0	8.89

(e) When the price of bond B turns out to be 943.4 at the end of one year, the one-year spot rate will be $1000/(1+{}_1r_2) = 943.4 \Rightarrow {}_1r_2 = 6\%$. When the price of bond B turns out to be 869.57 at the end of one year, the one-year spot rate will be $1000/(1+{}_1r_2) = 869.57 \Rightarrow {}_1r_2 = 15\%$. It follows that the expected one-year spot rate at the end of year 1 is $E[{}_1r_2] = 0.5\times6\%+0.5\times15\% = 10.5\%$. Since $f_2 = 11.09\% > E[{}_1r_2]$, the liquidity preference hypothesis provides a better description of the term structure of interest rates.

5. (a) The interest rate for a 6-month period is

$$
r^* = (1+r_m)^6 - 1 = \left(1+\frac{0.03}{12}\right)^6 - 1 = 0.01509.
$$

The present value of the 20 deposits is

$$
\mathrm{PV} = 100A_{r^*}^{20}(1+r_m)^3.
$$

The future value of the 20 deposits in 20 years is

$$
\mathrm{FV} = 100A_{r^*}^{20}(1+r_m)^3(1+r_m)^{240} = \frac{100}{r^*}\left[1-\frac{1}{(1+r^*)^{20}}\right]\left(1+\frac{0.03}{12}\right)^{243} = 3146.57.
$$

(b) (i) Let C be the amount of annual deposit. The present values of the deposits and the tuitions are given by

$$\begin{aligned}\text{PV(Deposits)} &= CA_{0.1}^{17} = \frac{C}{0.1}\left[1 - \frac{1}{(1.1)^{17}}\right] = 8.02155C,\\ \text{PV(Tuitions)} &= \frac{40000A_{0.1}^{4}}{(1.1)^{17}} = \frac{40000}{0.1}\left[1 - \frac{1}{(1.1)^{4}}\right]\frac{1}{(1.1)^{17}} = 25085.64.\end{aligned}$$

In order for the two present values to be equal, we need $C = \$25085.64/8.02155 = 3127.28$. Therefore, Joel and Julianne need to deposit \$3127.28 each year on the first 17 birthdays of Maria.

(ii) Let C_1 be the amount of the first deposit, the present value of the deposits is given by

$$\text{PV(Deposits)} = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^{17}\right] = \frac{C_1}{0.1-0.04}\left[1 - \left(\frac{1.04}{1.1}\right)^{17}\right] = 10.2436C_1.$$

In order for the present value of the deposits to be the same as the present value of the tuitions, we need $C_1 = 25085.64/10.2436 = 2448.90$. Therefore, the amount of the first deposit should be 2448.90.

(iii) The monthly interest rate is given by

$$r_m = (1.1)^{\frac{1}{12}} - 1 = 0.007974.$$

Let C be the monthly salary of Maria. The future value of the two years' salaries at her 22nd birthday is

$$CA_{r_m}^{24}(1.1)^6 = \frac{C}{r_m}\left[1 - \frac{1}{(1+r_m)^{24}}\right](1.1)^6 = 38.557C.$$

In order for the future value of her salaries to be \$40000, Maria's monthly salary must be equal to $C = 40000/38.557 = 1037.42$.

6. (a) The growth rate earnings (and dividends) is

$$g = (1 - \text{Dividend Payout Ratio}) \times \text{ROE} = (1-0.4)\times 0.15 = 0.09.$$

Using the Gordon's Model, we have

$$P_0 = \frac{D_1}{r-g} \Rightarrow r = \frac{D_1}{P_0} + g \Rightarrow r = 0.01 + 0.09 = 0.1.$$

Therefore, the discount rate is 10%.

(b) The ex-dividend price of Infosys at the end of year 3 is

$$\begin{aligned} P_3^{ex} &= \frac{D_4}{r - g_2} \\ &= \frac{D_0(1+g_1)^4}{r - g_2} \\ &= \frac{1.6(1.15)^4}{0.1 - 0.07} \\ &= 93.28. \end{aligned}$$

It follows that the current price of Infosys is given by

$$P_0 = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{D_3 + P_3^{ex}}{(1+r)^3} = \frac{1.6(1.15)}{1.1} + \frac{1.6(1.15)^2}{(1.1)^2} + \frac{1.6(1.15)^3 + 93.28}{(1.1)^3} = 75.33.$$

(c) The present value per share of Infosys growth opportunities is given by

$$PVGO = P_0 - \frac{E_1}{r} = 75.33 - \frac{4.5}{0.1} = 30.33.$$

(d) The possible reasons why a company has higher leading P/E ratio are

(a) Lower cost of equity capital (due to lower risk)

(b) Higher growth rate

(c) Higher dividend payout ratio

(d) The adoption of an accounting standard that leads to a lower accounting earnings

(e) Investors are overly optimistic about the future of the company

RSM332 Midterm Exam Solution - Spring 2014

1. (a) Since the mortgage interest rate in Canada is a semi-annually compounded rate, the monthly interest rate is given by

$$r_m = \left(1 + \frac{0.06}{2}\right)^{\frac{1}{6}} - 1 = 0.0049386.$$

We plan to repay the mortgage using $25 \times 12 = 300$ monthly payments. Therefore, the monthly mortgage payment is

$$C = \frac{300000}{A_{r_m}^{300}} = \frac{300000 r_m}{1 - \frac{1}{(1+r_m)^{300}}} = 1919.42.$$

(b) After the 50th payment, you still owe the bank 250 payments, so the outstanding balance is

$$CA_{r_m}^{250} = \frac{C}{r_m}\left[1 - \frac{1}{(1+r_m)^{250}}\right] = 275237.51.$$

(c) At the beginning of month t, you still owe the bank $301 - t$ payments, so the outstanding balance is

$$CA_{r_m}^{301-t} = \frac{C}{r_m}\left[1 - \frac{1}{(1+r_m)^{301-t}}\right].$$

It follows that the interest payment for month t is

$$CA_{r_m}^{301-t} r_m = C\left[1 - \frac{1}{(1+r_m)^{301-t}}\right],$$

and the principal payment for month t is

$$C - CA_{r_m}^{301-t} r_m = \frac{C}{(1+r_m)^{301-t}}.$$

Therefore, the interest payment for the 20th month is

$$C\left[1 - \frac{1}{(1+r_m)^{301-20}}\right] = 1438.62$$

and the principal repayment for the 20th month is

$$\frac{C}{(1+r_m)^{301-20}} = 480.80.$$

(d) From part (c), the present value of the principal repayment for month t is

$$\frac{1}{(1+r_m)^t}\frac{C}{(1+r_m)^{301-t}} = \frac{C}{(1+r_m)^{301}} = 435.68,$$

which is independent of t. Therefore, the present value of the principal repayments in the first 50 months is simply $50 \times 435.68 = 21784.08$.

(e) Since the present value of the interest payment for each month is equal to the difference between the present values of the mortgage payments and the principal repayment, the present value of the interest portion of the first 50 payments is

$$CA_{r_m}^{50} - 21784.08 = 63071.18,$$

where the first term is the present value of the first 50 mortgage payments, and the second term is the present value of the principal repayments in the first 50 months.

2. The possible reasons why some firms have higher P/E ratio are (one would get full marks for clearly pointing out any three of them):

(1) Lower cost of equity capital (due to lower risk)

(2) Higher growth rate

(3) Higher dividend payout ratio

(4) Investors are overly optimistic about company's future

(5) Adoption of an accounting standard that leads to a lower accounting earnings

3. We first figure out the effective quarterly interest rate, r_q. Since $(1+r_q)^4 = 1.08$, we have

$$r_q = (1.08)^{\frac{1}{4}} - 1 = 0.0194265.$$

We then convert the quarterly payments into annual payments with the same present value. Let C_1 be the annual payment at the end of year 1 such that it has the same present value as that of the quarterly payments from the beginning of the first quarter for year 1 to the beginning of the fourth quarter for year 1, we have

$$\begin{aligned} \frac{C_1}{1.08} &= 1 \times A_{r_q}^4 (1+r_q) \\ \Rightarrow \quad C_1 &= \frac{1.08 \times (1+r_q)}{r_q} \left[1 - \frac{1}{(1+r_q)^4}\right] \\ \Rightarrow \quad C_1 &= \$4.198076 \text{ (million)}. \end{aligned}$$

Similarly, let C_t be the equivalent annual payment at the end of year t that has the same present value of the quarterly payments for year t. We have $C_t = C_1(1+g)^{t-1}$, where $g = 0.03$. Therefore, C_1 to C_{20} is a growing annuity and its present value is given by

$$\text{PV} = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^{20}\right] = \frac{4.198076}{0.08-0.03}\left[1 - \left(\frac{1.03}{1.08}\right)^{20}\right] = \$51.426607 \text{ (million)}.$$

4. (a) In the absence of the capital market, Jim is facing two choices: either invest in the project, which will give him a utility of

$$(100 - 100) \times (100 + 200) = 0,$$

or simply consume his initial endowment, which will give him a utility of

$$100 \times 100 = 10000.$$

So he should not invest.

In the presence of the capital market: The NPV created by the investment opportunity is

$$\text{NPV} = \frac{200}{1+r} - 100 \ \Rightarrow \ \text{NPV} > 0 \text{ if } r < 1.$$

So he should invest in the project if the capital market's interest rate between time 0 and time 1 is less than 1 (i.e., r is less than 100%).

(b) After making the investment, Jim's endowments become $(100 - 100, 200 + 100) = (0, 300)$, and the budget constraint becomes

$$C_1 = 300 - (1 + 0.2)C_0.$$

The first order condition is MRS $= -(1 + r)$, which gives us

$$\frac{\partial U/\partial C_0}{\partial U/\partial C_1} = \frac{C_1}{C_0} = 1 + 0.2 \ \Rightarrow \ C_1 = 1.2C_0.$$

Combining the above two conditions, we have $C^* = 125$ and $C^* = 150$. Jim has to borrow 125 to finance his consumption at time 0.

Alternatively, we can find out the optimal consumption by substituting the budget constraint in the utility function to obtain

$$U(C_0, C_1) = C_0C_1 = C_0(300 - 1.2C_0) = 300C_0 - 1.2C_0^2.$$

Differentiating U with respect to C_0 and setting it equal to 0, we obtain

$$300 - 2.4C_0^* = 0 \ \Rightarrow C_0^* = \frac{300}{2.4} = 125.$$

It follows that $C_1^* = 300 - 1.2C_0^* = 150$.

(c) In the presence of financial markets, the optimal production decision is independent of consumers' preferences. So from part (a) and given that $r < 1$, Dan should also invest in the project because it has a positive NPV.

5. (a) Let x, y, and z be the number of units of Bonds X, Y, and Z in your synthetic portfolio. In order for the synthetic portfolio to replicate a 3-year, zero-coupon bond which pays off $5,000 at maturity, we need

$$\begin{aligned} 0 &= 100x + 1000z, \\ 0 &= 100x + 1000y, \\ 5000 &= 1100x \end{aligned}$$

Solving the three equations, we obtain

$$x = \frac{50}{11} = 4.5454545, \ \ y = z = -\frac{5}{11} = -0.45454545.$$

Therefore, to replicate the 3-year zero coupon bond, we need to short sell 5/11 units of Bond Z, short sell 5/11 units of Bond Y, and buy 50/11 units of Bond Z.

(b) In order to avoid arbitrage opportunities, the price of the 3-year zero-coupon bond should be the same as the cost of the synthetic bond, which is

$$P = -\frac{5}{11} \times 985 - \frac{5}{11} \times 950 + \frac{50}{11} \times 1180 = \$4484.090909.$$

The 3-year spot rate is given by

$$4484.090909 = \frac{5000}{(1 + r_3)^3}$$

$$\Rightarrow \quad r_3 = \left(\frac{5000}{4484.090909}\right)^{\frac{1}{3}} - 1 = 3.6967627\%.$$

(c) The theoretical price of Bond W is

$$P_W = \frac{1000}{(1 + r_3)^3} = \$896.81818$$

Since the actual price is $880 is lower than the theoretical price, there is an arbitrage opportunity. We buy Bond W, and short-sell the synthetic bond constructed in part (a).

Position	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Buy 5/11 units of Bond Z	−447.73	45.45	0	0
Buy 5/11 units of Bond Y	−431.82	0	45.45	0
Sell 50/11 units of Bond X	5363.64	−45.45	−45.45	−5000
Buy 5 units of Bond W	−4400	0	0	5000
Total	84.09	0	0	0

This trading strategy gives us an arbitrage profit.

6. (a) The growth rate of dividends is

$$\begin{aligned} g &= (1 - \text{Dividend Payout Ratio}) \times \text{ROE} \\ &= (1 - 0.50) \times 0.20 \\ &= 0.10. \end{aligned}$$

where "1 – Dividend Payout Ratio" is equal to the earnings retention ratio.

From the Gordon growth model, the discount rate for a financial company is

$$P_0 = \frac{D_1}{r-g} \quad \Rightarrow \quad r = \frac{D_1}{P_0} + g$$
$$\Rightarrow \quad r = 0.025 + 0.10 = 0.125$$

Therefore, the discount rate is 12.5%.

(b) The dividend for year 1 is $D_1 = 2 \times 1.12$.

The dividend for year 2 is $D_2 = 2 \times 1.12^2$.

The dividend for year 3 is $D_3 = 2 \times 1.12^3$.

Since the dividend will grow at $g_2 = 0.06$ from year 3 onward, the ex-dividend stock price at year 2 is

$$\begin{aligned} P_2^{ex} &= \frac{D_3}{r - g_2} \\ &= \frac{2 \times 1.12^3}{0.125 - 0.06} = 43.2286. \end{aligned}$$

Therefore, the current share price of BAC is

$$\begin{aligned} P_0 &= \frac{D_1}{1+r} + \frac{D_2 + P_2^{ex}}{(1+r)^2} \\ &= \frac{2 \times 1.12}{1.125} + \frac{2 \times 1.12^2 + 43.2286}{(1.125)^2} \\ &= 38.1293. \end{aligned}$$

(c) If the growth rate of dividend is constant, then we have

$$\begin{aligned} P_0 &= \frac{D_0(1+g)}{r-g} \\ \Rightarrow \quad r - g &= \frac{D_0(1+g)}{P_0} \quad \Rightarrow \quad g = \frac{P_0 r - D_0}{P_0 + D_0} \\ \Rightarrow \quad g &= \frac{38.1293 \times 0.125 - 2}{38.1293 + 2} = 0.0689. \end{aligned}$$

7. In practice, a convention is to quote interest rates on an annual basis. But interest may be calculated semi-annually, quarterly, monthly, or even daily. Thus, quoted interest rates and effective interest rates are generally different due to differences in the compounding frequency.

   For example, suppose that the interest rate is quoted 10% per year and the interest is compounded semi-annually. Then in one year \$100 becomes

$$\$100(1 + 0.05)^2 = \$110.25$$

   The *quoted annual interest rate* of 10% implies an interest of \$10, but the actual interest is \$10.25. Thus, the *effective annual interest rate* is 10.25% in this example.

8. (4 marks for each of the two arguments below)

   First, if $P_A < P_B$, we show there exists an arbitrage opportunity. In this case one buys 1 unit of bond A and short-sells 1 unit of bond B. This generates a positive profit now, since $P_A < P_B$. When receiving the face value F from bond A in three years, keep it in cash. Then use it to close out the short position on bond B one year later, i.e., paying the face value F to the buyer of bond B at the end of maturity for bond B. So this shows an arbitrage opportunity - earning a positive profit now without any risk.

   Second, if $P_A = P_B$, we show there also exists an arbitrage opportunity. In this case one buys 1 unit of bond A and short-sells 1 unit of bond B. This generates no profit and no loss now, since $P_A = P_B$. When receiving the face value F from bond A in three years, use all of it to buy one-year zero-coupon bond and let G be the payoff from this investment. Since all spot rates are positive, we have $G > F$ - we can use only part of G to close out the short position on bond B one year later, i.e., paying the face value F to the buyer of bond B at the end of maturity for bond B. So this shows an arbitrage opportunity - earning a positive profit $G - F$ at the end without any risk.