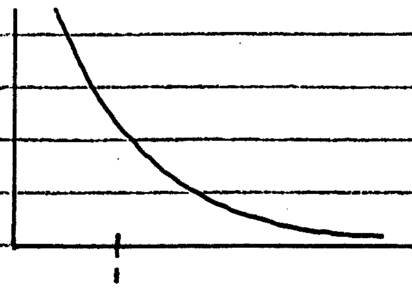
Chapter 5 : Integrals

§5.10 : Improper Integrals

consider $f(x) = 1/x^2$ and let's try to find $\int_1^\infty \frac{1}{x^2}\,dx$



this integral would represent the area under the curve from 1 to ∞

is this finite or not?

we'll look at this another way:

we'll consider $\int_1^t \frac{1}{x^2}\,dx$ and let t get bigger

we know that $\int \frac{1}{x^2}\,dx = \frac{-1}{x}$

so if $t = 100$ $\int_1^t \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{100} = -\frac{1}{100} + 1 = 0.99$

$t = 10\,000$ $\int_1^t \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{10000} = \frac{-1}{10\,000} + 1 = 0.9999$

$t = 1000\,000$ $\int_1^t \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{1000000} = \frac{-1}{1000\,000} + 1 = 0.999999$

it looks like $\int_1^\infty \frac{1}{x^2}\,dx$ should be equal to 1, but how do we verify this properly?

by formalizing what we just did above

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \frac{-1}{x}\Big|_1^t = \lim_{b\to\infty} -\frac{1}{t} + 1 = 1$$

what about $f(x) = 1/\sqrt{x}$?

then $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} 2\sqrt{x}\Big|_1^t = \lim_{t\to\infty} 2\sqrt{t} - 2 \to \infty$

but how does this happen? doesn't the graph of $f(x) = \frac{1}{\sqrt{x}}$

② look like the graph of $f(x) = \frac{1}{x^2}$?
how can one of them have finite area and the other not
REASON: $\frac{1}{x^2}$ goes to zero much faster than $\frac{1}{\sqrt{x}}$

let's look at the "general" problem $\int_1^\infty \frac{1}{x^p}\,dx$ (Ex 4 p 417)

if $p \neq 1$, then $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^t$

$$= \lim_{t\to\infty} \frac{1}{-p+1}\left(t^{-p+1} - 1\right)$$

so if $p > 1$, then $\lim_{t\to\infty} t^{-p+1} = 0$ and so

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$$

if $p < 1$, then $\lim_{t\to\infty} t^{-p+1} \to \infty$, so $\int_1^\infty \frac{1}{x^p}\,dx \to \infty$

if $p = 1$, $\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$
(Ex 1 p 416?)

$$= \lim_{t\to\infty} \ln t - \ln 1 = \infty$$

an integral of the form $\int_a^b f(x)\,dx$ is said to be improper (of type I) if $a = -\infty$ and/or $b = \infty$ (see page 414)

$\int_a^\infty f(x)\,dx$ (and $\int_{-\infty}^b f(x)\,dx$) are said to be convergent if the limits $\lim_{t\to\infty} \int_a^t f(x)\,dx$ (and $\lim_{t\to-\infty} \int_t^b f(x)\,dx$) exist (ie finite)

and divergent if limit does not exist

so we've shown that $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ if $p > 1$
and diverges if $p \leq 1$

what do we do with $\int_{-\infty}^\infty f(x)\,dx$?
use finite number a and split the integral the following way $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$

the only way that $\int_{-\infty}^{\infty} f(x)\,dx$ can be finite is if both $\lim_{t \to -\infty} \int_t^a f(x)\,dx$ and $\lim_{t \to \infty} \int_a^t f(x)\,dx$ converge 3

if either diverges, then $\int_{-\infty}^{\infty} f(x)\,dx$ diverges

see also Examples 2 and 3 on pages ~~425-6~~ 416-7

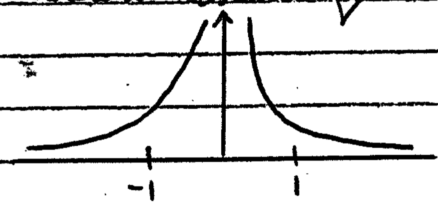
there is a second type of Improper Integral:

what's $\int_{-1}^{1} \frac{1}{x^2}\,dx$?

let's see $\int_{-1}^{1} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{-1}^{1} = -1-1 = -2$ fine?

no, how can the area under a positive curve be negative

so there's something else going on here



remember $\frac{1}{x^2} \to \infty$ as $x \to 0$!

this is the second type of Imp. Int.
(when $f(x)$ is unbounded on $[a,b]$)

so what do we do?

we have to split the integral into 2 pieces and look at each separately

ie $\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$

[now, by the symmetry of the function, $\frac{1}{x^2}$ is even, we only have to look at one of them since

$\int_{-1}^{1} \frac{1}{x^2}\,dx = 2\int_{0}^{1} \frac{1}{x^2}\,dx$ ($f(x)$ even $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$)]

but how do we do $\int_{0}^{1} \frac{1}{x^2}\,dx$?, the same way that we did the previous integrals, we'll look at

$\lim_{t \to 0} \int_{t}^{1} \frac{1}{x^2}\,dx$ but now, we're approaching 0 from the right, so we denote this by

$\lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x^2}\,dx$

④

$$= \lim_{t\to 0^+} \left. \frac{-1}{x} \right|_t^1 = \lim_{t\to 0^+} -1 + \frac{1}{t} = \infty$$

$\therefore$ we can say that $\int_{-1}^{1} \frac{1}{x^2}\,dx$ diverges

what about $\int_0^1 \frac{1}{\sqrt{x}}\,dx$?

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \left. 2\sqrt{x}\right|_t^1 = \lim_{t\to 0^+} 2 - 2\sqrt{t} = 2$$

so $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges

an integral $\int_a^b f(x)\,dx$ where $f(x)$ has a discontinuity (typically a vertical asymptote) somewhere on the interval of integration is called an <u>improper integral</u> (<u>of type II</u>) (p ~~427~~ 418

example : $\displaystyle\int_0^4 \frac{1}{(x-2)^4}\,dx \qquad f(x) = (x-2)^{-4} \to \infty$ as $x \to 2$

so we write $\displaystyle\int_0^4 \frac{1}{(x-2)^4}\,dx = \int_0^2 \frac{1}{(x-2)^4}\,dx + \int_2^4 \frac{1}{(x-2)^4}\,dx$

then $\displaystyle\int_0^2 \frac{1}{(x-2)^4}\,dx = \lim_{t\to 2^-} \int_0^t \frac{1}{(x-2)^4}\,dx$ (approaching 2 from the left)

$$= \lim_{t\to 2^-} \left. -\frac{1}{3}(x-2)^{-3} \right|_0^t$$

$$= \lim_{t\to 2^-} -\frac{1}{3}\left[(t-2)^{-3} - (-2)^{-3}\right]$$

$$= \lim_{t\to 2^-} -\frac{1}{3}\left[\frac{1}{(t-2)^3} + \frac{1}{8}\right] = \infty$$

$\therefore$ $\displaystyle\int_0^4 \frac{1}{(x-2)^4}\,dx$ diverges

see also Examples 5-8 on pages 427-9 418-20

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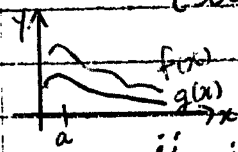
sometimes we would be unable to find the exact value of an improper integral (eg unable to find the antiderivative) but we would still like to know whether the integral is convergent or divergent

we can do this by making comparisons :

the Comparison Test for $\int_a^\infty f(x)\,dx$: (assuming $f(x) \geq 0$) 5

1. guess whether or not the integral converges or diverges by looking at the behavior of $f(x)$ for large x (ie do you think $f(x) \to 0$ fast enough to converge?)
2. confirm the guess by comparison:
   i, if $0 \leq g(x) \leq f(x)$ and $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges

Comparison theorem (p~~439~~ 420) (because functions are positive, $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$ ie the area under $f(x)$ is greater)



   ii, if $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)$ diverges

example: does $\int_2^\infty \frac{dx}{x^4+3x+7}$ converge or not?

as $x \to \infty$, x^4+3x+7 goes like x^4, so $\frac{1}{x^4+3x+7}$ goes to zero rapidly, so we'd expect the integral to converge

now, we need to find the appropriate function for comparison

for $x \geq 2$, $x^4+3x+7 > x^4 > 0$ (ie $3x+7>0$)

so we have $\frac{1}{x^4+3x+7} < \frac{1}{x^4}$

and since $\int_1^\infty \frac{1}{x^4}\,dx$ converges, $\int_2^\infty \frac{1}{x^4}\,dx$ must converge

and so $\int_2^\infty \frac{dx}{x^4+3x+7}$ converges by comparison

example 2: what about $\int_1^\infty \frac{2-\cos x}{x}\,dx$?

See also Ex 9 & 10 p~~430~~ 420-1

always have $-1 \leq \cos x \leq 1$, so suspect the integral diverges, let's see

since $-1 \leq \cos x \leq 1$:

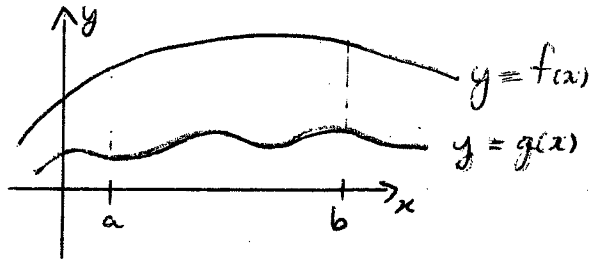
$$\frac{1}{x} \leq \frac{2-\cos x}{x} \leq \frac{3}{x}$$

and we know that $\int_1^\infty \frac{dx}{x}$ diverges, so our integral must diverge as well

Chapter 6: Applications of Integration

§6.1: More About Areas

what is the area of the region that lies between $y = f(x)$ and $y = g(x)$ (where $f(x) \geq g(x)$ in $[a,b]$) between $x = a$ & $x = b$?



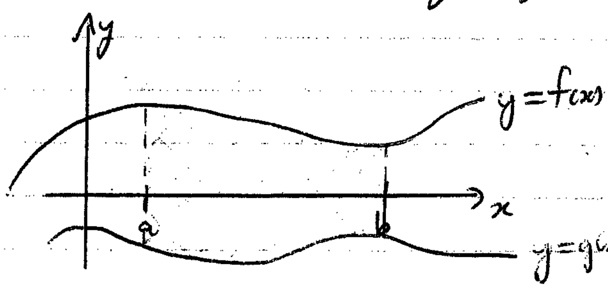
if $f(x) \geq 0$ and $g(x) \geq 0$ for x in $[a,b]$, then it's clear that

this area is the area under $f(x)$ − area under $g(x)$

so the area is $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$= \int_a^b (f(x) - g(x))\,dx$$

but it really doesn't matter that $f(x) \geq 0$ and $g(x) \geq 0$, integration still works the same way regardless of the signs of f and g

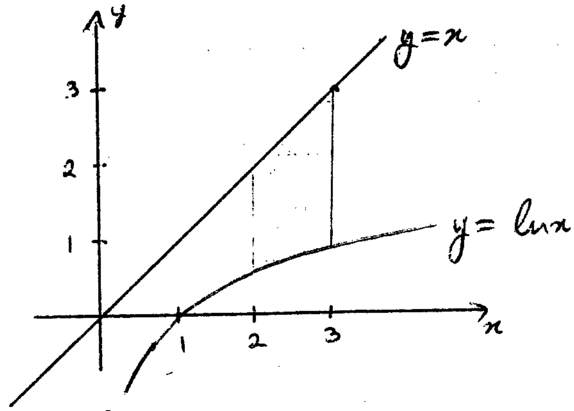


the area is $\int_a^b (f(x) - g(x))\,dx$ because we can move the curves up or down and it doesn't change anything

(p ~~442~~ 432)

examples:

i) find the area of the region bounded by the curves $y = \ln x$, $y = x$ and the lines $x = 2$ and $x = 3$



area $= \int_2^3 (x - \ln x)\,dx$

$= \frac{1}{2}x^2 \Big|_2^3 - \int_2^3 \ln x\,dx$

$= \left(\frac{1}{2}x^2 - (x\ln x - x)\right)\Big|_2^3$

$= \left(\frac{1}{2}(3)^2 - 3\ln 3 + 3\right) - \left(\frac{1}{2}(2)^2 - 2\ln 2 + 2\right)$

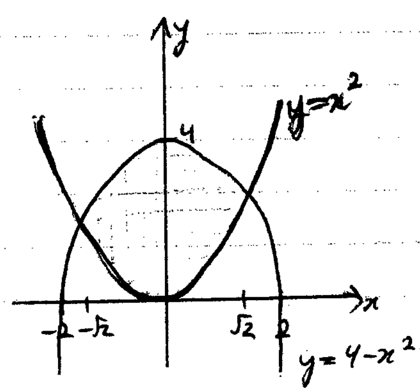
$\simeq 1.59$

$\int \ln x\,dx$ $\quad u = \ln x \quad du = \frac{1}{x}dx$ $\quad dv = dx \quad v = x$

$= x\ln x - \int\left(\frac{1}{x}\right)(x)\,dx$

$= x\ln x - x$

ii, find the area bounded by the curves $y = x^2$ and $y = 4 - x^2$



the parabolas intersect when

$4 - x^2 = x^2$

$4 = 2x^2$

$x^2 = 2$

$x = \pm\sqrt{2}$

so the area is $\int_{-\sqrt{2}}^{\sqrt{2}} \left((4 - x^2) - (x^2)\right)dx = \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2x^2)\,dx$

$= \left(4x - \frac{2}{3}x^3\right)\Big|_{-\sqrt{2}}^{\sqrt{2}}$

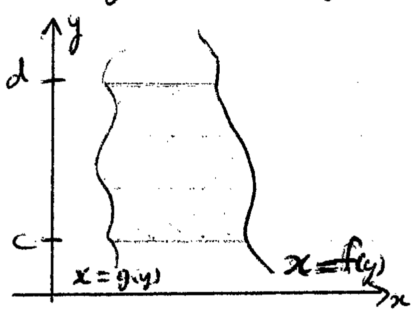
$= \left(4\sqrt{2} - \frac{2}{3}(2\sqrt{2})\right) - \left(-4\sqrt{2} + \frac{2}{3}(2\sqrt{2})\right)$

$= \frac{16}{3}\sqrt{2} \simeq 7.5$

⑧ ie the area of the region is the $\int(\text{top curve} - \text{bottom curve})\,dx$ (p448)

see also examples 1&2 (p 442-3) 433

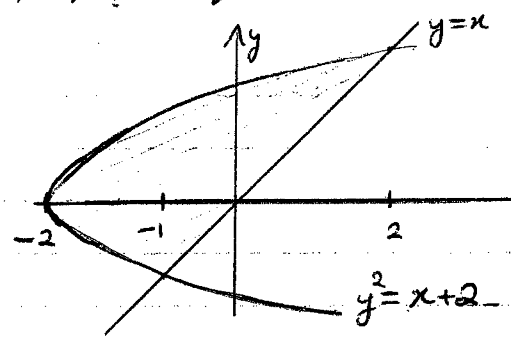
sometimes it's easier to consider x as a function of y
ie the area of the region bounded by $x = f(y)$, $x = g(y)$ between $y = c$ and $y = d$ where $f(y) \geq g(y)$ for all y in $[c,d]$



ie area $= \int_c^d (f(y) - g(y))\,dy$ (p [illegible]) 435

or $\int(\text{right curve} - \text{left curve})\,dy$

example: find the area enclosed by $y = x$ and $y^2 = x+2$



points of intersection of
$x=y$ and $x=y^2-2$

$y = y^2-2$

$y^2-y-2=0$

$(y-2)(y+1)=0$

$y=-1$ or $y=2$

$$\text{area} = \int_{-1}^{2} (y-(y^2-2))\,dy = \int_{-1}^{2}(y-y^2+2)\,dy$$

$$= \left(\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 + 2y\right)\Big|_{-1}^{2}$$

$$= \left(\tfrac{1}{2}(2)^2 - \tfrac{1}{3}(2)^3 + 2(2)\right) - \left(\tfrac{1}{2}(-1)^2 - \tfrac{1}{3}(-1)^3 + 2(-1)\right)$$

$$= 2 - \tfrac{8}{3} + 4 - \tfrac{1}{2} - \tfrac{1}{3} + 2$$

$$= 4\tfrac{1}{2}$$

see also example 5 on p ~~445~~ 435

What if we were given a table of values for the functions?

x	0	2	4	6	8	10	12
$f(x)$	7	8	7	7	9	8	8
$g(x)$	3	5	4	2	3	4	3

then we could use a numerical integration method, like Simpson's Rule, to approximate the area

$f(x)-g(x)$	4	3	3	5	6	4	5

recall Simpson's Rule : (p 422)

$$\int_\alpha^\beta \varphi(x)\,dx \approx \frac{\Delta x}{3}\left[\varphi(x_0)+4\varphi(x_1)+2\varphi(x_2)+4\varphi(x_3)+\cdots+4\varphi(x_{n-1})+\varphi(x_n)\right]$$

$$\Delta x = \frac{\beta-\alpha}{n}$$

so here $\Delta x = 2$, $n=6$, $x_0 = 0, x_1 = 2, \ldots, x_6 = 12$
and the function is $\varphi(x) = f(x)-g(x)$

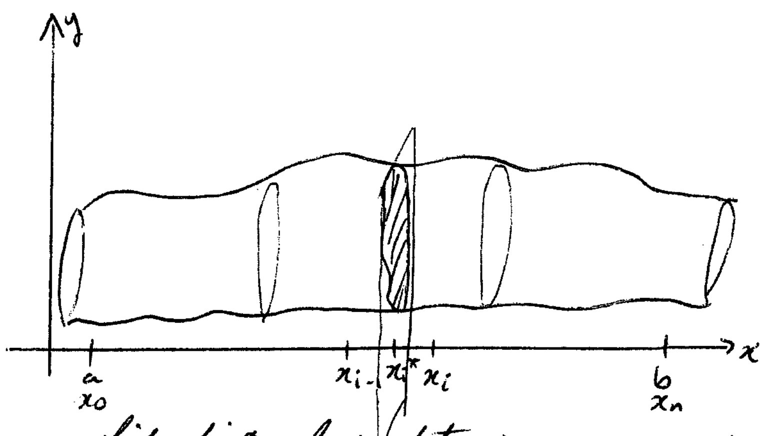
so $\int_0^{12}(f(x)-g(x))\,dx \approx \frac{2}{3}\left[4+4(3)+2(3)+4(5)+2(6)+4(4)+5\right]$

$= \frac{2}{3}(4+12+6+20+12+16+5)$

$= \frac{2}{3}(75) = 50$

see also Ex 4 p ~~449~~ 434-5

§ 6.2 : Volumes



consider a solid object lying between $x=a$ and $x=b$
what is its volume?
chop the interval $[a,b]$ into n pieces of width or thickness $\Delta x = \frac{b-a}{n}$

on each subinterval $[x_{i-1}, x_i]$, take a sample point x_i^*
at x_i^*, look at the cross-sectional area of the solid in

⑩

a plane perpendicular to the x axis, call it $A(x_i^*)$ then the volume of the solid on the subinterval is approximately $A(x_i^*)\Delta x$

so the total volume is approximately $\sum_{i=1}^{n} A(x_i^*)\Delta x$

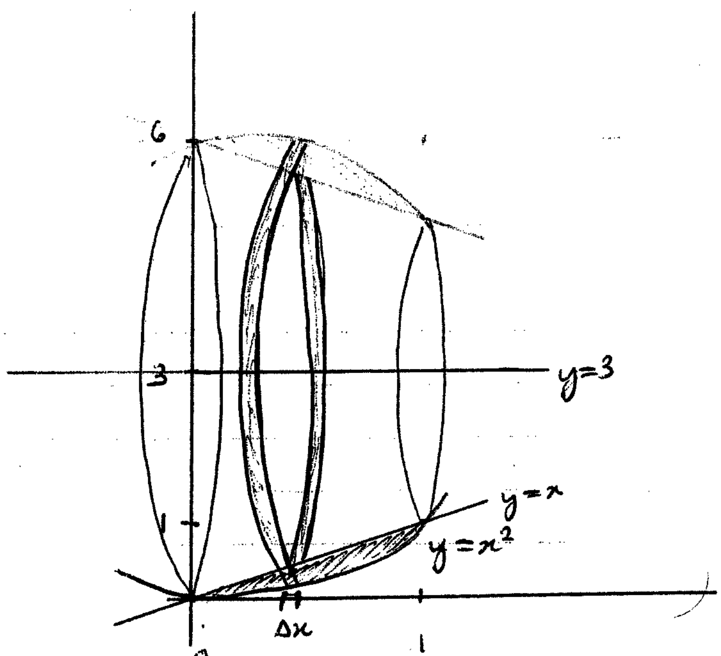
and the approximation gets better as $n \to \infty$ (or $\Delta x \to 0$)

so we have (p ~~449~~) : 439

if S is a solid that lies between $x=a$ and $x=b$ and the cross-sectional area of S in a plane through x and perpendicular to the x axis is $A(x)$, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

examples : i) find the volume of the solid obtained by rotating the region bounded by the curves $y=x$ and $y=x^2$ around the line $y=3$



the cross-sectional areas will be washers or rings

the inner radius is
$r_{in} = 3-x$

and the outer radius is
$r_{out} = 3-x^2$

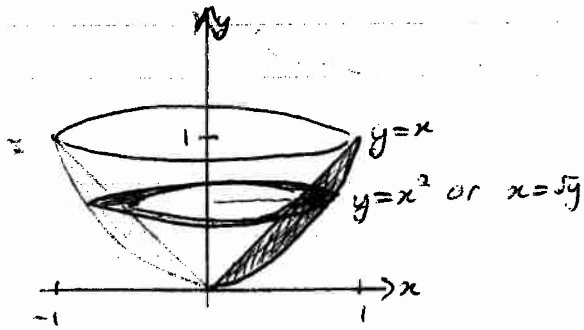
so the area of the washer is
$$\begin{aligned} A(x) &= \pi r_{out}^2 - \pi r_{in}^2 \\ &= \pi(3-x^2)^2 - \pi(3-x)^2 \\ &= \pi(9-6x^2+x^4-(9-6x+x^2)) \\ &= \pi(x^4-7x^2+6x) \end{aligned}$$

so the volume is $V = \int_0^1 \pi(x^4 - 7x^2 + 6x)\,dx$

$$= \pi\left(\tfrac{1}{5}x^5 - \tfrac{7}{3}x^3 + 3x^2\right)\Big|_0^1$$
$$= \pi\left(\tfrac{1}{5} - \tfrac{7}{3} + 3\right) = \tfrac{13}{15}\pi \simeq 2.7227$$

see also examples 1–6 on pages ~~444–454~~ 440–44
(solids of this sort are called solids of revolution) (p~~453~~) 443

ii) what if we had rotated the region around the line $x = 0$ (y axis)?
then we would change our approach slightly and integrate over y

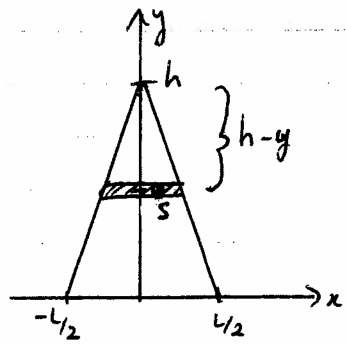


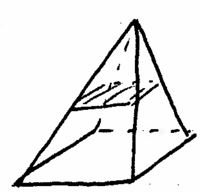
now, cross-sections in planes perpendicular to the y axis are washers
with $r_{in} = y$
and $r_{out} = \sqrt{y}$

so the cross-sectional area is
$$A(y) = \pi r_{out}^2 - \pi r_{in}^2 = \pi(y - y^2)$$

so the volume is $V = \int_0^1 \pi(y - y^2)\,dy = \pi\left(\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right)\Big|_0^1 = \pi/6$
$\simeq 0.5236$

iii) (Ex 8 p~~455~~ 445) find the volume of a pyramid of height h with square base with sides of length L





the cross-section at height y is a square

the lengths of the sides of the square are $2s$ where
$$\frac{s}{h-y} = \frac{L/2}{h} \quad \text{by similar triangles}$$

$$s = \frac{L}{2h}(h-y)$$

so $\quad 2s = \frac{L}{h}(h-y)$ and the square

has area $A = (2s)^2 = \frac{L^2}{h^2}(h-y)^2$

so $\quad V = \int_0^h \frac{L^2}{h^2}(h-y)^2\,dy = \frac{L^2}{h^2}\left[-\frac{1}{3}(h-y)^3\Big|_0^h\right] = \frac{L^2 h}{3}$

see also Ex 7 p ~~454-5~~ 444-5

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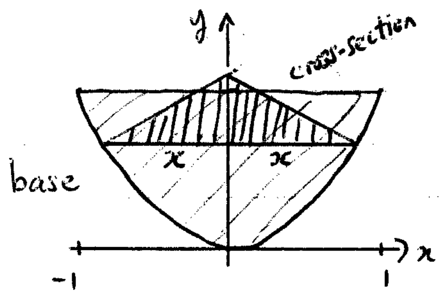
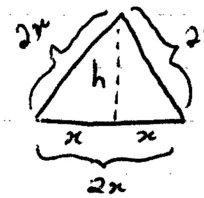
example: (p [illegible]) base is parabolic region $\{(x,y) \mid x^2 \le y \le 1\}$
cross-sections perpendicular to y axis are equilateral triang





but $x = \sqrt{y}$
base $2x = 2\sqrt{y}$
height $h = \sqrt{3}\,x$

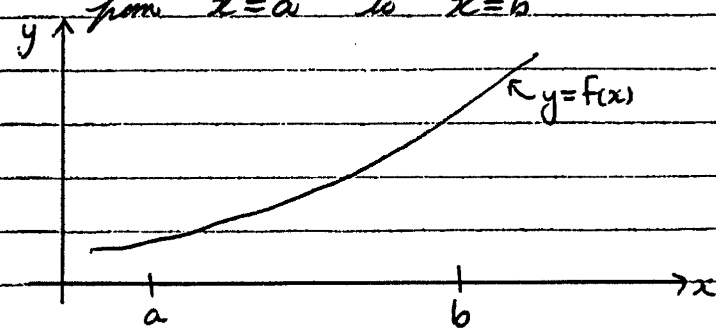
so the area of the cross section is $\quad A(y) = \frac{1}{2}bh = \frac{1}{2}2\sqrt{y}\,\sqrt{3}\sqrt{y} = \sqrt{3}\,y$

and so the volume is $\quad V = \int_0^1 A(y)\,dy = \int_0^1 \sqrt{3}\,y\,dy = \frac{\sqrt{3}}{2}y^2\Big|_0^1 = \sqrt{3}$

(§6.3: Volumes by Cylindrical Shells) A09

A10 **§6.~~3~~4: Arc Length**

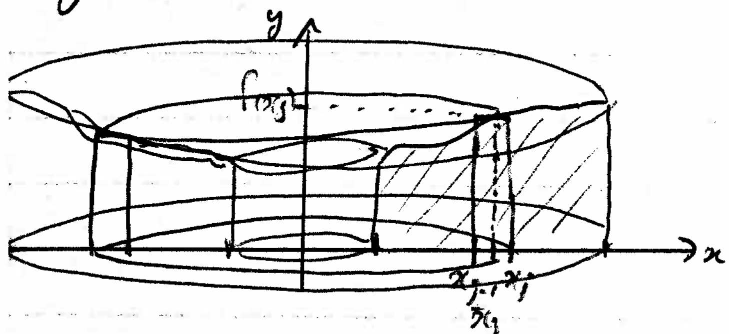
A11 suppose we're interested in the length of the curve $y = f(x)$ from $x = a$ to $x = b$



we'll chop the curve into little pieces and try to estimate the length of each little piece and then sum over the pieces

§ 6.3 : Volumes by Cylindrical Shells

Suppose we have the region bounded by $y = f(x)$, the x-axis (ie $y = 0$) and the lines $x = a$ and $x = b$



and we rotate the region around the y-axis

Chop the interval $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$ and let $\bar{x}_j$ be the midpoint of subinterval $[x_{j-1}, x_j]$

form the rectangle with height $f(\bar{x}_j)$ and rotate it around the y-axis to obtain a cylindrical shell with volume $V_j = 2\pi \bar{x}_j f(\bar{x}_j) \Delta x$

the volume of our solid is then approximately the sum of the volumes of these shells, ie

$$V \approx \sum_{j=1}^{n} V_j = \sum_{j=1}^{n} 2\pi \bar{x}_j f(\bar{x}_j) \Delta x \qquad (p\,450)$$

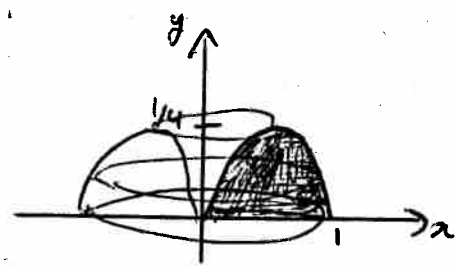
take the limit as $n \to \infty$ (or $\Delta x \to 0$) to get the true value:

$$V = \int_a^b 2\pi x f(x)\, dx \qquad (p\,451)$$

Example: find the volume of the solid obtained by rotating about the y axis the region bounded by $y = x - x^2$ and $y = 0$ (Cf Ex 2 p452)

$(y = x(1-x))$

12B



$$V = \int_0^1 2\pi x\,(x - x^2)\,dx$$

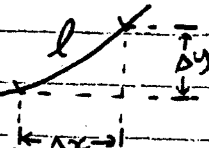
$$V = 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 \Big|_0^1\right)$$

$$= 2\pi\left(\tfrac{1}{3} - \tfrac{1}{4}\right)$$

$$= \frac{2\pi}{12} = \frac{\pi}{6}$$

see also Ex 1-4 p 451-3

when the pieces are small, they will be approximately straight (3)



and there will be the corresponding small changes in x and y, Δx and Δy

since the curve is approximately straight, we can use Pythagoras' theorem to approximate its length $l \approx \sqrt{(\Delta x)^2+(\Delta y)^2}$

now, we need to replace Δy by an expression in x only (because we'll want to integrate over x from a to b)

but recall that $f'(x) \approx \dfrac{\Delta y}{\Delta x}$ so $\Delta y \approx f'(x)\,\Delta x$

so $l \approx \sqrt{(\Delta x)^2+(f'(x)\,\Delta x)^2} = \sqrt{1+(f'(x))^2}\,\Delta x$

now, all we have to do is sum up over all of the little pieces to get the total arc length $L \approx \sum \sqrt{1+(f'(x))^2}\,\Delta x$

now take the limit as $\Delta x \to 0$ to get the formula for the arc length of curve $f(x)$ between $x=a$ and $x=b$:

$$\boxed{\text{arc length} \quad L = \int_a^b \sqrt{1+(f'(x))^2}\,dx}$$ (p463) 457

example: calculate the arc length of $y = 4x^{3/2}$ from $x=1$ to $x=4$

$$f(x) = 4x^{3/2} \Rightarrow f'(x) = 6x^{1/2}$$

so $$L = \int_1^4 \left(1+\left(6x^{1/2}\right)^2\right)^{1/2} dx$$

$$= \int_1^4 (1+36x)^{1/2}\,dx = \left(\frac{2}{3}\right)\left(\frac{1}{36}\right)(1+36x)^{3/2}\Big|_1^4$$

$$= \frac{1}{54}\left((145)^{3/2}-(37)^{3/2}\right)$$

$$\approx 28.17$$

NOTE: even though the formula for arc length is simple, it usually leads to integrals that have to be done numerically (as antiderivatives can't be found easily) (or at all)

See Ex 2 p464 457

if the curve is given as $x = g(y)$, $c \le y \le d$, then a similar derivation will lead to $L = \int_c^d \sqrt{(g'(y))^2+1}\, dy$ (p ~~466~~ 457)

see Ex 3 p ~~464~~ 457-8

another way that a curve can be described is parametrically (§1.7 p ~~74-6~~ 71-6) as $x = f(t)$ and $y = g(t)$ where the parameter t is specified over some interval
see Ex 1 - 4 p ~~74-6~~ 72-3

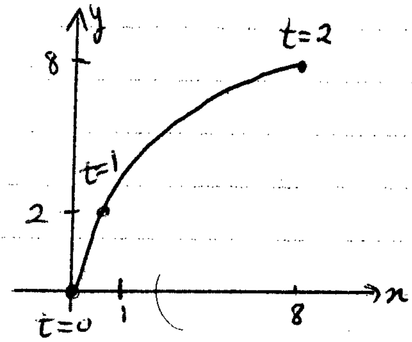
if the curve is specified by the parametric equations $x = f(t)$, $y = g(t)$ for $\alpha \le t \le \beta$

then $\ell \approx \sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t \approx \sqrt{(f'(t))^2+(g'(t))^2}\, \Delta t$

so $L = \int_\alpha^\beta \sqrt{(f'(t))^2+(g'(t))^2}\, dt$ (p ~~463~~ 456)

(can you see how the previous cases are just special cases of this?)

example: find the length of the curve given by $x = t^3$, $y = 2t^2$ from the points $(0,0)$ to $(8,8)$



we'll have $0 \le t \le 2$

arc length $L = \int_0^2 \sqrt{(3t^2)^2+(4t)^2}\, dt = \int_0^2 \sqrt{9t^4+16t^2}\, dt$

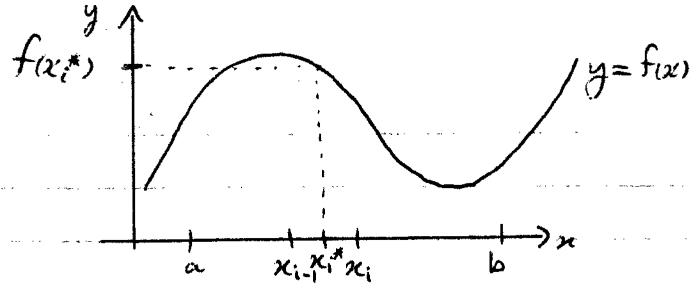
$= \int_0^2 t\sqrt{9t^2+16}\, dt$

$= \left(\frac{2}{3}\right)\left(\frac{1}{18}\right)(9t^2+16)^{3/2}\Big|_0^2 = \frac{1}{27}(9t^2+16)^{3/2}\Big|_0^2$

See also Ex 1 p ~~463~~ 456
and Ex 4 p ~~465~~ 458 $= \frac{1}{27}\left[(36+16)^{3/2} - (16)^{3/2}\right] \simeq 11.518$

§ 6.5: Average Value of a Function

what is the average value of a function $y = f(x)$ over an interval $a \le x \le b$?



chop the interval $[a,b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$

in each subinterval $[x_{i-1}, x_i]$, take a sample point x_i^*, then the average value can be approximated by

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{\sum_{i=1}^{n} f(x_i^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

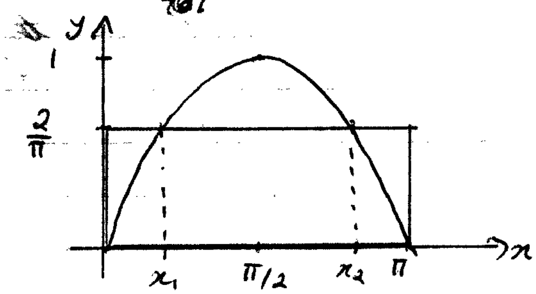
as $n \to \infty$, we'll get the true value, ie

$$f_{ave} = \lim_{\substack{n\to\infty \\ (\Delta x \to 0)}} \frac{1}{b-a} \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x}_{\text{Riemann sum}} = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad (p\ \cancel{468}\ 461)$$

example: find the average value of $f(x) = \sin x$ on $[0,\pi]$

$$f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}\left(-\cos x \Big|_0^{\pi}\right) = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$
$$\simeq 0.6366$$

see Ex 1 p ~~468~~ 461 and 3 p ~~469~~ 462



notice that since
$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$
we'll also have $\int_a^b f(x)\,dx = f_{ave}(b-a)$

ie the area under the curve is equal to the average value times the length of the interval (see p ~~468~~) 462

we could also notice that there are two values of x, say x_1 and x_2

where $f(x_1) = f(x_2) = 2/\pi = f_{ave}$

$$\sin(x_1) = \frac{2}{\pi} \Rightarrow x_1 = \arcsin(2/\pi) \simeq 0.6901 \text{ rad } (\simeq 39.54°)$$

$$\text{and } x_2 = \pi - x_1 \simeq 2.4515 \text{ rad } (\simeq 140.46°)$$

is this a coincidence? NO

<u>the Mean Value Thm for Integrals</u> (p ~~458~~) 462

if f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that $f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\left(\text{or } \int_a^b f(x)\,dx = f(c)\,(b-a) \right)$$

see also Ex 2 p ~~468-9~~ 462

A9 A10 A11

<u>§6.6: Applications to Physics and Engineering</u>

<u>Work</u> (p ~~471-4~~) 464-7

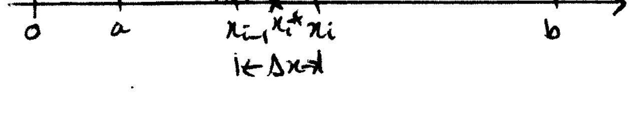
if an object is moved a distance d against a constant force F then the work done is $W = Fd$

but if the force varies with distance, then we have to chop the path of the object up into pieces over which the force is approx constant and sum over the pieces

more specifically, if we're moving the object along the x axis from $x=a$ to $x=b$ and the force is $f(x)$, then chop $[a,b]$ into n subintervals of length Δx, take sample point x_i^* from ith subinterval $[x_{i-1}, x_i]$ and the force is approx $f(x_i^*)$ over that subinterval so the work on the subinterval is

$W_i \simeq f(x_i^*)\,\Delta x$

then total is $W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$ (p ~~472~~ 465)

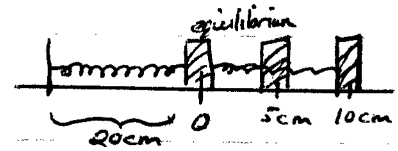


and then take the limit $W = \lim\limits_{\substack{n\to\infty \\ (\Delta x \to 0)}} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$

be careful about the difference between mass m and weight mg (p ~~471~~ 465)
and with the units

see Ex 1 p ~~472~~ 466

example: (p ~~471~~ 472 #6) (Hooke's Law)



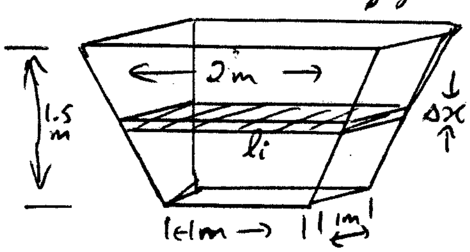
a spring has a natural length 20 cm
if a 25 N force is required to keep it stretched to a length of 30cm, how much work is required to stretch it from 20 to 25 cm?

<u>Hooke's Law</u> (p 472) states that the force required to maintain a spring stretched x units beyond its natural/equilibrium length is $f(x) = kx$ where k is the <u>spring constant</u>

30cm is 10 cm beyond equilibrium, so $x = 10\text{cm} = 0.1\text{m}$
so $25\text{N} = k(0.1\text{m}) \Rightarrow k = 250\text{ N/m}$ and $f(x) = 250x$
and so the work required to stretch $5\text{cm} = 0.05\text{ m}$ beyond equilibrium
is $W = \int_0^{0.05} f(x)\,dx = \int_0^{0.05} 250x\,dx = 125x^2\Big|_0^{0.05} = 0.3125\text{ J}$ (Nm)

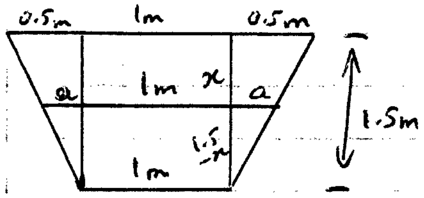
see also Ex 2 p ~~472-3~~ 466

example: a tank is filled with water and has the shape of an isosceles trapezoid (at the ends) and is 1m wide and 1.5m high



how much work is required to pump all of the water out?
($\rho = 1000\text{kg/m}^3$)

the work required to pump the water out is the same as that required to lift all the water to the top
consider a horizontal slice of the water at depth x_i of thickness Δx
the volume of this slice is $V_i = l_i\,\Delta x\,(1)$



$l_i = 1 + 2a$ (m)

by similar triangles $\frac{a}{0.5} = \frac{1.5-x_i}{1.5}$

so $a = 0.5\,\frac{(1.5-x_i)}{1.5}$

so $l_i = 1 + \frac{1.5-x}{1.5}$ and $V_i = \left(\frac{3-x_i}{1.5}\right)\Delta x = \left(2-\frac{2}{3}x_i\right)\Delta x$

$= \frac{1.5+1.5-x}{1.5}$

$= 2 - \frac{x}{1.5}$

$= 2 - \frac{x}{3/2}$

the mass of water in slice is

$m_i = \rho V_i = 1000\left(2-\frac{2}{3}x_i\right)\Delta x$

and then the work done to lift this slice to the top is

($g = 9.8\,m/s^2$) $W_i = \underbrace{m_i g}_{F_i}\, x_i = \left(1000\left(2-\frac{2}{3}x_i\right)\Delta x\right)(9.8)\,x_i$

$= 9800\left(2x_i - \frac{2}{3}x_i^2\right)\Delta x$

so ~~total~~ work done is $W = \int_0^{1.5} 9800\left(2x - \frac{2}{3}x^2\right)dx$

$W = \lim_{n\to\infty}\sum_{i=1}^{n} W_i$

$= 9800\left(x^2 - \frac{2}{9}x^3\right)\Big|_0^{1.5}$

$= 14700$ J

see also Ex 3, 4 p ~~473-4~~ 466-7

<u>Hydrostatic Pressure and Force</u> (p 474-9) 467-9

we'll look at the force exerted by water on dams or containers

the more fundamental concept is pressure, which is the force per unit area exerted by the water

some important things to know about pressure in any fluid, gas or liquid, are:

i, pressure is exerted equally in <u>all</u> directions

ii, pressure increases with depth (which is why the air is thinner at higher altitudes)

(but we'll stick with liquids, typically water) if the liquid has density ρ, the pressure at depth d will be $P = \rho g d$

and since pressure is force/area, $F = P \cdot \text{Area}$

if the pressure is not constant, we'll do our usual trick and divide the surface into pieces over which the pressure is approximately constant

example: ~~p332 #19 (international units)~~ ~~(p339 in...)~~

a lobster tank in a restaurant is 1.5 m long by 1 m wide by 0.7 m deep

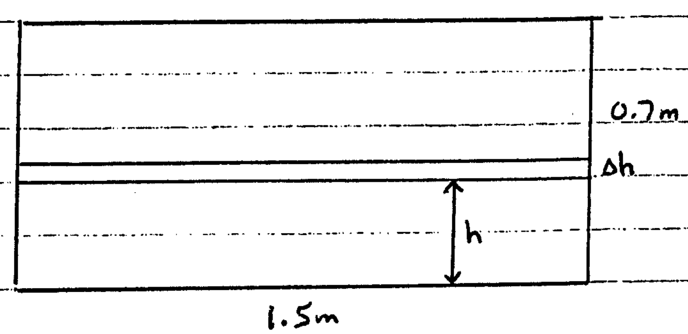
find the water force on the bottom and on each of the four sides ($\rho = 1000\ kg/m^3$)

on the bottom – the force is equal to the weight of all of the water in the tank

so $F = \rho g V = (1000\ kg/m^3)(9.8\ m/s^2)(1.5m)(1m)(0.7m)$

$= 10290\ N$

on the sides:



take a horizontal strip at height h with width Δh

the area of this strip will be $1.5\,\Delta h\ m^2$

the pressure at that height will be

$P = \rho g \cdot \text{depth} = \rho g (0.7 - h)$

so the force on this strip is $F = 1.5 \rho g (0.7-h)\Delta h$

so the total force is $F = \sum 1.5 \rho g (0.7-h)\Delta h$

taking the limit as $\Delta h \to 0$ gives

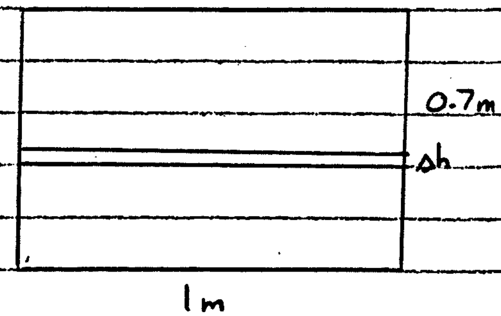
$$F = \int_0^{0.7} (1.5m)(1000\ kg/m^3)(9.8 m/s^2)(0.7-h)\,dh \qquad (N)$$

$$= 14700 \int_0^{0.7} (0.7-h)\,dh$$

$$= 147000 \left[0.7h - \tfrac{1}{2}h^2 \right] \Big|_0^{0.7}$$

$= 14700\left((0.7)(0.7) - \frac{1}{2}(0.7)(0.7)\right) = 3601.5\ N$

on the ends :



now the horizontal strip has area $\Delta h\ m^2$
and the pressure is $P = \rho g(0.7-h)$
so the force is $F = \rho g\,(0.7-h)\,\Delta h$

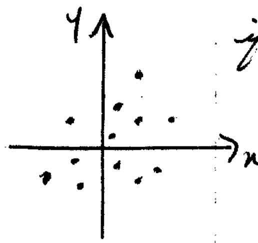
so the integral is $F = \int_0^{0.7} (1000)(9.8)\,(0.7-h)\,dh$

$= 9800\left((0.7)^2 - \frac{1}{2}(0.7)^2\right)$

$= 2401\ N$

see also Ex 5 p ~~475~~ 468

<u>Moments and Centres of Mass</u> (p ~~476-9~~) 469-72



if we have a system of n particles of masses $m_1, m_2, \dots, m_n$ and located at the points $(x_1,y_1), (x_2,y_2), \dots, (x_n,y_n)$, then the <u>centre of mass</u> of the system is located at the point $(\bar{x},\bar{y})$ where $\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$

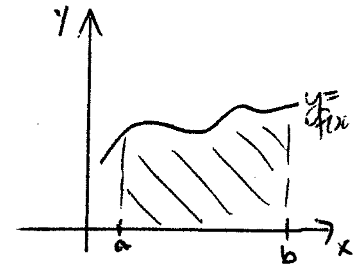
where $m = \sum_{i=1}^{n} m_i$ is the total mass and
$M_y = \sum_{i=1}^{n} m_i x_i$ is the <u>moment about the y axis</u> (tendency to rotate about the y axis)
and $M_x = \sum_{i=1}^{n} m_i y_i$ is the <u>moment about the x axis</u> (rotate around x)

see pages ~~476-7~~ 469-70 and Ex 6 p ~~477~~ 470

now if we have a <u>lamina</u> (a flat object) with constant density ρ that lies beneath the graph of $y = f(x)$ between $x = a$ and $x = b$, then the moments are (p 475)

$M_y = \rho \int_a^b x\,f(x)\,dx$ and $M_x = \rho \int_a^b \frac{1}{2}(f(x))^2\,dx$

using midpoints in approx. (see p ~~475~~ 471)



the mass of the region is $m = \rho A = \rho \int_a^b f(x)\,dx$

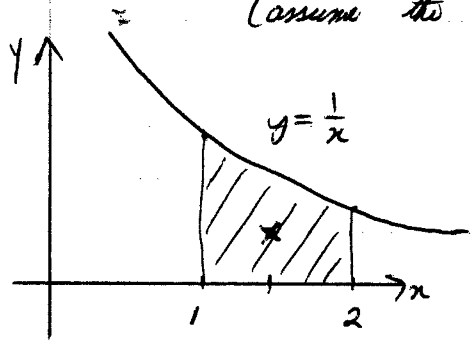
and so the location of the centre of mass or <u>centroid</u> is

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x\, f(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b x\, f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{1}{A}\int_a^b x\, f(x)\,dx$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}(f(x))^2\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}(f(x))^2\,dx}{\int_a^b f(x)\,dx} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\,dx$$

(p ~~472~~ 471)

example: (modified p456 #98) find the moments M_y and M_x and location of the centroid of the region bounded by $y = 1/x$, $y=0$, $x=1$, $x=2$ (assume the density is $\rho = 1$)



$$M_y = \rho \int_a^b x\, f(x)\,dx = (1)\int_1^2 x\left(\frac{1}{x}\right)dx = \int_1^2 dx = 2-1 = 1$$

$$M_x = \rho \int_a^b \frac{1}{2}(f(x))^2\,dx = (1)\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2 dx$$

$$= \frac{1}{2}\int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}\,\frac{-1}{x}\Big|_1^2 = \frac{-1}{2}\left(\frac{1}{2} - 1\right) = 1/4$$

the area is $A = \int_a^b f(x)\,dx = \int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

Mass $m = \rho A = A$

so the centroid is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{1}{(1)\ln 2}, \frac{1/4}{(1)\ln 2}\right) = \left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right)$

$\approx (1.44, 0.36)$

see also Ex 7 p ~~479~~ 472

A09
A11

Chapter 7: Differential Equations

§7.1 : Modelling With Differential Equations

we've been solving differential equations already - when we antidifferentiate to find $\int f(x)dx = F(x) + C$ (where $F'(x) = f(x)$) we've been solving the differential equation :

$$\frac{dy}{dx} = f(x)$$

just to show that this is so :

$$\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)\,dx$$

now integrate on both sides $y = \int dy = \int f(x)\,dx = F(x) + C$

<u>definitions</u> :

a <u>differential equation</u> is an equation involving an <u>unknown</u> function $y(x)$ and its derivatives and the independent variable x

examples : $\frac{dy}{dx} = y + x$, $y' = \frac{1}{2}x + 3$, $y'' + 3y' + y = x$

the <u>order</u> of a differential equation is the order of the highest-order derivative appearing in the equation

so $y' = x + 7$ has order 1

but $y'' + y = 0$ has order 2

the <u>solution</u> to a differential equation is the <u>function</u> $y(x)$

example : the differential equation $y' = x + 7$ has solution $y(x) = \frac{1}{2}x^2 + 7x + C$

to show that this is so, just substitute :

if $y = \frac{1}{2}x^2 + 7x + C$, then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x^2 + 7x + C\right)$

$$= x + 7$$

note that this solution contains an <u>arbitrary constant</u> C

where did it come from ?

from the antidifferentiation :

if $\frac{dy}{dx} = x+7$

then $y = \int dy = \int (x+7)\,dx = \frac{1}{2}x^2 + 7x + C$

what does this really mean ?

it means that there are infinitely many solutions to the diff. eq. $y' = x+7$ – one for each value of C

for this reason, $y = \frac{1}{2}x^2 + 7x + C$ is called the general solution because it represents the whole family of solutions – ie the whole family of curves that have slope $x+7$

in order to know the value of C, we need to have an initial condition $y(x_0) = y_0$

(essentially, this gives us a point that the curve must pass through – and only one member of the family will do this)

so say we had been given $y' = x+7$, $y(0) = 3$

(a diff. eq. together with an initial condition is called an initial-value problem)

we know the general solution is $y(x) = \frac{1}{2}x^2 + 7x + C$

but we need to satisfy $y(0) = 3$,

so $y(0) = \frac{1}{2}(0)^2 + 7(0) + C = 3 \Rightarrow C = 3$

so the solution that satisfies the initial condition is

$y(x) = \frac{1}{2}x^2 + 7x + 3$

and this is called a particular or unique solution

(it is the unique curve that has slope $x+7$ and passes through $(0,3)$)

where do differential equations come from ?

often when we are trying to model physical phenomena, we do not know the function that describes the system's behavior, but we can observe the behavior of its rate

see the discussion and examples on p ~~499-503~~ 494-499

§7.2: Direction Fields and Euler's Method

we'll look at a graphical method of solving first-order differential equations

idea: if we're trying to solve $y' = f(x,y)$, then we know that at the point (x,y), the slope of the solution curve is $f(x,y)$ – ie we know it

so we can draw a ~~(slope)~~ direction field which is a graph of the general behavior of y'

at each point (x,y), calculate $y' = f(x,y)$ and draw a small line segment that indicates this slope

do this for many points to produce the slope field

(show example)

see the examples on p. ~~505–508~~. p ~~510~~ 500–3

Euler's Method

this is a numerical method that approximates the solution to the differential equation

essentially, what the slope field did "graphically", Euler's Method will do "numerically"

how it works: say we have differential equation $y' = f(x,y)$ and initial condition $y(x_0) = y_0$

from what we've just done with slope fields, we know that y' gives us a direction to head in

so take small step (Δx) h, the corresponding change in y is $\Delta y = f(x_0, y_0)(\Delta x)$ h (since $f(x,y) = y' \approx \Delta y/\Delta x$)

so our new y value, call it y_1, is

$$y_1 = y_0 + \Delta y = y_0 + f(x_0, y_0)(\Delta x)\ h$$

and this the value that corresponds with $x_1 = x_0 + (\Delta x)$ h

at our new point, (x_1, y_1), we have slope $f(x_1, y_1)$

so if we take another step of (Δx) h $\Delta y = f(x_1, y_1)(\Delta x)$, and so $y_2 = y_1 + f(x_1, y_1)(\Delta x)$

you keep doing this for as many steps as required / desired and you generate a set of points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

where y_j is a numerical approximation of the <u>true</u> or <u>exact</u> value $y(x_j)$

the general formula for Euler's Method is :

$$\boxed{y_{n+1} = y_n + f(x_n, y_n)\underbrace{(\Delta x)}_{h}}$$

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the <u>error</u> of Euler's Method is the difference between the calculated approximation and the exact value

ie $\quad$ error $= y(x_j) - y_j$

and if the number of steps used is n, the error will be approximately proportional to $1/n$

reason : the larger n is, the smaller $\overset{h}{(\Delta x)}$ will be for a given problem

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example : use ten steps of Euler's Method to approximate the solution of $y' = \frac{1}{x}$, $y(1) = 0$ with step size $h = 0.1$

$$y(1) = 0 \quad \Rightarrow \quad x_0 = 1, \; y_0 = 0$$

then $x_1 = 1.1$, $x_2 = 1.2$, etc ...

$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + \frac{0.1}{x_n}$$

so

$$y_1 = y_0 + \frac{0.1}{x_0} = 0 + \frac{0.1}{1} = 0.1$$

$$y_2 = y_1 + \frac{0.1}{x_1} = 0.1 + \frac{0.1}{1.1} = 0.190909$$

$$y_3 = y_2 + \frac{0.1}{x_2} = 0.190909 + \frac{0.1}{1.2} = 0.274242$$

$$y_4 = y_3 + \frac{0.1}{x_3} = 0.274242 + \frac{0.1}{1.3} = 0.351165$$

$$y_5 = y_4 + \frac{0.1}{x_4} = 0.351165 + \frac{0.1}{1.4} = 0.422594$$

$$y_6 = y_5 + \frac{0.1}{x_5} = 0.422594 + \frac{0.1}{1.5} = 0.489261$$

$$y_7 = y_6 + \frac{0.1}{x_6} = 0.489261 + \frac{0.1}{1.6} = 0.551761$$

$$y_8 = y_7 + \frac{0.1}{x_7} = 0.551761 + \frac{0.1}{1.7} = 0.610585$$

$$y_9 = y_8 + \frac{0.1}{x_8} = 0.610585 + \frac{0.1}{1.8} = 0.666141$$

$$y_{10} = y_9 + \frac{0.1}{x_9} = 0.666141 + \frac{0.1}{1.9} = 0.718773$$

exercise - compare with the true solution $y = \ln x$

see the examples on pages ~~503-~~ 510 504-5

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§7.3 : Separable Equations

(p ~~513~~ 508)

a DE is called <u>separable</u> if it can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

because we can then separate the variables as $g(y)\,dy = f(x)\,dx$
(y on one side, x on the other)

to solve, all we need to do is integrate on both sides

$$\int g(y)\,dy = \int f(x)\,dx + C$$

if $G(y)$ and $F(x)$ are antiderivatives of $g(y)$ and $f(x)$, we'll have $\quad G(y) = F(x) + C$ (only need one constant)

examples :

i, $\frac{dP}{dt} = kP$ or $\frac{dP}{P} = k\,dt$ so $\int \frac{dP}{P} = \int k\,dt + C$

so $\ln|P| = kt + C$ or $P(t) = e^{kt+C} = e^C e^{kt} = Q e^{kt}$ P_0

ii, $\frac{dy}{dx} = x y^2 \cos(x^2)$ $\quad y(0) = 1$

separate to get $\quad \frac{dy}{y^2} = x\cos(x^2)\,dx$

$$\int \frac{dy}{y^2} = \int x \cos(x^2)\,dx + C$$

$$-\frac{1}{y} = \frac{1}{2}\sin(x^2) + C$$

SEE ALSO
EX 1–4
p 504–6
509–11

so the general solution is $y = \dfrac{-2}{\sin(x^2) + K}$

$y(0) = 1 \Rightarrow K = -2$ so unique solution is $y = \dfrac{-2}{\sin(x^2) - 2} = \dfrac{2}{2 - \sin(x^2)}$

Example: the velocity of a falling body ~~[illegible]~~

think of an object falling from a great height

we know that gravity will act on the object, causing it to accelerate

the force of gravity on the object is mg (we'll take downwards as positive – then velocity will be positive)

but air resistance will also act on the object, causing it to slow down

like all frictional forces, air resistance, at least to a good approximation, is proportional to velocity (ie the faster the object goes, the more strongly the resistance)

so the force of air resistance will be of the form $-kv$ for proportionality constant $k > 0$ (recall that upwards is negative)

so the total force on the body is $F =$ gravity + air resistance $= mg - kv$

by Newton's 2nd Law $F = ma$, but $a = \dfrac{dv}{dt}$, so we have $mg - kv = m\dfrac{dv}{dt}$ (a differential equation in velocity v)

rewrite as $\dfrac{dv}{dt} = g - \dfrac{k}{m}v = -\dfrac{k}{m}\left(v - \dfrac{m}{k}g\right)$

separate the variables $\dfrac{dv}{v - mg/k} = -\dfrac{k}{m}\,dt$

integrate $\displaystyle\int \frac{dv}{v - mg/k} = \int -\frac{k}{m}\,dt$

to get $\ln|v - mg/k| = -\dfrac{k}{m}t + C$

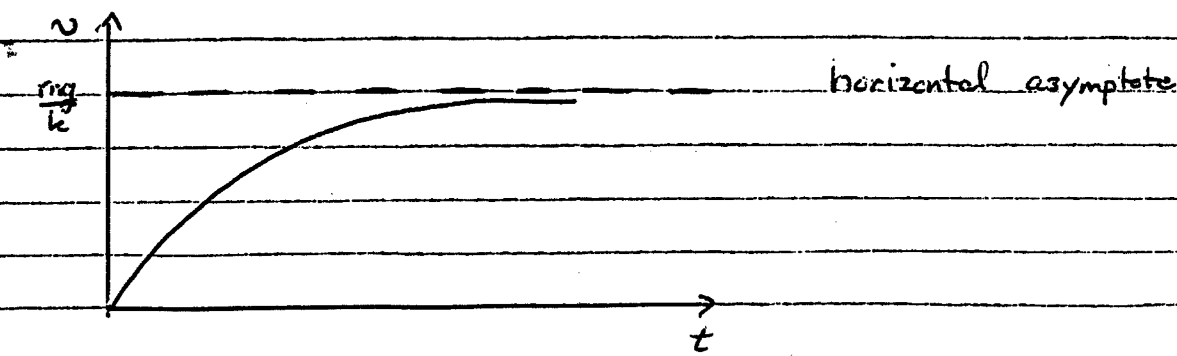
exponentiate both sides to get $|v - mg/k| = e^{-kt/m + C}$
$= e^C e^{-kt/m}$
$= K e^{-kt/m}$

which is $v - mg/k = A e^{-kt/m}$ (general solution)
A is determined by the initial conditions
we'll assume that $v(0) = 0$, ie the object starts at rest at time 0, then

$$0 - mg/k = A e^0 \Rightarrow A = -mg/k$$

so our particular solution is $v - mg/k = -mg/k \, e^{-kt/m}$
or $v = mg/k \left(1 - e^{-kt/m}\right)$

notice that, as $t \to \infty$, $v \to mg/k$
the solution looks like:



so this means that there is a maximum velocity that the object can attain, $v = mg/k$ (though in this model it is never actually reached)
mg/k is called the <u>terminal velocity</u>
how does it come about?
recall that $F = mg - kv$ — we can have the situation where gravity has accelerated the object to such a velocity v that air resistance will balance gravity and the total force is zero, ie $F = mg - kv = 0 \Rightarrow v = mg/k$
but if $F = 0$, $\frac{dv}{dt} = 0$ and the object's velocity is no longer changing
so, in conclusion, if the object falls from a point high enough (so that it falls for a long enough time), it will accelerate up to its terminal velocity mg/k and then fall at that constant speed

Example: salt concentration (~~p524-526~~) (~~[illegible]~~)

suppose there is a reservoir holding 100 million litres of water (ie 100 Ml) that supplies a city with 1 million litres per day

the reservoir is fed by two sources – a spring which provides 0.9 million litres per day and run-off from the surrounding land which provides 0.1 million litres per day

the spring is clean, but the run-off water contains salt with a concentration of 0.0001 kg per litre (it's winter time and salt is being used on the roads)

assume the following: i, there is no salt in the reservoir initially

ii, the reservoir is well-mixed

iii, the reservoir stays full

find the concentration of salt in the reservoir as a function of time

first of all, concentration is quantity / volume but we'll work with the quantity Q

at all times, we have salt entering through the run-off water and leaving through consumption by the city, so

rate of change of Q = rate salt entering − rate salt leaving

so we need to work out these two rates on the RHS:

rate salt entering:

this is easy because it's constant

rate salt entering = concentration of salt in run-off water × volume of run-off water entering reservoir per day

$= 0.0001 \text{ kg}/\text{l} \times 100\,000 \text{ l}/\text{day}$

$= 10 \text{ kg}/\text{day}$

rate salt leaving:

the city uses water of concentration $Q/100$ Ml, so

rate salt leaving = concentration of salt in reservoir × volume of water used by city per day

$$= \frac{Q \text{ kg}}{100 \text{ Ml}} \times 1 \text{ Ml}/\text{day}$$

$$= \frac{Q}{100} \text{ kg}/\text{day}$$

∴ the differential equation for Q is :

$$\frac{dQ}{dt} = \text{rate of change of } Q$$
$$= \text{rate entering} - \text{rate leaving}$$
$$= 10 - \frac{Q}{100}$$

so $\frac{dQ}{dt} = 10 - \frac{Q}{100} = \frac{-1}{100}(Q - 1000) = -0.01(Q-1000)$

separate variables $\frac{dQ}{Q-1000} = -0.01\,dt$

integrate $\int \frac{dQ}{Q-1000} = \int -0.01\,dt$

to get $\ln|Q-1000| = -0.01t + C$

so the general solution is $Q - 1000 = Ae^{-0.01t}$

and we know that $Q(0) = 0$, so

$0 - 1000 = Ae^0 \Rightarrow A = -1000$

so the particular solution is $Q = 1000(1 - e^{-0.01t})$ (kg)

and, therefore, the concentration is :

$$C = \frac{Q}{100\,Ml} = \frac{1000(1-e^{-0.01t})\,kg}{100 \times 10^6\,l} = 10^{-5}(1-e^{-0.01t})\ kg/l$$

again, our solution will have a horizontal asymptote

see also Ex 6 p. ~~518~~ 513

p. 506

§ 7.4: Exponential Growth and Decay

the solution of the DE $\frac{dy}{dt} = ky$ is $y = Ae^{kt}$

(we have done $\frac{dP}{dt} = kP$) , where $A = y(0)$

if $k > 0$, this represents exponential growth (p ~~524~~ 520)

if $k < 0$, this represents exponential decay

for population P (p 525-8) 520, we'll have DE $\frac{dP}{dt} = kP$

then $P(t) = P_0 e^{kt}$ $(P_0 = P(0))$

note that we can write $\frac{1}{P}\frac{dP}{dt} = k$

for this reason, we call k the <u>relative growth rate</u> (p ~~525~~ 520)

example: if a population grows at a rate of 3% per year,

i) $k = 0.03$ and $P(t) = P_0 e^{0.03t}$

if the initial population is $P(0) = P_0 = 1000$,

we'll have $P(t) = 1000 e^{0.03t}$

what is the population 10 years later?

$$P(10) = 1000 e^{0.03(10)} = 1000 e^{0.3} \simeq 1350$$

ii) if $\frac{dP}{dt} = kP$, $P(0) = 250$ and $P(5) = 400$

then
$$P(t) = P(0) e^{kt} = 250 e^{kt}$$
$$P(5) = 250 e^{5k} = 400$$
$$e^{5k} = \frac{400}{250} = 1.6$$

so $5k = \ln(1.6) = 0.47$

so $k = \frac{1}{5}(0.47) = 0.094$

$\therefore P(t) = 250 e^{0.094t}$

see also examples 1 and 2 on pages ~~525-8~~ 521-3

what if we are talking about a radioactive element? (p ~~522-4~~)

then the mass decays exponentially (p ~~523~~ 523-4)

ie $\frac{dm}{dt} = km$ $(k<0)$

so $m(t) = m_0 e^{kt}$

it is usual to talk about the <u>half-life</u> $(t_{1/2})$ of the element

ie the time required to reduce the amount present by $\frac{1}{2}$

$$m(t_{1/2}) = \frac{1}{2} m_0 \Rightarrow e^{kt_{1/2}} = 0.5$$
$$kt_{1/2} = \ln(0.5) = -\ln 2$$

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(relationship between k and $t_{1/2}$) $\quad t_{1/2} = -\frac{1}{k} \ln 2$

or $k = -\frac{1}{t_{1/2}} \ln 2$

example: ${}^{14}_{6}C$ has a half-life of 5730 years
carbon dating works by comparing the fraction of ${}^{14}_{6}C$ left in a "dead" thing to the amount that is present in living things (which is constantly replenished)

a fossil was found to have only 15% ${}^{14}_{6}C$ left
how old is it ?

$$m(t) = m_0 e^{kt} = m_0 e^{t(-\ln 2/t_{1/2})} = m_0 e^{-t \ln 2/5730}$$

so if $\frac{m(t)}{m_0} = 0.15 \Rightarrow e^{-t\ln 2/5730} = 0.15$

or $-t\frac{\ln 2}{5730} = \ln(0.15) = -1.89712$

then $t = \frac{(1.89712)(5730)}{(\ln 2)} \simeq 15\,683$ years

see also example 3 on p ~~528-9~~ 523-4

Newton's Law of Heating and Cooling (p ~~529~~-30) 524-5
the temperature of an object increases or decreases at a rate that is proportional to the difference between its temperature and the temperature of its surroundings
(and the constant of proportionality depends on the materials comprising the object in question)

say that we have a cup of coffee initially at 40°C and a can of coke originally at 8°C and we leave them in a room with temperature 21°C
what happens ?
well obviously the coffee cools and the coke warms
but, let's look at the details

for the coffee:

let $H(t)$ represent the temperature of the coffee, so $H(0) = 40$

the coffee cools, so $\frac{dH}{dt} < 0$

the temperature difference $H - 21 > 0$, so we'll write

$$\frac{dH}{dt} = -\alpha(H - 21) \quad (\text{for } \alpha > 0)$$

separate the variables $\frac{dH}{H-21} = -\alpha\, dt$

integrate both sides $\int \frac{dH}{H-21} = \int -\alpha\, dt$

and we get $\ln|H-21| = -\alpha t + C$

which we can rewrite as $H - 21 = Ke^{-\alpha t}$ (why?)

and the general solution is $H(t) = Ke^{-\alpha t} + 21$

let $H(0) = 40°C$ so $K = 19$ (why?)

and the particular solution is $H(t) = 21 + 19e^{-\alpha t}$

for the coke:

let $C(t)$ represent the temperature of the coke, so $C(0) = 8°C$

using reasoning similar to that above, we'll have

$$\frac{dC}{dt} = \beta(21 - C) \quad (\text{for } \beta > 0)$$ (can you explain this?)

which has general solution $C(t) = 21 - Ke^{-\beta t}$ (why?)

and particular solution $C(t) = 21 - 13e^{-\beta t}$ (why?)

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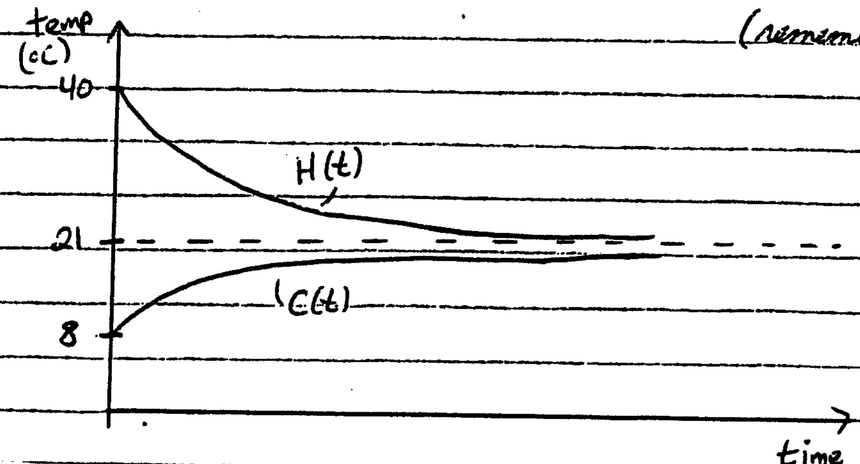
if we plot these solutions, we have:

(remember that $\alpha, \beta > 0$)



so we see that as $t \to \infty$, $H(t) \to 21$ and $C(t) \to 21$ (expected)

notice that in both cases, we were solving equations of the form $\frac{dT}{dt} = -k(T - 21)$,

which have solutions of the form $T = 21 + Ae^{-kt}$
notice also that $T(t) = 21$ (constant) is also a solution to the differential equation (where $A = 0$)
(ie if something starts at the temperature of the room, it will stay that temperature)
this solution is called the equilibrium solution and can be found by solving for $\frac{dT}{dt} = 0$

and the equilibrium is called stable, because all other solutions tend towards it as $t \to \infty$

see also Ex 4 p ~~530~~ 525

A10 Compounded Interest (p. 531–2) 526-7

if an investment pays interest at a rate of r and the interest is compounded n times per year, then after t years, the value is $A_0 (1 + \frac{r}{n})^{nt}$, where A_0 is the amount of the original investment

example: if we invest \$1000 at 5% interest, then after one year, it will be worth:

$1000 (1.05) = $1050 annual compounding
$1000 (1.025)^2 = $1050.63 semiannual "
$1000 (1.0125)^4 = $1050.95 quarterly "
$1000 (1.004167)^{12} = $1051.16 monthly "
$1000 (1.00096154)^{52} = $1051.25 weekly "
$1000 (1.000136986)^{365} = $1051.27 daily "

what if the interest were compounded continuously? then we'd have to take the limit as $n \to \infty$

$$\lim_{n\to\infty} A_0 (1 + \tfrac{r}{n})^{nt} = A_0 \lim_{n\to\infty} \left((1 + \tfrac{r}{n})^{n/r} \right)^{rt}$$

$$= A_0 \lim_{m\to\infty} \left((1 + \tfrac{1}{m})^m \right)^{rt} = A_0 e^{rt}$$

then we'd have $\$1000\, e^{0.05} = \$1051.27$ (ie daily interest can be modelled this way)

see Ex 5 p ~~531-2~~ 526-7

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A11

§7.5 = the Logistic Equation

when we talk about the growth of populations, there are two growth rates that we can speak of:

the absolute growth rate $\frac{dP}{dt}$ (people/yr)

or the relative growth rate $\frac{1}{P}\frac{dP}{dt}$ (%/yr)

the simplest model of population growth is the one that we've already seen, ie assume that the relative growth rate is a constant, $\frac{1}{P}\frac{dP}{dt} = k$

but this is $\frac{dP}{dt} = kP$, which has solution $P = P_0 e^{kt}$

this model works really only for small populations or for populations of simple organisms where there are no constraints on growth

see §7.1 p. ~~499-500~~ 495ff

the logistic model of population growth assumes that the relative growth rate is not a constant, but a decreasing, linear function of the population,

$$\frac{1}{P}\frac{dP}{dt} = k - aP \qquad a, k > 0$$

for small P, we'll have $\frac{1}{P}\frac{dP}{dt} \approx k$, so the logistic model reduces to the exponential model in this case

also note that if $P = \frac{k}{a}$, then $\frac{dP}{dt} = 0$, ie the relative growth rate decreases to zero, so this model predicts a limiting population, $K = \frac{k}{a}$, also called the carrying capacity

replace $a = k/K$ and rewrite the D.E. as

$$\frac{1}{P}\frac{dP}{dt} = k - \frac{k}{K}P$$

use M instead of K

$$\frac{dP}{dt} = kP - \frac{k}{K}P^2 = kP\left(1 - \frac{P}{K}\right) \quad (p\ \cancel{530}\ 531)$$

so have $\frac{dP}{dt} = 0$ if $P = 0$ or $P = K$ — these are equilibrium or constant solutions

$$\frac{d}{dP}\left(\frac{dP}{dt}\right) = k\left(1 - \frac{P}{K}\right) + kP\left(-\frac{1}{K}\right) = k - 2\frac{kP}{K} = k\left(1 - \frac{2P}{K}\right)$$

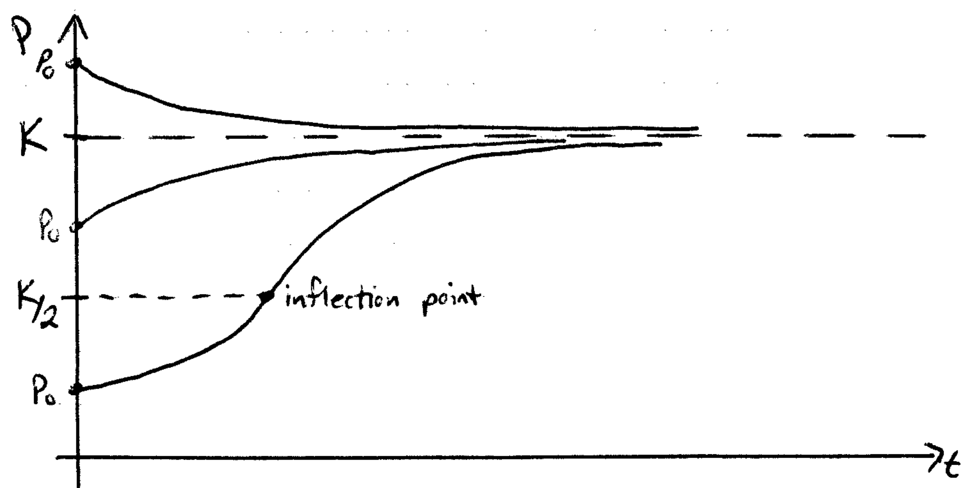
so the absolute growth rate, $\frac{dP}{dt}$, has a max at $P = \frac{K}{2}$ (see p ~~500~~ 495, fig. 3 p ~~500~~ and p ~~530-7~~ 531-2),

so for $0 < P < \frac{K}{2}$, $\frac{dP}{dt}$ is positive and increasing

for $\frac{K}{2} < P < K$, $\frac{dP}{dt}$ is positive and decreasing

and for $P > K$, $\frac{dP}{dt}$ is negative – ie the population will shrink if it is above the carrying capacity K

also, there are two equilibrium solutions:
$P = 0$ (unstable) and $P = K$ (stable)
(notice how much information we've gotten without actually solving the D.E.!)



see figure ~~2~~ 3 p ~~500~~ ~~532~~ 495
see Ex 1 p ~~530~~ 531-2 (which shows the direction field for the Logistic Equation)

now, let's solve it:

$$\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right) = kP\left(\frac{K-P}{K}\right) = \frac{k}{K}P(K-P)$$

separate $\quad \dfrac{dP}{P(K-P)} = \dfrac{k}{K}\,dt$

but $\quad \dfrac{1}{P(K-P)} = \dfrac{1}{K}\left(\dfrac{1}{P} + \dfrac{1}{K-P}\right)$

so have $\quad \dfrac{1}{K}\left(\dfrac{1}{P} + \dfrac{1}{K-P}\right)dP = \dfrac{k}{K}\,dt$

or $\quad \left(\dfrac{1}{P} + \dfrac{1}{K-P}\right)dP = k\,dt$

integrate $\quad \displaystyle\int\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = \int k\,dt$

to get $\quad \ln|P| - \ln|K-P| = kt + C$

which is $\quad \ln|K-P| - \ln|P| = -kt - C$

or $\quad \ln\left|\dfrac{K-P}{P}\right| = -kt - C$

exponentiate $\quad \left|\dfrac{K-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt} = Be^{-kt}$

so our general solution is $\quad \dfrac{K-P}{P} = Ae^{-kt} \qquad (A = \pm B)$

but $\quad \dfrac{K-P}{P} = \dfrac{K}{P} - 1 = Ae^{-kt}$

so $\quad \dfrac{K}{P} = 1 + Ae^{-kt}$ or $\boxed{P = \dfrac{K}{1+Ae^{-kt}}}$ (p ~~537~~) 534

what's A? let $P(0) = P_0$, then $\dfrac{K-P_0}{P_0} = Ae^0 = A$ (p ~~539~~) 534

so the population is $\quad P = \dfrac{K}{1+\left(\dfrac{K-P_0}{P_0}\right)e^{-kt}} = \dfrac{KP_0}{P_0 + (K-P_0)e^{-kt}}$

notice that with either form of the solution, we'll have that
$\lim_{t\to\infty} P(t) = K$ (regardless of A/P_0)

because $\quad e^{-kt} \to 0$ as $t \to \infty$ $(k > 0)$

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example: suppose that $\frac{dP}{dt} = 0.05\,P\left(1 - \frac{P}{2000}\right)$ $\quad P(0) = 1500$

then we know that the population follows the Logistic Model where $k = 0.05$
and the carrying capacity is $K = 2000$

so $$P(t) = \frac{K P_0}{P_0 + (K - P_0)e^{-kt}} = \frac{(2000)(1500)}{1500 + 500\,e^{-0.05t}} = \frac{6000}{3 + e^{-0.05t}}$$

how long does it take for the population to reach 1800?

$$P(t) = 1800 \Rightarrow \frac{6000}{3 + e^{-0.05t}} = 1800$$

so $$3 + e^{-0.05t} = \frac{6000}{1800} = 3.\overline{3333}$$

or $$e^{-0.05t} = 1/3$$

so $$-0.05t = \ln(1/3) = -1.0986123$$

$$t \approx 22 \text{ years}$$

what is the population at time $t = 50$ years?

$$P(50) = \frac{6000}{3 + e^{-0.05(50)}} \approx 1947$$

notice how the growth slows down as the carrying capacity is approached

see also examples
1 p ~~536-7~~ 531-2
2 p ~~537-8~~ 532-3
3 p ~~539-40~~ 534-5
4 p ~~540-1~~ 535-6

example: ~~[illegible]~~ (~~[illegible]~~)
the population of a species of elk on Reading Island
when the population was 600, the relative birth rate was 35%
and the relative death rate was 15%
at 800, they were 30% and 20%
the island is isolated – no hunting or migration

a) write a differential equation to model the population as a function of time

assume that the relative growth rate is a linear function of population

let the population at time t be $P(t)$, we're told that $\frac{1}{P}\frac{dP}{dt}$ is a linear function of P, so $\frac{1}{P}\frac{dP}{dt} = \alpha - \beta P$

when $P = 600$, $\frac{1}{P}\frac{dP}{dt} = 35\% - 15\% = 20\%$

when $P = 800$, $\frac{1}{P}\frac{dP}{dt} = 30\% - 20\% = 10\%$

so we really have that ① $\alpha - \beta(600) = 20\% = 0.20$

② $\alpha - \beta(800) = 10\% = 0.10$

① − ② gives $200\beta = 0.10 \Rightarrow \beta = \frac{1}{2000}$

then, from ②, $\alpha = \frac{1}{10} + \frac{800}{2000} = \frac{1}{2}$

so the diff. eq is $\frac{1}{P}\frac{dP}{dt} = \frac{1}{2} - \frac{1}{2000}P = \frac{1}{2000}(1000 - P)$

or $\frac{dP}{dt} = \frac{1}{2000}P(1000 - P)$

b) find the equilibrium size of the population

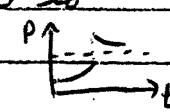
Today, there are 900 elk on the island, how do you expect the population to change in the future?

$\frac{dP}{dt} = \frac{1}{2000}P(1000 - P)$ is a logistic equation, so

we can read the equilibrium population directly from it

the equilibrium population is $P = 1000$

so we'd expect the current population of 900 to increase to about 1000



c) what if 450 more elk were moved to the island?

then the population would be 1350, which is above the equilibrium of 1000, so the population would decrease to about 1000

Chapter 8: Infinite Sequences and Series

§8.1: Sequences

a real-valued function defined on the set of positive integers is called a <u>sequence of real numbers</u>

example: $a(n) = \dfrac{n}{n+1}$ $n = 1, 2, 3, 4, \ldots$

so $a(1) = \dfrac{1}{1+1} = \dfrac{1}{2}$, $a(2) = \dfrac{2}{2+1} = \dfrac{2}{3}$, $a(3) = \dfrac{3}{3+1} = \dfrac{3}{4}$, ...

we usually denote $a(n)$ as a_n
and write the sequence as the infinite list of real numbers $\{a_1, a_2, a_3, \ldots\}$ or as $\{a_n\}$ (p ~~557~~) 554

given sequence $\{a_n\}$, (p ~~559~~) if it has a <u>limit</u>, ie $\lim\limits_{n\to\infty} a_n = L$

it means that as $n \to \infty$, $a_n \to L$, ie the a_n's get arbitrarily close to L and stay arbitrarily close to L

if a sequence has a limit, it is unique
a sequence that has a limit is called <u>convergent</u> (p ~~559~~ 556)
a sequence without a limit is called <u>divergent</u>

note: a sequence can be divergent in a couple of ways:
$\{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, -1, \ldots\}$ has no limit
$\{a_n\} = \{n\} = \{1, 2, 3, \ldots\}$ is unbounded ie $\lim\limits_{n\to\infty} n = \infty$

examples:

i, $a_n = 2^n$, $\lim\limits_{n\to\infty} 2^n = \infty$, divergent

ii, $a_n = \sin(\pi/2n)$, $\lim\limits_{n\to\infty} \sin(\pi/2n) = 0$, converges to 0

iii, $a_n = \dfrac{4n}{\sqrt{n^2+1}}$, $\lim\limits_{n\to\infty} \dfrac{4n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \dfrac{4n/n}{\sqrt{n^2/n^2 + 1/n^2}} = \lim\limits_{n\to\infty} \dfrac{4}{\sqrt{1+\frac{1}{n^2}}} = 4$ converges

iv, $a_n = \dfrac{e^n}{n}$, $\lim\limits_{n\to\infty} \dfrac{e^n}{n}$ is $\dfrac{\infty}{\infty}$, L'Hôpital's Rule tells us that

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty, \text{ so } \lim_{n\to\infty} \frac{e^n}{n} = \infty$$

see the Limit Laws on page ~~500~~ 557 and examples 1 - 8 on p ~~557-62~~ 554-8

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§ 8.2: Series

what if we wanted to sum an infinite number of numbers?

ie $a_1 + a_2 + \cdots + a_j + \cdots$

we'd be led naturally to consider the infinite series

$\sum_{k=1}^{\infty} a_k$ (p ~~567~~ 565)

how can we sum an infinite number of numbers?

well, we can't really

but, we can sum a finite number of numbers:

let

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots$

$S_n = a_1 + \cdots + a_n = S_{n-1} + a_n = \sum_{k=1}^{n} a_k$

these S_n's are called partial sums (p ~~568~~) and we generate an infinite sequence of them $S_1, S_2, S_3, \ldots S_j, \ldots$

now, what would $\lim_{n\to\infty} S_n$ correspond to?

well, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k$

in other words, the convergence of the infinite series is tied to the convergence of its sequence of partial sums:

definition: given infinite series $\sum_{k=1}^{\infty} a_k$, if its sequence of partial sums $\{s_n = \sum_{k=p}^{n} a_k\}$ converges to finite number L, then the series is said to converge to L, $\sum_{k=p}^{\infty} a_k = L$ and we call L the sum of the series (p ~~568~~) 566

if the sequence of partial sums diverges, then the series diverges

example: ~~p 567 #4~~

$\sum_{k=3}^{\infty} \frac{1}{k^2-k}$ we need to find the sum of the series we do this by finding the limit of the sequence of partial sums

now $\frac{1}{k^2-k} = \frac{1}{k(k-1)} = \frac{-1}{k} + \frac{1}{k-1} = \frac{1}{k-1} - \frac{1}{k}$

so $s_n = \sum_{k=3}^{n} \frac{1}{k^2-k} = \sum_{k=3}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)$

this is an example of a telescoping series – all of the terms, except the first and last cancel (see Ex 6 p 571) 568-9

$= \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{n-1}} - \frac{1}{n}$

$= \frac{1}{2} - \frac{1}{n}$

so $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n} = \frac{1}{2}$ $\therefore \sum_{k=3}^{\infty} \frac{1}{k^2-k} = \frac{1}{2}$

example: $\sum_{k=0}^{\infty} 3^k = 1 + 3 + 9 + 27 + \cdots$ diverges because the sequence of partial sums is unbounded

but $\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges because the sequence of partial sums has no limit

(p ~~569~~) 567

a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$ $(a \neq 0)$ is called a geometric series

look at nth partial sum $s_n = a + ar + \cdots + ar^{n-1}$

$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

then $s_n - rs_n = (1-r)s_n = a - ar^n = a(1-r^n)$

ie $s_n = \frac{a(1-r^n)}{1-r}$

then if $|r| < 1$, $r^n \to 0$ and $s_n \to \frac{a}{1-r}$

ie the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r|<1$ and its sum is $\frac{a}{1-r}$

if $|r| \geq 1$, the series will be divergent (see p ~~568~~ 567)

examples: i, $\sum_{n=1}^{\infty} \frac{3^{n+1}}{7^n} = \sum_{n=1}^{\infty} \frac{3^2\, 3^{n-1}}{7^1\, 7^{n-1}} = \sum_{n=1}^{\infty} \frac{9}{7}\left(\frac{3}{7}\right)^{n-1} = \frac{9/7}{1-3/7} = \frac{9/7}{4/7} = \frac{9}{4}$

ii, $\sum_{n=1}^{\infty} 2^{4n}\, 3^{-n} = \sum_{n=1}^{\infty} \frac{2^{4n}}{3^n} = \sum_{n=1}^{\infty} \frac{2^4\, 2^{4n-4}}{3 \cdot 3^{n-1}} = \sum_{n=1}^{\infty} \frac{16}{3}\left(\frac{16}{3}\right)^{n-1}$ diverges since $|r|>1$

see also Examples 1–5 on pages ~~569–70~~ 566–8

some useful results:

i, if $\sum_{k=1}^{\infty} a_k = L$ (ie converges) and $\sum_{k=1}^{\infty} b_k = M$ (ie converges),

then $\sum_{k=1}^{\infty} (a_k + b_k) = L + M$ (ie the sum of convergent series is convergent)

and $\sum_{k=1}^{\infty} \alpha a_k = \alpha L$ for any $\alpha \in \mathbb{R}$ (Thm p ~~573~~) 570

ii, if $\sum_{k=1}^{\infty} a_k$ converges, then $a_n \to 0$ as $n \to \infty$ (Thm p ~~572~~ 570)

iii, if $a_n \not\to 0$ as $n \to \infty$, $\sum_{k=1}^{\infty} a_k$ diverges (Divergence Test) (p ~~572~~)

BUT, $a_n \to 0$ as $n \to \infty$ is NOT enough to guarantee convergence of $\sum_{k=1}^{\infty} a_k$

example 7 p ~~571~~ 569: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

it can be shown that $s_{2^n} > 1 + \frac{n}{2}$ (see p 577)

and so $s_{2^n} \to \infty$ as $n \to \infty$ and the series diverges

see also Examples 8 and 9 on pages ~~572–3~~ 570–1

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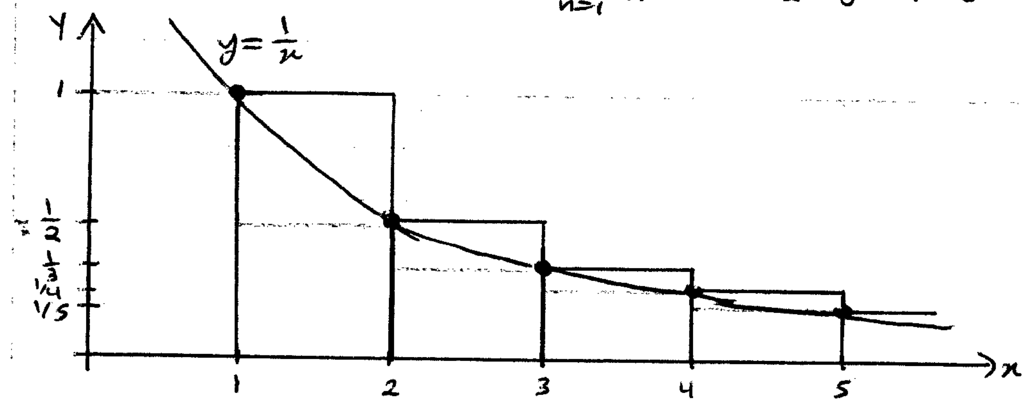
§ 8.3 : the Integral and Comparison Tests

sometimes it is difficult to work with the sequence of partial sums and so we'd like to know if there are ways to determine whether or not a series converges without necessarily having to determine its sum

consider the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$



notice that $\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx$ which diverges

and so we must have that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

see page ~~577~~ 575 for $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$

and page ~~578~~ 576 for $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$

so we can see that there is a relationship between infinite series and improper integrals :

the Integral Test (p ~~578~~) : 577

suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent

and so we know that the p series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

and diverges if $p \le 1$ (just like $\int_1^\infty \frac{1}{x^p}\,dx$) (Ex 2 p ~~579~~ 577-8)

example: does $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$ converge?

take $f(x) = \frac{2}{x(\ln x)^2}$, which is positive, decreasing and continuous for $x \ge 2$

(we can take $x \ge 2$, since $k \ge 2$ in the original series)

and $\int_2^\infty \frac{2}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{2}{x(\ln x)^2}\,dx$ (substitution: let $u = \ln x$ then $du = \frac{1}{x}dx$)

$$= \lim_{b\to\infty} \left.\frac{-2}{\ln x}\right|_2^b$$

$$= \lim_{b\to\infty} \frac{-2}{\ln b} + \frac{2}{\ln 2}$$

$$= \frac{2}{\ln 2} \quad \text{(converges)}$$

and so the series $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$ converges (since the integral does)

see also Example 1 p ~~579~~ 577

example: does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge?

let $f(x) = \frac{x}{x^2+1}$ which is positive and continuous for $x \ge 1$

is it decreasing? $f'(x) = \frac{(1)(x^2+1)-(x)(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0$ yes

then $\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \left.\frac{1}{2}\ln(x^2+1)\right|_1^t$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\ln(t^2+1) - \ln(2)\right) = \infty$$

so the improper integral diverges and hence the series does as well what about $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+1}$?

since $n^2+7n+1 > n^2$ for $n \geq 1$, $\frac{3}{n^2+7n+1} < \frac{3}{n^2}$

and so $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+1} < \sum_{n=1}^{\infty} \frac{3}{n^2}$, which is known to be

convergent and hence the series must converge

this is the basic idea of the Comparison Test (p ~~580~~): 579

suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

i, if $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent

ii, if $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent

example: $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

for $n \geq 1$, $n^2+1 \leq 2n^2$ (since $n^2 \geq 1$)

and so $\frac{1}{n^2+1} \geq \frac{1}{2n^2} \Rightarrow \frac{n}{n^2+1} \geq \frac{1}{2n}$

and then $\sum_{n=1}^{\infty} \frac{n}{n^2+1} \geq \sum_{n=1}^{\infty} \frac{1}{2}\frac{1}{n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges

and so $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges

see also Examples 3 and 4 on page ~~581~~ 579

what would we do with $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$?

the inequality $\frac{1}{3^n-2} > \frac{1}{3^n}$ does not help with the Comparison Test

because our series is greater than the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$

but we would think that $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ should converge because as $n \to \infty$ $\frac{1}{3^n-2} \to 0$ just like $\frac{1}{3^n}$ does

this is where the Limit Comparison Test (p ~~582~~ 580) comes in:

suppose that Σa_n and Σb_n are series with positive terms
if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ where $0 < c < \infty$, then either both series converge or both diverge

so for $\sum\limits_{n=1}^{\infty} \frac{1}{3^n-2}$, $a_n = \frac{1}{3^n-2}$, $b_n = \frac{1}{3^n}$

then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{1/3^n-2}{1/3^n} = \lim\limits_{n\to\infty} \frac{3^n}{3^n-2} = \lim\limits_{n\to\infty} \frac{1}{1-2/3^n} = 1$

and since $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum\limits_{n=1}^{\infty} \frac{1}{3^n-2}$ converges as well

example: $\sum\limits_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^3+1}}$

for large k, $\frac{2k+1}{\sqrt{k^3+1}} \to \frac{2k}{k^{3/2}} = \frac{2}{\sqrt{k}}$, so we'd expect this series to diverge

we'll do the limit comparison with $\Sigma \frac{1}{\sqrt{k}}$ (which diverges)

let $a_k = \frac{2k+1}{\sqrt{k^3+1}}$, $b_k = \frac{1}{\sqrt{k}}$

so $\frac{a_k}{b_k} = \frac{2k+1}{\sqrt{k^3+1}} \Big/ \frac{1}{\sqrt{k}} = \frac{(2k+1)\sqrt{k}}{\sqrt{k^3+1}} = \frac{2k^{3/2}+\sqrt{k}}{\sqrt{k^3+1}} = \frac{2+1/k}{\sqrt{1+1/k^3}}$

so, as $k\to\infty$, $\frac{a_k}{b_k} \to \frac{2k^{3/2}}{k^{3/2}} = 2$

$\therefore$ by the Limit Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^3+1}}$ diverges

see also Example 5 on page ~~582~~ 580 and examples 6 & 7 p 581-2 for how we can estimate sums

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§ 8.4: Other Convergence Tests

now we'll see what to do with series with both positive and negative terms

let's say we're given series Σa_k

now, since $a_k \le |a_k|$, we'll have $\Sigma a_k \le \Sigma |a_k|$

so, if $\sum |a_k|$ converges, $\sum a_k$ must converge (Thm p ~~594~~) 588

definition: a series $\sum a_n$ is called <u>absolutely convergent</u> if the series of absolute values $\sum |a_n|$ converges (p ~~589~~) 588

definition: a series $\sum a_n$ is called <u>conditionally convergent</u> if it converges, but $\sum |a_n|$ diverges

series in which consecutive terms have opposite signs are called <u>alternating series</u> (p ~~580~~) 585

examples: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ or $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$

$$0 - 1 + \sqrt{2} - \sqrt{3} + 2 - \sqrt{5} + \cdots = \sum_{n=0}^{\infty} (-1)^n \sqrt{n}$$

<u>Alternating Series Test</u>: (p ~~587~~) 585

Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $(b_n > 0)$

if i) $b_{k+1} \leq b_k$ for all k, ie $\{b_k\}$ is a decreasing sequence

and ii) $b_k \to 0$ as $k \to \infty$

then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges

so then $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$ converges

since $b_k = \frac{1}{k^2} > 0$, $b_{k+1} = \frac{1}{(k+1)^2} < \frac{1}{k} = b_k$ and $b_k \to 0$

furthermore, since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is also convergent,

we have $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ is absolutely convergent

but $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$ is only conditionally convergent as it converges, but $\sum_{k=1}^{\infty} \frac{1}{k}$ does not

see also Examples 1-3 p ~~587-8~~ 586 and 5-7 p ~~589-91~~ 588-9

more examples: i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n}$ conditionally convergent (just like ↑)

ii) $\sum_{n=1}^{\infty} \sin(n\pi/4) = \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0 + \left(-\frac{1}{\sqrt{2}}\right) + (-1) + \left(-\frac{1}{\sqrt{2}}\right) + 0 + \cdots$

(not strictly alternating)

this series diverges as $b_n \not\to 0$

iii) $\sum_{n=1}^{\infty} \frac{(-1)^n 4n^2}{7n^2+3n+1}$ $\quad b_n = \frac{4n^2}{7n^2+3n+1} \to \frac{4}{7} \neq 0$ $\therefore$ series diverges

iv) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ $\quad b_n = \frac{n}{n^2+1}$ we know that $b_{n+1} \le b_n$ (SF.3)

and that $b_n \to 0$

and so this series is convergent by A.S.T.

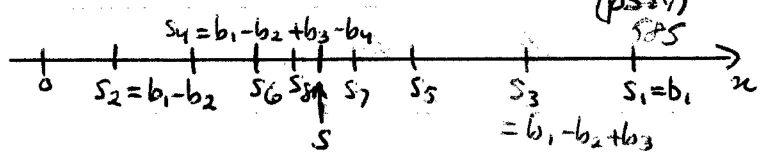
but $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent and so our series is not absolutely convergent

Consider the convergent alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$

ie $b_{n+1} \le b_n, \quad b_n \to 0$

(p ~~587~~ 585)

look at the partial sums:



the partial sums will oscillate around the sum of the series s (p ~~587~~ 585)

the remainder R_n (p ~~588~~ 587) is the error in using s_n to estimate the sum of the series s

for an alternating series, $|R_n| = |s - s_n| \le b_{n+1}$ (p ~~588~~ 587)

example: let's find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ correct to 2 decimal places

(we know that this series converges by the AST and actually absolutely)

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

$b_n = \frac{1}{n^2}$ need $b_{n+1} \le 0.005$ ie $\frac{1}{(n+1)^2} < 0.005$

which means $\frac{1}{0.005} \le (n+1)^2 \Rightarrow 200 \le (n+1)^2$

$\Rightarrow \quad 14.14 \le n+1 \quad \Rightarrow \quad n = 14$

so we use $S_{14} = \sum_{n=1}^{14} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \frac{1}{64} - \frac{1}{81}$
$+ \frac{1}{100} - \frac{1}{121} + \frac{1}{144} - \frac{1}{169} + \frac{1}{196} \cong 0.84$

see also Example 4 p ~~589~~ 587

the <u>Ratio Test</u> (p ~~591~~): 589

i, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

ii, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, the series diverges

iii, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the test cannot tell us

To see why part iii, is true, see the NOTE ~~at the bottom~~ or of page ~~591~~ 590 with $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

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examples:

i, $\sum_{n=1}^{\infty} \frac{10^n}{n!}$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{10^{n+1}/(n+1)!}{10^n/n!} = \lim_{n\to\infty} \frac{10^{n+1}}{10^n} \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{10}{n+1} = 0 \quad \therefore$ the series converges (absolutely)

ii, $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$, $\lim_{n\to\infty} \frac{2^{n+1}(n+1)!/(n+1)^{n+1}}{2^n n!/n^n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}}$

$= \lim_{n\to\infty} \frac{2(n+1)\;n^n}{(n+1)(n+1)^n} = \lim_{n\to\infty} 2\frac{n^n}{(n+1)^n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n$

$= \lim_{n\to\infty} 2\left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \frac{2}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{2}{(1+\frac{1}{n})^n} = \frac{2}{e} < 1$

$\therefore$ the series converges

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See also Ex 829 p ~~592~~ 590

iii, $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^2}$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)^2}{2^n/n^2} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{n^2}{(n+1)^2} = 2 > 1$

$\therefore$ this series diverges

§8.5: Power Series

a <u>power series centred at $x=a$</u> has the form (p ~~594~~ 593):

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

notice that when $x=p$; ie a given value, we have

$\sum_{n=0}^{\infty} c_n(p-a)^n = c_0 + c_1(p-a) + c_2(p-a)^2 + \cdots$ which is just a numerical series like the ones we've been studying

so a power series may converge for some values of x

notice that if $x=a$, we'll have $\sum_{n=0}^{\infty} c_n(a-a)^n = c_0$

ie all power series converge at their <u>centre</u> a

in general, the convergence of a power series will depend on the <u>coefficients</u>, the c_j's

if we have $a=0$, then we'll have

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

a power series defines a function whose domain is the set of x for which it converges

example: if $c_j = 1$ for all j (and $a=0$), we'll have

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \begin{cases} \dfrac{1}{1-x} & \text{if } |x|<1 \quad (-1<x<1) \\ \text{diverges if} & |x|\geq 1 \end{cases}$$

a geometric series (p ~~594~~ 593)

we can test a power series for convergence using the Ratio Test:

example: for what values of x is $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ convergent?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^2 2^{n+1}}}{\frac{x^n}{n^2 2^n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\ \frac{n^2 2^n}{(n+1)^2 2^{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{2}\frac{n^2}{(n+1)^2} = \frac{|x|}{2}$$

for convergence, we need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

ie $\frac{|x|}{2} < 1 \Rightarrow -2 < x < 2$ or $|x| < 2$

if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series diverges, so our power series will diverge if $\frac{|x|}{2} > 1$ or $|x| > 2$ or $x < -2$, $x > 2$

what if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$?

then the Ratio Test cannot tell us – ie we have to check

$\frac{|x|}{2} = 1 \Rightarrow |x| = 2 \Rightarrow x = -2$ or $x = 2$

if $x = -2$, $\sum_{n=0}^{\infty}\frac{x^n}{n^2 2^n} = \sum_{n=0}^{\infty}\frac{(-2)^n}{n^2 2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$ converges

if $x = 2$, $\sum_{n=0}^{\infty}\frac{x^n}{n^2 2^n} = \sum_{n=0}^{\infty}\frac{2^n}{n^2 2^n} = \sum_{n=0}^{\infty}\frac{1}{n^2}$ converges

$\therefore$ the power series converges for $|x| \le 2$ or $-2 \le x \le 2$

see Examples 1–3 on pages ~~594~~-5 593-4

<u>Thm</u> (p ~~596~~ 595) : for a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are 3 possibilities :

i, the series converges only when $x = a$ (Ex 1 p ~~594~~) 593
ii, the series converges for all x (Ex 3 p 595) 594
iii, the series converges for $|x-a| < R$ and diverges for $|x-a| > R$ (Ex 2 p ~~594-5~~ 593 and our example)

R is called the <u>radius of convergence</u> of the power series in our example, we had $R=2$

we use the convention that case i, means $R=0$, and case ii, $R=\infty$

R defines the <u>interval of convergence</u> $|x-a|<R$ or $a-R<x<a+R$

on which the power series converges

but we must check the endpoints $a-R$ and $a+R$

in our example, the interval of convergence was $-2 \le x \le 2$ or $[-2,2]$

in case i, we have $\{a\}$ and case ii, $(-\infty,\infty)$

examples: find the radius and intervals of convergence of:

i, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}(x+1)^n$ (the centre is $a=-1$)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\ln(n+1)}{(n+1)}(x+1)^{n+1}\Big/\frac{\ln n (x+1)^n}{n}\right|$$

$$= \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}\,\frac{n}{n+1}\,|x+1| = |x+1|$$

so we need $|x+1|<1$ ie the radius is $R=1$

or $-2<x<0$

if $x=-2$, $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}(-2+1)^n = \sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$ converges (by AST) (give details)

if $x=0$, $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}(0+1)^n = \sum_{n=1}^{\infty}\frac{\ln n}{n}$ diverges (Comparison with $\sum\frac{1}{n}$)

$\therefore$ the interval of convergence is $-2\le x<0$ or $[-2,0)$

ii, $\displaystyle\sum_{n=0}^{\infty}\frac{n!\,x^n}{n^3}$ (the centre is $a=0$)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{(n+1)^3}\Big/\frac{n!\,x^n}{n^3}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!}\,\frac{n^3}{(n+1)^3}\,|x|$$

$$= \lim_{n\to\infty}(n+1)\left(\frac{n}{n+1}\right)^3|x|$$

$$= \lim_{n\to\infty}\frac{n^3}{(n+1)^2}|x| = \infty$$

ie this series diverges for all x

except $x=0$

so $R=0$ and $\{0\}$

iii) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\qquad (a=0)$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \Big/ \frac{(-1)^n x^{2n}}{(2n)!} \right|$$
$$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0 \text{ for all } x$$

ie this series converges for all x, so $R=\infty$ and $(-\infty,\infty)$

see also Examples 4 & 5 on pages ~~597-8~~ 596-7

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§8.6: Representations of Functions as Power Series

we have already met the idea that a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ defines a function on its interval of convergence

now, let's look at this from the other way around – ie given a function $f(x)$, is there a power series representation for it?

consider $f(x) = \frac{1}{1-x}$, then we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$
$= 1+x+x^2+x^3+\dots$

examples: i) then what is $g(x) = \frac{1}{1+x}$?

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n = 1-x+x^2-x^3+\dots$$

and this would be good for $|-x|<1$ or $|x|<1$
(ie same interval of convergence)

ii) how about $h(x) = \frac{1}{x+3}$?

$$\frac{1}{3+x} = \frac{1}{3(1+\frac{x}{3})} = \frac{1}{3} \frac{1}{1-(-\frac{x}{3})} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}} \qquad \text{good for } \left|-\frac{x}{3}\right|<1$$
or $|x|<3$

iii) and for $p(x) = \dfrac{x^4}{x+3}$?

$$\frac{x^4}{x+3} = x^4\left(\frac{1}{x+3}\right) = x^4 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+4}}{3^{n+1}}, \text{ good for } |x|<3$$

see also Examples 1-3 on page ~~600~~ 599

if the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$

and if $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$

then i, $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

$\left(\text{ie } \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n(x-a)^n\right) \quad \text{termwise differentiation}\right)$

and ii, $\int f(x)\,dx = C + c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3 + \cdots$

$$= C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} \quad (\text{ie termwise integration})$$

Thm (p ~~601~~ 600) : the radii of convergence of the power series in i, and ii, is still R

but : the intervals of convergence are not necessarily the same
when we differentiate, we can lose convergence at an endpoint
and when we integrate, we can gain convergence at an endpoint

example : (Ex 5 p ~~602~~ 601) since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+1) x^n$$

radius is $R=1$
and interval of convergence is still $|x|<1$ (why?)

example: we know that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x|<1$

so $\ln(1+x) = \int \frac{1}{1+x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ also for $|x|<1$

if $x=0$, $\ln(1) = 0 \Rightarrow C=0$, so $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

if $x=-1$, $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-1}{n+1}$ diverges

if $x=1$, $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges (AST)

so $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is actually good for $-1 < x \le 1$

(ie we have gained convergence at an endpoint)

[so $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$]

see also Example 6 p ~~602~~ 601

example: (Ex 7 p ~~602-3~~) 601-2

since $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x|<1$

we'll have $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ also for $|x|<1$ (Ex 1)

so then $\arctan x = \int \frac{1}{1+x^2}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

if $x=0$ $\arctan(0)=0 \Rightarrow C=0$ and $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

and we know that $R=1$, ie series good for $|x|<1$

if $x=-1$ $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges (AST)

if $x=1$ $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges (by AST) and so $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ is valid for $-1 \le x \le 1$

notice that if $x=1$, we have $\arctan(1) = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

ie $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

example: $\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}$ $\quad |-x^4|<1 \Rightarrow |x|<1$

so $\int \frac{1}{1+x^4}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1}$ good for $|x| \le 1$ (why?) $\longleftarrow$ (notice what we have here)

and then
$$\int_0^1 \frac{1}{1+x^4}\,dx = \left(C + \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{4n+1}}{4n+1}\right) - \left(C + \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{4n+1}}{4n+1}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} = 1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \frac{1}{21} + \cdots$$

so we can estimate $\int_0^1 \frac{1}{x^4+1}\,dx$ to any accuracy we like since we have an alternating series with $b_n = \frac{1}{4n+1}$, so $b_{n+1} = \frac{1}{4(n+1)+1}$

or $b_{n+1} = \frac{1}{4n+5}$

so if we'd like an error of 0.1 or less, we'd need
$\frac{1}{4n+5} < 0.1$ or $10 < 4n+5 \Rightarrow n > \frac{5}{4} \Rightarrow n=2$

ie $\int_0^1 \frac{1}{1+x^4}\,dx \approx 1 - \frac{1}{5} + \frac{1}{9} \approx 0.9$

see also Example 8 on page ~~603~~ 602

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§8.7: Taylor and Maclaurin Series

we've seen that a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ may represent a function $f(x)$ on its interval of convergence
but is this representation unique? are these coefficients c_j special in any way? (ie predetermined)

the answer to both questions is yes and here's why:

if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

so $f(a) = c_0$

then $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots$

so $f'(a) = c_1$

and $f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + 20c_5(x-a)^3 + \cdots$

so $f''(a) = 2c_2 \Rightarrow c_2 = \dfrac{f''(a)}{2}$

and $f'''(x) = 6c_3 + 24c_4(x-a) + 60c_5(x-a)^2 + \cdots$

so $f'''(a) = 6c_3 \Rightarrow c_3 = \dfrac{f'''(a)}{6} = \dfrac{f'''(a)}{3!}$

we can see that the pattern will be $c_n = \dfrac{f^{(n)}(a)}{n!}$

ie these coefficients are determined by the function (and hence they are unique)

and so $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

this is called the <u>Taylor series of f centred at a</u> (p ~~607~~ 605)

in the special case where $a=0$, we'll have the <u>Maclaurin series</u> of f $\quad f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$

examples: i) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\;_{-1<x<1}$ is a Maclaurin series

(Ex 1 p ~~607~~ 606) ii) if $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$

and the Maclaurin series for e^x is

$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

$\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty} \dfrac{|x|}{n+1} = 0 \Rightarrow R = \infty$ and converges for all x

so, in particular if $x=1$, we have another (and nice) way of defining e,
ie $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$

we have seen some other Maclaurin series:

i, $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad -1 < x < 1$

ii, $\frac{1}{x+3} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^{n+1}} \qquad -3 < x < 3$

iii, $\frac{x^4}{x+3} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+4}}{3^{n+1}} \qquad -3 < x < 3$

iv, $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \qquad -1 < x < 1$

v, $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad -1 < x \leq 1$

vi, $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad -1 < x < 1$

v, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad -1 \leq x \leq 1$

more examples:

i, we can use the Maclaurin series for e^x to get a Taylor series centred somewhere else

ie $e^x = e^{x+1-1} = e^{-1} e^{x+1} = e^{-1} \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!} (x+1)^n$

or we could get it directly, ie if $f(x) = e^x$, $f^{(n)}(x) = e^x$,
so $f^{(n)}(-1) = e^{-1}$ and so the Taylor series centred at $a=-1$
is $e^x = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!} (x+1)^n$

see also Ex 3 p ~~610~~ 609

ii, (Ex 5 p ~~611~~ 610) $f(x) = \cos x$, find Maclaurin series
$f(0) = 1$, $f'(x) = -\sin(x) \Rightarrow f'(0) = 0$,
$f''(x) = -\cos x \Rightarrow f''(0) = -1$, $f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(0) = 1$

this cycle will repeat, ie $f^{(n)}(0) = \begin{cases} 1 & n = 0, 4, 8, 12, \ldots \\ 0 & n = 1, 3, 5, 7, 9, \ldots \\ -1 & n = 2, 6, 10, 14, \ldots \end{cases}$

so we see that the odd terms will disappear and we'll have

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

which we can write as $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

and from our work in §8.5, we know that $R = \infty$ and this series converges for all x

Example 4 on pages ~~610-1~~ 609-10 shows that $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

which converges for all x (ie $R = \infty$)

we can do even more:

i, the Maclaurin series for $f(x) = x^2 e^x$ would be

$$x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \quad (\text{for all } x)$$

ii, $\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots$

(take $\frac{\sin x}{x} = 1$ at $x = 0$) (for all x, $x \neq 0$)

iii, $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right) = 1$ (ie without L'Hôpital's Rule)

want alternating

iv, $\int e^{x^2} dx = \int \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \right) dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)}$ (ie antiderivative does exist)

show 5

v, $\int_0^1 e^{x^2} dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)} = 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \frac{1}{216} + \cdots$

(p617 35 ~~p616 #29~~)

vi, $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$ (for all x)

vii, $\int \cos(x^2)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!\,(4n+1)}$ (for all x)

so $\int_0^1 \cos(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!\,(4n+1)} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(4n+1)} = 1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} + \frac{1}{685440}$

$\simeq 0.90452425$

the error is less than $\frac{1}{(2\times5)!\,(4\times5+1)} = \frac{1}{(10!)(21)} = \frac{1}{76204800} \simeq 1.3\times10^{-8}$

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viii, $e^x \arctan x = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{k=0}^{\infty} (-1)^n \frac{x^{2k+1}}{2k+1}\right)$

$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$

$= x + x^2 + \left(\frac{x^3}{2!} - \frac{x^3}{3}\right) + \left(\frac{x^4}{3!} - \frac{x^4}{3}\right) + \left(\frac{x^5}{5} + \frac{x^5}{4!} - \frac{x^5}{6}\right) + \cdots$

$= x + x^2 + \frac{x^3}{6} - \frac{x^4}{6} + \frac{3}{40}x^5 + \cdots$

see also Examples ~~6–10~~ on pages ~~611–15~~
6+7 610–11
10–13 613–6

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§8.8: the Binomial Series (Binomial Series in §8.7)

if k is a positive integer, then the Binomial theorem tells us that

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3$$

$$+ \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots + kx^{k-1} + x^k$$

$$= \sum_{n=0}^{k} \binom{k}{n} x^n$$

$\frac{k!}{(k-n)!\,n!}$

where $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$ $n = 1, 2, \ldots, k$

and $\binom{k}{0} = 1$ (note that this is a Maclaurin Series)

what if k is not a positive integer?

if $f(x) = (1+x)^k$, then $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$, so $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-1}$, so $f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$, so $f'''(0) = k(k-1)(k-2)$

ie $f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$

so $f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$

and thus the Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

which is called the <u>Binomial series</u> (p ~~618~~) 612

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k-1)(k-2)\cdots(k-n+1)x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|k-n|}{n+1}|x|$$

$$= \lim_{n\to\infty}\frac{\left|\frac{k}{n}-1\right|}{\left|1+\frac{1}{n}\right|}|x| = |x|$$

so if $|x|<1$, the series will converge

and if $|x|>1$, the series will diverge

at the endpoints, convergence will depend on k:

if $-1 < k \le 0$, the series will converge at $x = 1$

if $k \ge 0$, the series will converge at $x = \pm 1$

examples:

(p ~~620 #1~~) p 617 # 21

i, $\sqrt{1+x}$ so $k = 1/2$

$$\binom{1/2}{n} = \frac{(1/2)(-1/2)(-3/2)(-5/2)\cdots(\frac{1}{2}-n+1)}{n!} \quad (n\ge 2) \quad = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(\frac{3}{2}-\frac{2n}{2})}{n!}$$

$$= \frac{(-1)^{n-1}\, 3\cdot 5\cdot 7\cdots(2n-3)}{2^n\, n!}$$

$\binom{1/2}{0} = 1$

$\binom{1/2}{1} = \frac{1}{2}$ fits pattern so $\sqrt{1+x} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{3\cdot 5\cdot 7 \cdots (2n-3)}{2^n\, n!} x^n$ (for $-1 \le x \le 1$)

ii) $\sqrt{1-x^2} = 1 + \frac{1}{2}(-x^2) + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{3\cdot 5\cdot 7 \cdots (2n-3)\,(-x^2)^n}{2^n\, n!}$ $(-1 \le x \le 1)$

$$= 1 - \frac{1}{2}x^2 - \sum_{n=2}^{\infty} \frac{3\cdot 5\cdot 7 \cdots (2n-3)}{2^n\, n!} x^{2n}$$

iii) $\frac{1}{\sqrt{2+x}} = \frac{1}{\sqrt{2(1+\frac{x}{2})}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+x/2}} = \frac{1}{\sqrt{2}} \left(1+\frac{x}{2}\right)^{-1/2}$

$$= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x}{2}\right)^n$$

$\binom{-1/2}{0} = 1$, $\binom{-1/2}{n} = \frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-n+1)}{n!}$

$$= \frac{(-1/2)(-3/2)(-5/2)\cdots(1/2-n)}{n!}$$

$$= \frac{(-1/2)(-3/2)(-5/2)\cdots((1-2n)/2)}{n!}$$

$$= \frac{(-1)^n\,(1\cdot 3\cdot 5 \cdots (2n-1))}{2^n\, n!}$$

so $\frac{1}{\sqrt{2+x}} = \frac{1}{\sqrt{2}}\left(1 + \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdots (2n-1)}{2^n\, n!} \frac{x^n}{2^n}\right)$

$$= \frac{1}{\sqrt{2}}\left(1 + \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdots (2n-1)}{4^n\, n!} x^n\right)$$

good for $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$, ie $R = 2$

see also Examples 1 and 2 on page G19-20
§ 9.9 p G11-3

A10

Chapter 11: Partial Derivatives

§11.1 (and 9.6): Functions of Several Variables

we've used the concept of a function of two variables, say $f(x,y)$, before

for example, when we were talking about differential equations, we wrote them as $y' = f(x,y)$

and there was no problem with doing this

why not?

because the extension from function of one variable $y = f(x)$ to functions of two variables $z = f(x,y)$ (independent x & y) is intuitive and obvious

you already have intuitive experience with functions of more than one variable

for example, what factors determine the temperature you can expect outside?

location and time of year (ie position and time)

so the temperature T is a function of position and time, ie $T = f(x,t)$ (see also Ex 1 p~~739~~ 738)

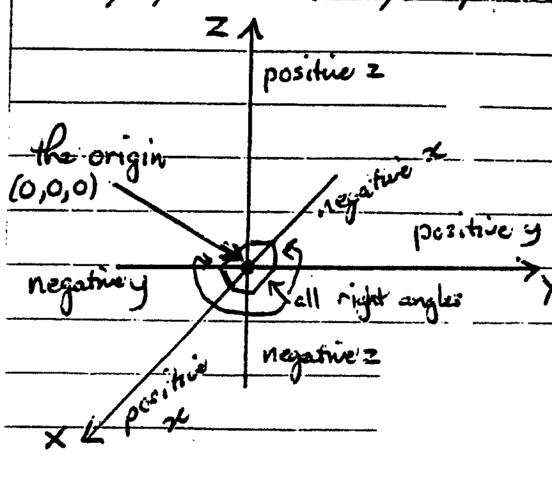
(this is, of course, an over-simplification of real weather, but it illustrates the point)

we're going to be interested in plotting functions of two variables $z = f(x,y)$, so we need to understand

Cartesian Three-Space

we specify points by ordered triples (x,y,z)

and we take our x and y axes and add a third perpendicular axis, which we call z



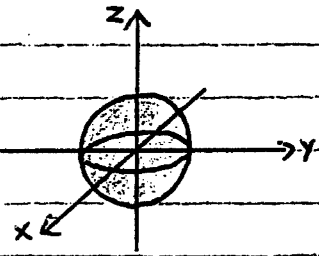
this is the standard way of drawing the axes: think of the xy plane as being horizontal and then z is vertical here, the yz plane is the plane of the paper and x is coming out at you

the graph of a function $f(x,y)$ (of two variables) is the set of all points (x,y,z) such that $z = f(x,y)$ (p~~678~~ 675)
ie the set of points $(x, y, f(x,y))$
and, in general, this is a <u>surface</u> in 3space (p~~679~~ 675)

the easiest way to draw these graphs is to draw <u>cross-sections</u> or <u>traces</u> in which one of the variables is fixed and the other varies

examples:

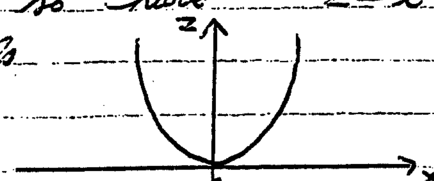
i) $x^2+y^2+z^2=1$ we know this is the sphere of radius 1 centred around the origin, so we can easily draw it:



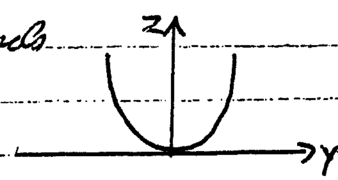
ii) $z=x^2+y^2$ we know that we'll always have $z \geq 0$ and $z=0$ only if $x=y=0$

look at what happens in the xz and yz planes:

in the xz plane, $y=0$ so have $z=x^2$, a parabola opening upwards

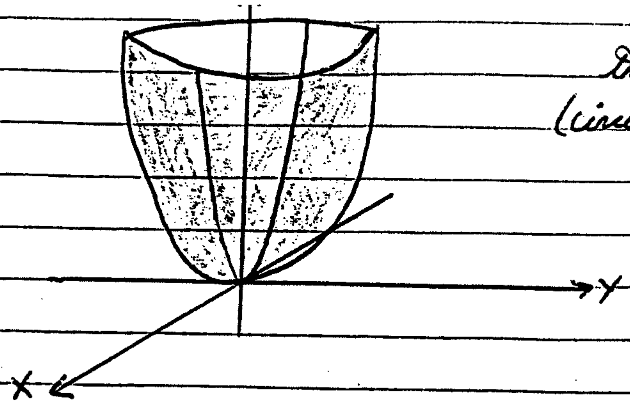


in the yz plane, $x=0$ and we have $z=y^2$, also a parabola opening upwards



and in a plane $z=c$, $x^2+y^2=c$, which is a circle so put all of this information together to get that the graph $z=f(x,y)=x^2+y^2$ looks like:

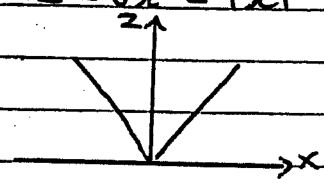
(66)

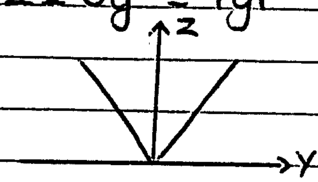


this is called a (circular) paraboloid

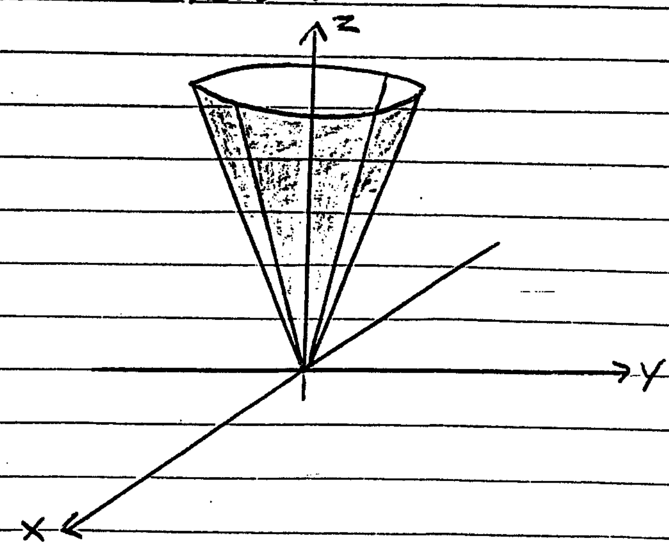
iii, $z=\sqrt{x^2+y^2}$ because $\sqrt{x^2+y^2}$ is the positive root by definition, we'll have $z\geq 0$ again but now our cross-sections are the following

$z=\sqrt{x^2}=|x|$ and $z=\sqrt{y^2}=|y|$



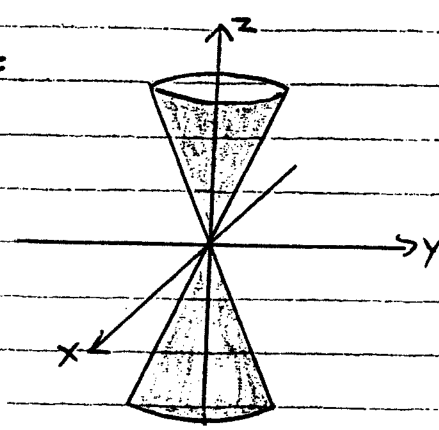


and if $z=c$, $\sqrt{x^2+y^2}=c \Rightarrow x^2+y^2=c^2$ still circles, so we have:



which is a (half) (right circular) cone

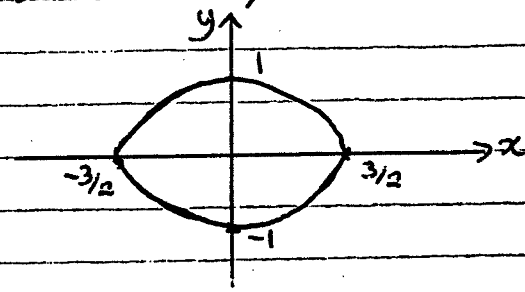
iv, $z^2=x^2+y^2$, so now $z=\pm\sqrt{x^2+y^2}$, and we now have both halves of the cone:



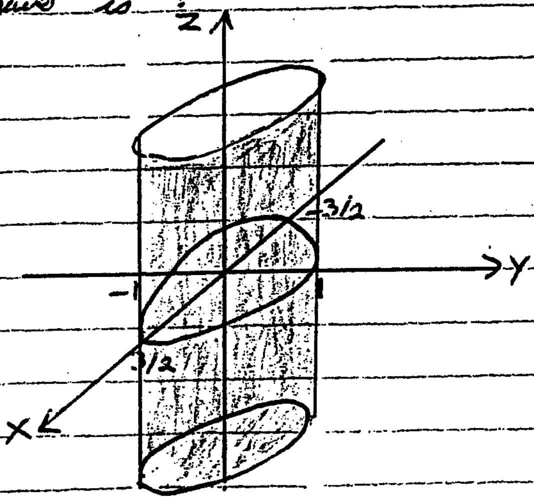
v) $4x^2 + 9y^2 = 9$

we can rewrite this as $\frac{4}{9}x^2 + y^2 = 1$ or as $\left(\frac{2}{3}x\right)^2 + y^2 = 1$

which is an ellipse in the xy plane:



but, we are in 3 space, so z not appearing in the equation means that z is free to be anything, so what we really have is:



in any plane $z = c$, have the same ellipse

this is called an (elliptical) cylinder

(show overhead)

see also Examples 5-9 on pages ~~678-83~~ 675-80

a function of two variables is a linear function if its graph is a plane (or a straight line)

so, if we have $z = f(x,y)$, a linear function must be a linear function in both variables x and y

ie it must have the form $z = ax + by + c$

for constants a, b, c

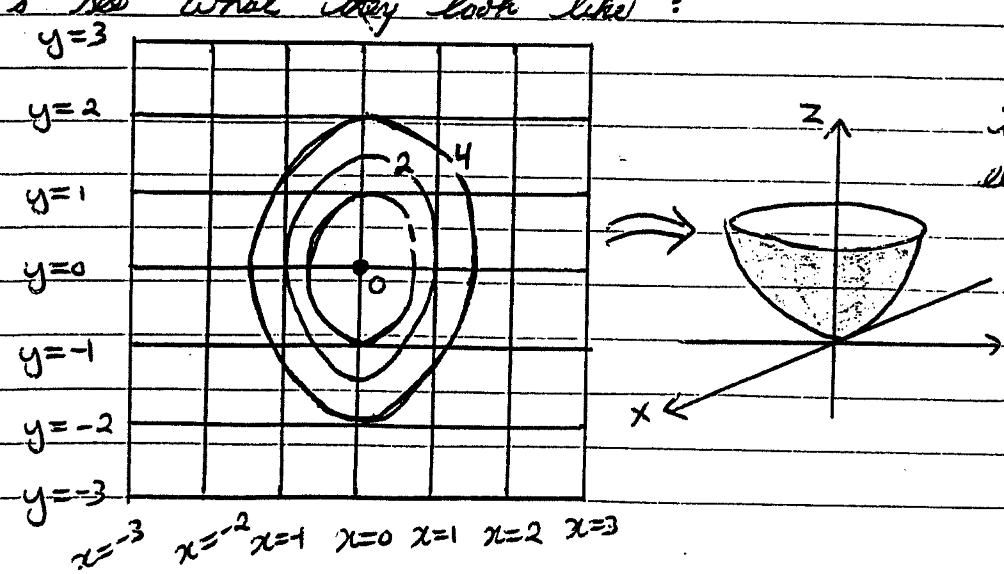
see Example 4 on page ~~678~~ 675

contour diagrams are especially useful in applications like topographical maps and weather maps
what you do is represent the third variable by joining (x,y) points that have the same $z=f(x,y)$ values by contour lines or level curves (p 741) 740
this way, $z=f(x,y)$ can be represented two-dimensionally

some general rules:
- contour lines representing different levels cannot cross
- the closer the contour lines, the steeper the surface

a contour line is the result of intersecting the surface $z=f(x,y)$ with the plane $z=c$, so it represents the set of (x,y) points where $f(x,y)=c$

example: let's look at the contours of $f(x,y)=2x^2+y^2$
now, $2x^2+y^2=c$ is an ellipse if $c>0$
if $c=0$, $2x^2+y^2=0 \Rightarrow (x,y)=(0,0)$
and if $c<0$, $2x^2+y^2=c$ is not defined
for $c=1$, the contour line is the ellipse $2x^2+y^2=1$
" $c=2$ " " " " " " $2x^2+y^2=2$
and so on...
let's see what they look like:



this is an elliptical paraboloid

see also Examples 4 p 741 740 and 6-9 p 743-4 742-3
and the plots on page 745 744

the graph of a function of one variable is a curve in 2 space
the graph of a function of two variables is a surface (or a curve) in 3 space
the graph of a function of three variables, $g(x,y,z)$, would, in general, be a solid in 4 space, which we can't draw
so we need to have ways of visualizing functions of 3 or more variables
we do this by using the techniques we've already developed

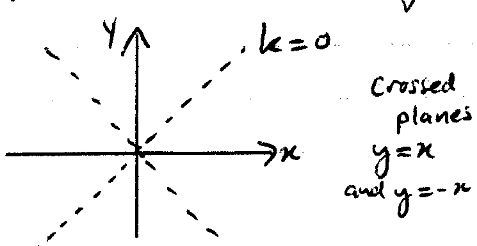
say we have function $g(x,y,z)$; what if we were to look at

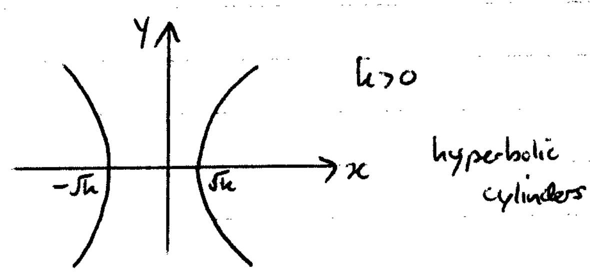
what happens when $z = \text{constant}$, ie look at $g(x,y,c)$
then we'd have a function of two variables, which we can draw as a surface in 3 space

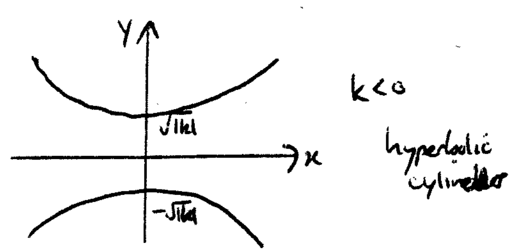
a function of three variables, $f(x,y,z)$ can be represented by a family of level surfaces of the form $f(x,y,z)=c$
then we can look at these level surfaces themselves (directly) or study them (indirectly) by looking at their contour diagrams

example: (p [illegible]) describe the level surfaces of $f(x,y,z) = x^2 - y^2$

if we look at $f(x,y,z) = k$, we have $x^2 - y^2 = k$, which is a hyperbola (as a curve)
but we must realize that this curve "repeats" itself for all z
ie we have something like







see also Example 12 page 746 745

§11.3 : Partial Derivatives

imagine we are standing at some point (x,y) on a surface $z = f(x,y)$ and we are interested in describing how the surface changes in the x and y direction (ie the slopes)

if we want to estimate $\frac{\Delta z}{\Delta x}$, what do we need to be sure of?

we need to be sure that the Δz we measure has been caused by Δx alone, ie that there is no influence from a change in the y variable

so, to estimate $\frac{\Delta z}{\Delta x}$, we need to keep y constant

let's say we take a step of $\Delta x = h$, then we move from the point $(x, y, f(x,y))$ to the point $(x+h, y, f(x+h,y))$

so $\frac{\Delta z}{\Delta x} = \frac{f(x+h,y) - f(x,y)}{h}$

then the derivative with respect to x is the limit of this ratio as $h \to 0$, and we call it a partial derivative because we are differentiating with respect to only one of the variables

definition: the partial derivative of $f(x,y)$ with respect to x at the point (x,y) is $\boxed{f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}}$

similarly, when we try to get $\frac{\Delta z}{\Delta y}$, we need to be sure that there is no influence from a change in x, so we need to hold x constant and only move in the y direction

if $\Delta y = h$, then $\frac{\Delta z}{\Delta y} = \frac{f(x, y+h) - f(x,y)}{h}$

and the partial derivative of $f(x,y)$ with respect to y at the point (x,y) is $\boxed{f_y(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}}$

Notation: since $z = f(x,y)$

$$f_x(x,y) = z_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$

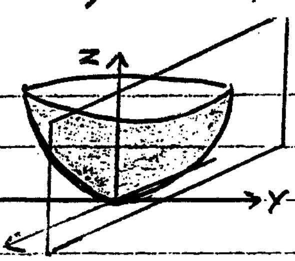
$$f_y(x,y) = z_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$$ (p 758) 758

$$f_x(a,b) = z_x(a,b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

$$f_y(a,b) = z_y(a,b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

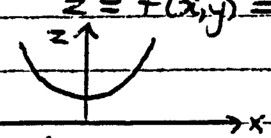
To distinguish a partial derivative $\frac{\partial f}{\partial x}$ from a "regular" derivative $\frac{df}{dx}$, we pronounce $\frac{\partial f}{\partial x}$ as "die f die x" and not as "d f d x"

how can we be sure that all of this will work?
think about what happens if we take $z = f(x,y)$ (a surface) and hold one of the variables constant, say $y = c$, what do we have?
we have the curve that results from the intersection of the surface $z = f(x,y)$ with the plane $y = c$
and this curve will be a function of x only (see page 760)



for example, if $f(x,y) = x^2 + y^2$

and we cut at the plane $y = 2$,
we'll have: $z = f(x,y) = x^2 + 4$, which looks like:



(just a parabola)

(See Ex 2 p 760 -1)

759-60

so talking about the derivative with respect to x is fine
we said that to get $\frac{\partial f}{\partial x}$, we were differentiating with respect to x and holding y constant (p 759) 758
and this is exactly how we do it
also, all of the rules from single-variable calculus still hold

above, we had $f(x,y) = x^2+y^2$

so $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) = 2x$

(holding y constant $\Rightarrow y^2$ constant)

and $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 2y$ (now holding x constant)

another example:

if $f(x,y) = 2xy + 3y^2 - x^2$

then $\frac{\partial f}{\partial x} = 2y - 2x$ and $\frac{\partial f}{\partial y} = 2x + 6y$

note: even though we like to use $z = f(x,y)$, the x, y and z are just labels, we could just as easily be talking about $w = f(u,v)$ and ask about $\frac{\partial f}{\partial u}$ or $\frac{\partial f}{\partial v}$

e.g. if $f(u,v) = u^3 - v^2$, $\frac{\partial f}{\partial u} = 3u^2$ and $\frac{\partial f}{\partial v} = -2v$

what about a function of 3 (or more) variables, like $f(x,y,z)$?

then $\frac{\partial f}{\partial x}$ means differentiate with respect to x, holding y and z constant

$\frac{\partial f}{\partial y}$ means differentiate with respect to y, holding x and z constant

and we'll also have $\frac{\partial f}{\partial z}$, which means differentiate with respect to z, holding x and y constant

example: $f(x,y,z) = e^{ax+by+cz}$

then $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{ax+by+cz}) = e^{ax+by+cz}\frac{\partial}{\partial x}(ax+by+cz)$

$= a e^{ax+by+cz}$

ie the Chain Rule works as before

so $\frac{\partial f}{\partial y} = b e^{ax+by+cz}$ and $\frac{\partial f}{\partial z} = c e^{ax+by+cz}$

more examples:

i) $f(x,y) = x\cos(xy) + 3$

$$\frac{\partial f}{\partial x} = \left(\frac{\partial}{\partial x}(x)\right)\cos(xy) + x\left(\frac{\partial}{\partial x}(\cos(xy))\right) + \frac{\partial}{\partial x}(3)$$ ie the Product Rule works, too

$$= \cos(xy) - xy\sin(xy)$$

and $\frac{\partial f}{\partial y} = -x^2\sin(xy)$

ii, $f(x,y,z) = xyz\ln\left(\frac{xy}{z}\right)$

$$\frac{\partial f}{\partial x} = yz\ln\left(\frac{xy}{z}\right) + xyz\left(\frac{z}{xy}\right)\left(\frac{y}{z}\right) = yz\ln\left(\frac{xy}{z}\right) + yz$$

$$\frac{\partial f}{\partial y} = xz\ln\left(\frac{xy}{z}\right) + xyz\left(\frac{z}{xy}\right)\left(\frac{x}{z}\right) = xz\ln\left(\frac{xy}{z}\right) + xz$$

and $\frac{\partial f}{\partial z} = xy\ln\left(\frac{xy}{z}\right) + xyz\left(\frac{z}{xy}\right)\left(\frac{-xy}{z^2}\right) = xy\ln\left(\frac{xy}{z}\right) - xy$

iii, $f(x,y) = 3x + 7xy^2 + 3y^2$

$$\frac{\partial f}{\partial x} = 3 + 7y^2 \qquad \frac{\partial f}{\partial y} = 14xy + 6y$$

$$f_x(1,1) = 10 \qquad f_y(1,1) = 20$$

iv, $g(x,y) = \ln(xy\cos(xy))$

then $\frac{\partial g}{\partial x} = \frac{1}{xy\cos(xy)}\frac{\partial}{\partial x}\left(xy\cos(xy)\right)$

$$= \frac{1}{xy\cos(xy)}\left[y\cos(xy) - xy\sin(xy)\frac{\partial}{\partial x}(xy)\right]$$

$$= \frac{y\cos(xy) - xy^2\sin(xy)}{xy\cos(xy)}$$

and $\frac{\partial g}{\partial y} = \frac{x\cos(xy) - x^2y\sin(xy)}{xy\cos(xy)}$

see also the examples 1,3,4,5 p 759, 761-2 6th)

when we looked at functions of a single variable, $y = f(x)$, we defined the derivative $\frac{dy}{dx} = y'(x) = f'(x) = \frac{df}{dx}$

and then we could define the second derivative as the derivative of the derivative

ie $\quad y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

we can also define <u>second partial derivatives</u> : (p 762)

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$

example: $\quad f(x,y) = x^2 y + 3\ln(xy) - \cos(x+y)$

$$\frac{\partial f}{\partial x} = 2xy + \frac{3}{xy}y + \sin(x+y) = 2xy + \frac{3}{x} + \sin(x+y)$$

$$\frac{\partial^2 f}{\partial x^2} = 2y - \frac{3}{x^2} + \cos(x+y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x + \cos(x+y)$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{3}{xy}x + \sin(x+y) = x^2 + \frac{3}{y} + \sin(x+y)$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{3}{y^2} + \cos(x+y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x + \cos(x+y)$$

notice that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ this is no fluke:

<u>Clairaut's theorem</u> : (p 763)

suppose f is defined on a disk D that contains the point (a,b)

if f_{xy} and f_{yx} are both continuous on D, then
$$f_{xy}(a,b) = f_{yx}(a,b)$$

see also example 6 p 762 (ok)

we can define even further partial derivatives like
$$f_{xyx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$
$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x}, \text{ etc...}$$

if $f(x,y) = x^2y^2 + xy^3$

then
$f_x = 2xy^2 + y^3$ $\qquad$ $f_y = 2x^2y + 3xy^2$
$f_{xy} = 4xy + 6y^2$ $\qquad$ $f_{yx} = 4xy + 3y^2$
$f_{xyx} = 4y$ $\qquad$ $f_{yxx} = 4y$

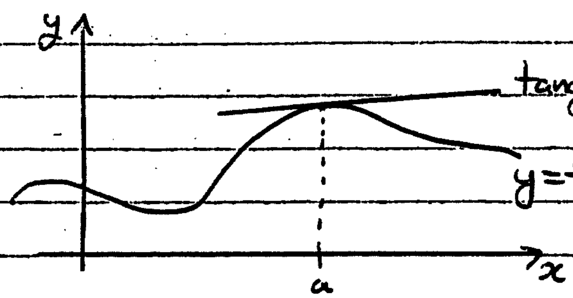
$f_{xx} = 2y^2$
$f_{xxy} = 4y$ $\qquad$ ie $f_{xyx} = f_{yxx} = f_{xxy}$

See Ex 7 p 764 (ok)

and so on...

§11.4: Tangent Planes and Linear Approximations

with a function of one variable, $y = f(x)$, we know that if we zoom in close enough, the curve will be approximately linear (ie it'll look like a straight line)
this is what allows us to make approximation by tangent lines



near $x=a$, the curve will be close to tangent line

now, imagine what happens if we zoom in on the graph of a surface $z = f(x,y)$.

"as we zoom in, the graph will get flatter and flatter
ie the surface will look like a plane close up
we know from our study of partial derivatives, that the surface has two tangent lines - one for x and one for y and these two tangent lines can be used to define the <u>tangent plane</u> to the surface

(see pages 770 (a))

if we use the fact $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ give us the slopes in the x and y directions and also that we know the equation for a plane from the x and y slopes, we have:

the <u>tangent plane</u> to the surface $z = f(x,y)$ at the point (a,b) is

(p 770 (a)) $$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

example: find the tangent plane to the surface $z = \frac{1}{2}x^2 + y^3$ at the point $(1,1)$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = x \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = 3y^2$$

so at $(1,1)$ (ie $x=y=1$), $f_x(1,1) = 1$, $f_y(1,1) = 3$

$\therefore$ the tangent plane is
$$\begin{aligned} z &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &= \tfrac{1}{2}(1)^2 + (1)^3 + (x-1) + 3(y-1) \\ &= \tfrac{1}{2} + 1 + x - 1 + 3y - 3 \\ &= x + 3y - \tfrac{5}{2} \end{aligned}$$

now, just as we used tangent lines to approximate $y = f(x)$ near $x = a$, we can use the tangent plane to approximate $z = f(x,y)$ near (a,b):
(this is called (<u>local</u>) <u>linearization</u>)

(p 772) (a)

for (x,y) near (a,b),
$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

example: let's see how good the tangent plane approximation is for our function $z = f(x,y) = \frac{1}{2}x^2 + y^3$ near $(1,1)$

at the point $(1.05, 0.95)$, $z = \frac{1}{2}(1.05)^2 + (0.95)^3 = 1.408625$
and the tangent plane approximation would give us

$$f(1.05, 0.95) \approx 1.05 + 3(0.95) - 5/2 = 1.4 \quad \text{(not bad)}$$

another example: find the tangent plane to the surface
$z = x^2y^2 + xy - 3x$ at the point $(-1, 1, 3)$

$$z_x = f_x(x,y) = 2xy^2 + y - 3 \qquad f_x(-1,1) = -4$$
$$z_y = f_y(x,y) = 2x^2y + x \qquad f_y(-1,1) = 1$$

so $z = 3 + (-4)(x - (-1)) + (1)(y - 1)$
$= -4x + y - 2$

see also Ex 1-3 pages 771 + 773-4 (de)

the Differential:
let's look at our expression for local linearization again:
$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
or $f(x,y) - f(a,b) \approx f_x(a,b)(x-a) + f_y(a,b)(y-b)$

define $\Delta x = x - a$ and $\Delta y = y - b$
then $\Delta f = f(x,y) - f(a,b)$
so the above expression can be rewritten as
$$\Delta f \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$
now replace the Δ's with d's to get the differential:

the differential (df or dz) of f at the point (a,b) is given by

$$\boxed{df = f_x(a,b)\,dx + f_y(a,b)\,dy} \quad \text{(p 774)}$$

or, at the point (x,y)

$$\boxed{\begin{aligned} df &= f_x\,dx + f_y\,dy \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \end{aligned}}$$

examples: i, (p 779 #24) $z = x^2 - xy + 3y^2$

(x,y) changes from $(3,-1)$ to $(2.96, -0.95)$

compare Δz and dz

$$\begin{aligned}\Delta z &= f(2.96,-0.95) - f(3,-1)\\ &= \left((2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2\right) - \left(3^2 - 3(-1) + 3(-1)^2\right)\\ &= 14.2811 - 15\\ &= -0.7189\end{aligned}$$

$$\begin{aligned}dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\\ &\approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = (2x-y)\Delta x + (6y-x)\Delta y\\ &= (2(3)-(-1))(2.96-3) + (6(-1)-3)(-0.95--1)\\ &= -0.73\end{aligned}$$

ii, a spherical metal ball of radius 1 m is to be covered in paint of thickness 1 mm

how much paint is required?

$$\Delta V \approx dV = \frac{dV}{dr}dr = 4\pi r^2 dr \approx 4\pi r^2 \Delta r = 4\pi(1)^2(0.001)$$
$$\simeq 0.01257\ m^3$$

see also Examples 4–6 on pages 775–6 (di)

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§11.5: the Chain Rules

in single variable calculus, the derivative of the composite function $y = f(g(x))$ is always $\frac{dy}{dx} = f'(g(x))\,g'(x)$

or $\frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dx}$

for functions of two variables, the formula for the Chain Rule will depend on what the <u>intermediate variables</u> depend on

(p 782)

the best way to see the Chain Rule is to use tree diagram (p782)

consider $z = f(x,y)$ where x and y are functions of t
then we have $z = f(x(t), y(t))$
so z depends on t and we could ask about the rate of change of z wrt t ie $\frac{dz}{dt}$

draw a tree diagram

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & & \text{dependent variable} \\ x \arrow[dr, "\frac{dx}{dt}"'] & & y \arrow[dl, "\frac{dy}{dt}"] & \text{intermediate variables} \\ & t & & \text{independent variable} \end{tikzcd}$$

over each branch, write the appropriate derivative
from z to t, there are 2 paths – one through x, the other through y
to get the derivative, take the products of the derivatives along each path and then sum over the paths

ie $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$ (case I) (p750)

or $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
the differential

examples:

i) $z = x^2 + y^2$ where $x = \cos(2t)$, $y = \sin(2t)$

then $\frac{\partial z}{\partial x} = 2x \qquad \frac{dx}{dt} = -2\sin(2t)$

$\frac{\partial z}{\partial y} = 2y \qquad \frac{dy}{dt} = 2\cos(2t)$

so $$\begin{aligned}\frac{dz}{dt} &= (2x)(-2\sin(2t)) + (2y)(2\cos(2t)) \\ &= (2\cos(2t))(-2\sin(2t)) + (2\sin(2t))(2\cos(2t)) \\ &= 0\end{aligned}$$

not surprising since $z = x^2 + y^2 = (\cos(2t))^2 + (\sin(2t))^2 = 1$

ii) find $\frac{dz}{dt}$ at $t=1$ if $z = \cos(x^2) + xy^2$,
$x = \ln t$, $y = t^2+3t$

$$\frac{\partial z}{\partial x} = -2x\sin(x^2) + y^2 \qquad \frac{dx}{dt} = \frac{1}{t}$$

$$\frac{\partial z}{\partial y} = 2xy \qquad \frac{dy}{dt} = 2t+3$$

if $t=1$ $x=0$, $y=4$

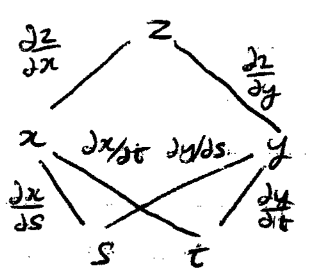
so $\frac{\partial z}{\partial x} = 16$, $\frac{\partial z}{\partial y} = 0$ and $\frac{dz}{dt} = (16)(1) + (0)(5) = 16$

$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 5$

see also Examples 1 & 2 p 781

another possibility is that we could have $z = f(x,y)$
where $x = g(s,t)$ and $y = h(s,t)$
ie x and y are functions of 2 variables themselves

then the tree diagram looks like



(all derivatives will be partials)

and
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
(case II)
(p 782)
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

examples:

i) $z = 3x^2y^3 + x - \ln y$ where $x = e^s - t$
$y = s^2t - \cos t$

$\frac{\partial z}{\partial x} = 6xy^3 + 1$
$= 6(e^s - t)(s^2t - \cos t)^3 + 1$

$\frac{\partial z}{\partial y} = 9x^2y^2 - \frac{1}{y}$
$= 9(e^s - t)^2(s^2t - \cos t)^2 - \frac{1}{s^2t - \cos t}$

$\frac{\partial x}{\partial s} = e^s$, $\frac{\partial x}{\partial t} = -1$, $\frac{\partial y}{\partial s} = 2ts$, $\frac{\partial y}{\partial t} = s^2 + \sin t$

so $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left[6(e^s - t)(s^2t - \cos t)^3 + 1\right](e^s)$
$+ \left[9(e^s - t)^2(s^2t - \cos t)^2 - \frac{1}{s^2t - \cos t}\right](2st)$

and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left[6(e^s - t)(s^2t - \cos t)^3 + 1\right](-1)$
$(s^2 + \sin t)\left[9(e^s - t)^2(s^2t - \cos t)^2 - \frac{1}{s^2t - \cos t}\right]$

ii, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ if $s = t = 0$

then $x = 1$, $y = -1$, $\frac{\partial z}{\partial x} = -5$, $\frac{\partial z}{\partial y} = 10$,

$\frac{\partial x}{\partial s} = 1$, $\frac{\partial x}{\partial t} = -1$, $\frac{\partial y}{\partial s} = 0$, $\frac{\partial y}{\partial t} = 0$

so $\frac{\partial z}{\partial s} = (-5)(1) + (10)(0) = -5$

$\frac{\partial z}{\partial t} = (-5)(-1) + (10)(0) = 5$

iii, $z = x + y^2$ $\quad x = s^2t^2$, $y = st^3 - t$

$\frac{\partial z}{\partial x} = 1 \quad \frac{\partial z}{\partial y} = 2y = 2(st^3 - t)$

$\frac{\partial x}{\partial s} = 2st^2$, $\frac{\partial x}{\partial t} = 2s^2t$, $\frac{\partial y}{\partial s} = t^3$, $\frac{\partial y}{\partial t} = 3st^2 - 1$

$\frac{\partial z}{\partial s} = (1)(2st^2) + 2(st^3 - t)(t^3) = 2st^2 + 2st^6 - 2t^4$

$\frac{\partial z}{\partial t} = (1)(2s^2t) + 2(st^3 - t)(3st^2 - 1) = 2s^2t + 6s^2t^5 - 6st^3$
$- 2st^3 + 2t$

or $z = x + y^2 = s^2t^2 + (st^3 - t)^2 = s^2t^2 + s^2t^6 - 2st^4 + t^2$

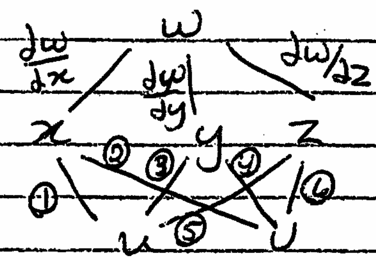
so $\dfrac{\partial z}{\partial s} = 2st^2 + 2st^6 - 2t^4$

$\dfrac{\partial z}{\partial t} = 2s^2t + 6s^2t^5 - 8st^3 + 2t$

see also Example 3 p 782

iii, find $\dfrac{\partial w}{\partial u}$ for $w = f(x(u,v), y(u,v), z(u,v))$

draw the diagram:



w depends on x, y, z

and these depend on u, v

① $\dfrac{\partial x}{\partial u}$ ② $\dfrac{\partial x}{\partial v}$

③ $\dfrac{\partial y}{\partial u}$ ④ $\dfrac{\partial y}{\partial v}$

⑤ $\dfrac{\partial z}{\partial u}$ ⑥ $\dfrac{\partial z}{\partial v}$

from w to u, there are three paths through x, y and z

(see the generalized rule p 783)

so $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

similarly, $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

so if $w = x^2y^2 + y^2z^2$ where $x = uv$, $y = u^2+v^2$ and $z = u - v^3$,

then $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= 2xy^2(v) + (2x^2y + 2yz^2)(2u) + (2y^2z)(1)$$

$$= 2(uv)(u^2+v^2)^2(v) + \left[2u^2v^2(u^2+v^2) + 2(u^2+v^2)(u-v^3)^2\right](2u) + 2(u^2+v^2)^2(u-v^3)$$

see also Examples 4-7 on pages 783-4

if we have an implicit function $F(x,y) = 0$, then y is being defined as an (implicit) function of x, ie we have $F(x, y(x)) = 0$ using case 1 (p 780) of the Chain Rule, we'd have (if we

(p 785)

differentiate wrt x) : $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = \frac{-F_x}{F_y}$

example : $x^2y^2 + 6xy + y^3 = e^x$

rewrite as $F(x,y) = x^2y^2 + 6xy + y^3 - e^x = 0$

then $\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(2xy^2 + 6y - e^x)}{2x^2y + 6x + 3y^2}$

see Ex 8 p 785 (ch)

if we have $F(x,y,z) = 0$, where $z = f(x,y)$, we'd have

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cancel{\frac{\partial y}{\partial x}}^{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

(p 795)

and $$\frac{\partial F}{\partial x}\cancel{\frac{\partial x}{\partial y}}^{0} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

example : $x^2y + xyz + x = 2z^2$

or $F(x,y,z) = x^2y + xyz + x - 2z^2$

so $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(2xy + yz + 1)}{xy - 4z}$ $\quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(x^2 + xz)}{xy - 4z}$

see Ex 9 p 786

§ 11.6 : Directional Derivatives and the Gradient Vector

consider the function $z = f(x,y)$

assume we are at the point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$ on the surface

we know that the partial derivatives

$$f_x(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

and $$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

can tell us about the rates of change in the x and y directions

but what if we want to know the rate of change of $z = f(x,y)$ in an arbitrary direction specified by the unit vector $\hat{u}$?

then we need the <u>directional derivative</u> (p 790)

we take a small step in the direction of $\hat{u}$ and look at the rate of change by taking the limit

$\hat{u} = (a,b)$

ie if $\hat{u} = a\hat{\imath} + b\hat{\jmath}$, then $h\hat{u} = ha\hat{\imath} + hb\hat{\jmath}$

so we move from point (x_0, y_0) to $(x_0 + ha, y_0 + hb)$

then the change in the value of the function is

$$f(x_0 + ha, y_0 + hb) - f(x_0, y_0)$$

and we have moved a distance of $h\|\hat{u}\| = h$

so the average rate of change over the step is

$$\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

and then take the limit to get:

the directional derivative of f at (x_0, y_0) in the direction of the unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$ is

$$D_{\hat{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

(p 790)

note: this formula reduces to f_x and f_y under appropriate conditions

to actually do calculations, we typically use the formula:

$$D_{\hat{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b \qquad \text{(Thm 3 p 791)}$$

notice that this can be written as

$$\begin{aligned} D_{\hat{u}} f(x,y) &= (f_x(x,y), f_y(x,y)) \cdot (a,b) \\ &= (f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}) \cdot (a\hat{\imath} + b\hat{\jmath}) \\ &= (f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath}) \cdot \hat{u} \end{aligned}$$

define $f_x(x,y)\,\hat{\imath} + f_y(x,y)\,\hat{\jmath} = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = \nabla f = \text{grad } f$

to be the <u>gradient vector</u> of $f(x,y)$

the gradient vector has some special properties:

the direction of grad f is: i) perpendicular to the contours
ii) in direction of $\overset{\text{max}}{\text{increasing}}$ f

the magnitude $\|\text{grad} f\|$ is: i) the maximum rate of change of f at (x,y)
ii) large when contours close together and small when they're not

(see pages 792-3)

so we have that $D_{\hat{u}} f(x,y) = \nabla f \cdot \hat{u}$ (p 792)

or at the point (x_0, y_0), $D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$

examples:

i) find the directional derivative of $f(x,y) = x \sin y$ in the direction of $2\hat{i} + \hat{j}$ at the point $(1, \pi)$

the gradient is $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \sin y\, \hat{i} + x \cos y\, \hat{j}$

$$\nabla f(1,\pi) = \sin\pi\, \hat{i} + (1)\cos\pi\, \hat{j} = -\hat{j}$$

(dot product: $\vec{u} = u_1\hat{i} + u_2\hat{j}$, $\vec{v} = v_1\hat{i} + v_2\hat{j}$, $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2$)

the unit vector in the direction of $\vec{u} = 2\hat{i} + \hat{j}$ is $\hat{u} = \frac{2\hat{i}+\hat{j}}{\|2\hat{i}+\hat{j}\|} = \frac{2\hat{i}+\hat{j}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\hat{i} + \frac{\hat{j}}{\sqrt{5}}$

so the directional derivative is

$$D_{\hat{u}} f(1,\pi) = \nabla f(1,\pi) \cdot \hat{u}$$
$$= (-\hat{j}) \cdot \left(\frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}\right) = \frac{-1}{\sqrt{5}}$$

so the function is decreasing in this direction

ii) $f(x,y) = e^x y - x^2 y^3 + x$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = (e^x y - 2xy^3 + 1)\hat{\imath} + (e^x - 3x^2y^2)\hat{\jmath}$$

at $(1,1)$, $\nabla f(1,1) = (e-1)\hat{\imath} + (e-3)\hat{\jmath}$

the directional derivative in the direction of $\hat{u} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$

is
$$D_{\hat{u}} f(1,1) = ((e-1)\hat{\imath} + (e-3)\hat{\jmath})\cdot\left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right)$$
$$= \frac{1}{\sqrt{2}}(e-1) + \frac{1}{\sqrt{2}}(e-3)$$
$$= \sqrt{2}e - 2\sqrt{2}$$

in the direction of $\hat{u} = \frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$

$$D_{\hat{u}} f(1,1) = \frac{1}{\sqrt{2}}(e-1) - \frac{1}{\sqrt{2}}(e-3) = \sqrt{2}$$

see also examples 1 – 4 on pages 790 – 3

for a function of three variables $f(x,y,z)$, we'd have

(p793) $D_{\hat{u}} f(x,y,z) = \nabla f \cdot \hat{u}$ where $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$

example: find the directional derivative of $f(x,y,z) = xyz^2$ at $(1,2,-1)$ in the direction of $2\hat{\imath} + \hat{\jmath} - 3\hat{k}$

$\nabla f = yz^2\,\hat{\imath} + xz^2\,\hat{\jmath} + 2xyz\,\hat{k}$, $\nabla f(1,2,-1) = 2\hat{\imath} + \hat{\jmath} - 4\hat{k}$

$$\hat{u} = \frac{2\hat{\imath} + \hat{\jmath} - 3\hat{k}}{\sqrt{(2)^2 + (1)^2 + (-3)^2}} = \frac{2\hat{\imath}}{\sqrt{14}} + \frac{1}{\sqrt{14}}\hat{\jmath} - \frac{3}{\sqrt{14}}\hat{k}$$

so
$$D_{\hat{u}} f(1,2,-1) = \nabla f(1,2,-1)\cdot\hat{u} = \frac{4 + 1 + 12}{\sqrt{14}} = \frac{17}{\sqrt{14}} \simeq 4.5434$$

see also Ex 5 p 794

from the properties of the gradient, we know that the maximum value of $D_{\hat{u}} f$ is $|\nabla f|$ and it occurs when $\hat{u}$ and ∇f have the same direction (Thm p794

so in an example above, the max value of $D_{\hat{u}} f$ is
$|\nabla f(1,2,-1)| = |2\hat{\imath} + \hat{\jmath} - 4\hat{k}| = \sqrt{21}$
and it occurs in the direction of $2\hat{\imath} + \hat{\jmath} - 4\hat{k}$

see Examples 6 and 7 on pages 795-6

if $F(x,y,z) = k$ is a level surface to a function of three variables $F(x,y,z)$ and $P_0 = (x_0, y_0, z_0)$ is a point on this surface, then the tangent plane to the level surface at this point is

(p.796) $F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

and the normal line (perpendicular to the plane) passing through the point is
$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

these equations follow from the fact that the gradient vector $\nabla F(x_0,y_0,z_0)$ will be perpendicular to the level surface (see pages 805-6)

example: (p 800 # 37) $\quad z+1 = xe^y\cos z, \ (1,0,0)$
$F(x,y,z) = z - xe^y\cos z = 1$
$\nabla F = (-e^y\cos z)\hat{\imath} + (-xe^y\cos z)\hat{\jmath} + (1+xe^y\sin z)\hat{k}$
$\nabla F(1,0,0) = -1\hat{\imath} - 1\hat{\jmath} + \hat{k}$
so the tangent plane is $\quad -(x-1) - (y-0) + (z-0) = 0$
or $\quad z = x+y-1$

and the normal line is $\quad \dfrac{x-1}{-1} = \dfrac{y-0}{-1} = \dfrac{z-0}{1}$

or $\quad 1-x = -y = z \quad$ or $\quad x-1 = y = -z$

see also Ex 8 p 797