Midterm Exam II

November 18, 2011

No books. No notes. No calculators. No electronic devices of any kind.

Name ____________________________

Student Number __________________

Problem 1. (3 points)
Find the determinant of the matrix

$$\begin{pmatrix} 0 & 0 & x & 0 & 0 \\ 2 & 3 & 7 & 0 & 2 \\ 9 & 1 & 0 & 0 & 2 \\ x & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 5 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 & 0 & x & 0 & 0 \\ 2 & 3 & 7 & 0 & 2 \\ 9 & 1 & 0 & 0 & 2 \\ x & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 5 & 0 \end{vmatrix} \leftarrow \text{expand Row I}$$

expand Col III

$$= x \begin{vmatrix} 2 & 3 & 0 & 2 \\ 9 & 1 & 0 & 2 \\ x & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \end{vmatrix} = -5x \begin{vmatrix} 2 & 3 & 2 \\ 9 & 1 & 2 \\ x & 0 & 0 \end{vmatrix} \leftarrow \text{expand Row III}$$

$$= -5x^2 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = -5x^2(6-2) = \underline{\underline{-20x^2}}$$

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Problem ~~1~~ 2. (3 points)
Find the inverse of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ -\mathrm{II} \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \cdot(-1) \end{array} \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \begin{array}{l} -\mathrm{III} \\ -\mathrm{III} \\ \\ \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\boxed{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}}$$

1	2	3	4	5	6	total/25

3

Problem ~~2~~. (4 points)
The matrix of a linear transformation T is

$$\begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 4 & 2 & 1 & 2 \end{pmatrix}$$

(a) Find a basis for the range of T.

(b) The dimension of the domain of T is # colums(A) = 5.

(c) The dimension of the null space of T is # free vars = 3.

$$\begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 4 & 2 & 1 & 2 \end{pmatrix}\begin{matrix} \\ -\mathrm{I} \\ -2\mathrm{I} \end{matrix} \to \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}\begin{matrix} \\ \\ -\mathrm{II} \end{matrix} \to \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

row echelon form.

pivot columns = II and IV, so a basis for the range of T is $\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(remember: range(T) = Column space of [T].)

Problem 4. (4 points)
Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transormation.

The scalar λ is called an *eigenvalue* of T, and the non-zero vector $\vec{v}$ is called an *eigenvector* of T, *corresponding to the eigenvalue* λ, if

$$T(\vec{v}) = \lambda\vec{v}.$$

Suppose that T has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$, and that $\vec{v}_1 = \binom{-1}{1}$ is an eigenvector of T corresponding to the eigenvalue 3 and $\vec{v}_2 = \binom{2}{-1}$ is an eigenvector of T corresponding to the eigenvalue -1.

(a) Find the matrix of T with respect to the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$.

(b) Find the standard matrix of T.

(a) $T(\vec{v}_1) = 3\vec{v}_1 = 3\vec{v}_1 + 0\vec{v}_2$

$T(\vec{v}_2) = -\vec{v}_2 = 0\vec{v}_1 - 1\vec{v}_2$

so $[T]_{\mathcal{B}} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) the change of basis matrix is $P = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$.

$$[T] = P[T]_{\mathcal{B}}P^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -8 \\ 4 & 7 \end{pmatrix}$$

Problem 5. (5 points)
Find the steady state vector of the Markov process with transition matrix A an initial state $\vec{x}_0$.

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{2}{3} \end{pmatrix} \qquad \vec{x}_0 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

Space of fixed points is $Nul(A - I)$:

$$A - I = \begin{pmatrix} -1 & 1/2 & 1/3 \\ 1/2 & -3/4 & 0 \\ 1/2 & 1/4 & -1/3 \end{pmatrix} \begin{matrix} \cdot(-1) \\ \cdot 2 \\ \cdot 2 \end{matrix} \rightarrow \begin{pmatrix} 1 & -1/2 & -1/3 \\ 1 & -3/2 & 0 \\ 1 & 1/2 & -2/3 \end{pmatrix} \begin{matrix} \\ -I \\ -I \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1/2 & -1/3 \\ 0 & -1 & 1/3 \\ 0 & 1 & -1/3 \end{pmatrix} \begin{matrix} \\ \\ +II \end{matrix} \rightarrow \begin{pmatrix} 1 & -1/2 & -1/3 \\ 0 & -1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \cdot(-1) \\ \\ \end{matrix} \rightarrow \begin{pmatrix} 1 & -1/2 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} +\frac{1}{2}II \\ \\ \\ \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{gen'l solution} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} z \\ 1/3\, z \\ z \end{pmatrix} = z \begin{pmatrix} 1/2 \\ 1/3 \\ 1 \end{pmatrix}.$$

total amount $x + y + z = \text{const.} = 2 + 4 + 5 = 11$.

need z s.t. $\quad 1/2\, z + 1/3\, z + z = 11$

$$\frac{3}{6} z + \frac{2}{6} z + \frac{6}{6} z = 11$$

$$\frac{11}{6} z = 11$$

$$\underline{\underline{z = 6}}$$

steady state vector is $\underline{\underline{\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}}}$

so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6 \begin{pmatrix} 1/2 \\ 1/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$

Problem 6. (6 points)
The following dynamical system is given.

$$\begin{aligned} x_{n+1} &= y_n & x_0 &= 3 \\ y_{n+1} &= 2x_n + y_n & y_0 &= 0 \end{aligned}$$

(a) Find explicit formulas for x_n, and y_n.
(b) Find the limiting growth rate of x_n, i.e., find $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$.
(c) Find the limiting proportion of x_n to y_n, i.e., find $\lim_{n\to\infty} \frac{x_n}{y_n}$.
(d) Can you find a non-trivial initial condition that will assure that the state vector stays bounded, i.e., does not grow beyond all bounds?

$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$. transition matrix $= A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

Charpol. $= \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda(\lambda-1) - 2 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$

eigenvalues are $\lambda = 2$ and $\lambda = -1$.

eigenvector for $\lambda = 2$: $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

eigenvector for $\lambda = -1$: $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

general solution: $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 2^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 (-1)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

initial condition: $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$ $\quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 2^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1)^n \begin{pmatrix} 2 \\ -2 \end{pmatrix}$.

(a) $\boxed{\begin{aligned} x_n &= 2^n + 2(-1)^n \\ y_n &= 2 \cdot 2^n - 2(-1)^n \end{aligned}}$

(b) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2^{n+1} - 2(-1)^{n+1}}{2^n - 2(-1)^n} = \lim_{n\to\infty} \frac{2\cdot 2^n + 2(-1)^n}{2^n - 2(-1)^n}$

$= \lim_{n\to\infty} 2 \frac{1 + 2(-\frac{1}{2})^n}{1 - 2(-\frac{1}{2})^n} = 2 \frac{1+0}{1-0} = 2.$

(c) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{1}{2}$

(d) $c_1 = 0$ $c_2 = 1$ works. this would be

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

with initial condition $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ the system oscillates between the two states $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

So the state vectors stay bounded.

(only the eigenvalue $\lambda = -1$ is used because the state vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are in the eigenspace E_{-1}, so they stay in E_{-1}.)