MAT 1322, MIDTERM 1, VERSION A

1. [2 points, 7.2 #19] Use Euler's method with step size 0.2 to estimate y(0.4), where y is the solution of the initial value problem y'= 2x+y , y(0) =1.

A. 1.2 B. 1.46 C. 1.56 D. 1.48 **E. 1.52** F.1.44

Solution: We are given $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = 2x + y$. We then have

$$\begin{aligned} y_1 &= y_0 + hF(x_0, y_0) = 1 + 0.2(0 + 1) = 1.2 \simeq y(0.2) \\ y_2 &= y_1 + hF(x_1, y_1) = 1.2 + 0.2(0.4 + 1.2) = 1.52 \simeq y(0.4). \end{aligned}$$

2. [2 points, 7.3 #9] Find y(2) if y is the solution of the initial value problem $\frac{dy}{dx}$= xy$^2 + x$, $y(0) = 0$

A. -2.185 B. 2.321 C. 2 D. 4 E. 6 F. -6.80

Solution: Writing the separable equation $\frac{dy}{dx}$= xy$^2 + x$ in differential form and integrating both sides, we have

$$(y^2+1)dy = xdx \Leftrightarrow \int (y^2+1)dy = \int xdx \Leftrightarrow \arctan(y) = \frac{x^2}{2}+C \Leftrightarrow y = \tan\left(\frac{x^2}{2} + C\right)$$

Since $y(0) = 0$, we have $C = 0$. Therefore $y = \tan\left(\frac{x^2}{2}\right)$, so $y(2) = \tan 2 = -2.185$.

3. [2 points, 7.4 #19] Find P(ln2) if $\frac{dP}{dt}$= 2P - 6 and P(0)=4

A. 7 B. C. 5 D. 3ln(2) E. 2ln2 +6 F. 1

Solution: Writing the separable equation $\frac{dP}{dt} = 2P - 6$ in differential form and integrating both sides, we have

$$\frac{dP}{2P-6} = dt \Leftrightarrow \int \frac{dP}{2P-6} = \int dt \Leftrightarrow \frac{\ln|2P-6|}{2} = t + C.$$

Since $P(0) = 4$, we get $C = \frac{\ln 2}{2}$. Hence $\ln|2P - 6| = 2t + \ln 2$ so $|2P - 6| = e^{2t+\ln 2} = 2e^{2t} \Rightarrow |P - 3| = e^{2t}$. Then $|P(\ln 2) - 3| = e^{2\ln 2} = 4$ and since $P > 0$, we obtain $P(\ln 2) = 7$.

4. [2 points, 8.2 #15] Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$\sum_{n=1}^{\infty} 6^{n+1}\ 7^{-n}$

A.∞ B. C.7 D.6 **E**. 36 F. 25.51

Solution: We have

$$\sum_{n=1}^{\infty} 6^{n+1}7^{-n} = \sum_{n=1}^{\infty} 6 \cdot \left(\frac{6}{7}\right)^n = 6 \cdot \frac{6}{7} + 6 \cdot \left(\frac{6}{7}\right)^2 + 6 \cdot \left(\frac{6}{7}\right)^3 + \dots$$

This series is a geometric series with first term $a = \frac{36}{7}$ and common ratio $r = \frac{6}{7}$. Since $|r| < 1$, the series converges and we have

$$\sum_{n=1}^{\infty} 6 \cdot \left(\frac{6}{7}\right)^n = \frac{a}{1-r} = \frac{\frac{36}{7}}{\frac{1}{7}} = 36.$$

5. [2 points, 8.4 #11,21,23] Which series among three series below are convergent?

(1) $\sum_{n=1}^{\infty} 2^n$ n^{-4} (2) $\sum_{n=1}^{\infty}$ (-1)$^{n+1}$ $/\sqrt{n}$ (3) $\sum_{n=1}^{\infty}$ $\mathrm{n}^{-1.1}$
A. *all* B. only (1) C. only (2) D. (1) and (2) **E. (2) and (3)** F.(1) and (3)

Solution:
(1) The series $\sum_{n=1}^{\infty} 2^n n^{-4}$ diverges. Indeed, let $a_n = \frac{2^n}{n^4}$. Then

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^4}}{\frac{2^n}{n^4}} = \lim_{n\to\infty} 2\frac{(n+1)^4}{n^4} = 2$$

Since $L > 1$, the series diverges by the Ratio Test.
(2) The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges. Indeed, if $b_n = \frac{1}{\sqrt{n}}$ for all n, then we have:

$$b_{n+1} = \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} = b_n, \text{ for all } n, \text{ and } \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Therefore the series converges by the Alternating Series Test.
(3) The series $\sum_{n=1}^{\infty} n^{-1.1}$ converges. Indeed, this series is the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 1.1$. Since $p = 1.1 > 1$, this series converges.

6. [2 points, 8.5 #6,10,17] Find the radius of the convergence R of the following power series.

(1) $\sum_{n=1}^{\infty} n^2(2x-1)^n/ 3^n$
A. ∞ B.1 C.3 **D.** 3/2 E 2/3 F 1/2

Solution: The series

$$\sum_{n=1}^{\infty} \frac{n^2(2x-1)^n}{3^n} = \sum_{n=1}^{\infty} \frac{n^2 2^n}{3^n} \left(x - \frac{1}{2}\right)^n$$

is a power series about $a = \frac{1}{2}$. Let $a_n = \frac{n^2 2^n}{3^n}\left(x - \frac{1}{2}\right)^n$. Then

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|(n+1)^2 \cdot \left(\frac{2}{3}\right)^{n+1} \cdot \left(x - \frac{1}{2}\right)^{n+1} \cdot \frac{1}{n^2} \cdot \left(\frac{3}{2}\right)^n \cdot \frac{1}{\left(x-\frac{1}{2}\right)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{2}{3}\left(x - \frac{1}{2}\right)\frac{(n+1)^2}{n^2}\right| = \left|\frac{2}{3}\left(x - \frac{1}{2}\right)\right|.
\end{aligned}$$

By applying the Ratio test, the series converges if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Leftrightarrow \left|x - \frac{1}{2}\right| < \frac{3}{2}.$$

It then follows that the radius of convergence is $\frac{3}{2}$.

7. [3 points, 7.5 #3,8] Some fish population P(t) in a lake can be modelled by the logistic equation. The maximal population
for the fish of this species in that lake is 4000. The lake was stocked with 500 fish of this species and this number doubled in 6 months
Knowing that the solution of the logistic equation has a form: P(t)= K/(1+A e $^{-kt}$) :
a) Find the constants K and A.
b) Find the constant k using a year as a unit of time .
c) Find the estimated fish population in the lake after two years rounded to the closest integer.
(*Provide a detailed solution and draw a box around the final answers.*)

Solution:
a)K is the carrying capacity, so $K = 4000$. Since $P(0) = 500$, we have

$$500 = \frac{4000}{1+A} \Leftrightarrow A = 7.$$

b) We know that

$$P\left(\frac{1}{2}\right) = 1000 \Leftrightarrow \frac{4000}{1+7e^{-\frac{k}{2}}} = 1000 \Leftrightarrow e^{-\frac{k}{2}} = \frac{3}{7} \Leftrightarrow k = -2\ln\left(\frac{3}{7}\right) = 1.694$$

c)The fish population in the lake after two years will be

$$P(2) = \frac{4000}{1+7e^{-3.388}} \simeq 3235$$

MAT 1322, MIDTERM 1, VERSION B

1. [2 points, 7.2 #19] Use Euler's method with step size 0.2 to estimate y(0.4), where y is the solution of the initial value problem y'= 3x+y , y(0) =1.

A. 1.2 B. 1.46 **C. 1.56** D. 1.48 E. 1.52 F.1.44

Solution: We are given $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = 3x + y$. We then have

$$\begin{aligned} y_1 &= y_0 + hF(x_0, y_0) = 1 + 0.2(0 + 1) = 1.2 \simeq y(0.2) \\ y_2 &= y_1 + hF(x_1, y_1) = 1.2 + 0.2(0.6 + 1.2) = 1.56 \simeq y(0.4). \end{aligned}$$

2. [2 points, 7.3 #9] Find y(4) if y is the solution of the initial value problem $\frac{dy}{dx}$= xy$^2 + x$, $y(0) = 0$

A. -2.185 B. 2.321 C. 2 D. 4 E. 6 **F. -6.80**

Solution: Writing the separable equation $\frac{dy}{dx}$= xy$^2 + x$ in differential form and integrating both sides, we have

$$(y^2+1)dy = xdx \Leftrightarrow \int (y^2+1)dy = \int xdx \Leftrightarrow \arctan(y) = \frac{x^2}{2}+C \Leftrightarrow y = \tan\left(\frac{x^2}{2} + C\right)$$

Since $y(0) = 0$, we have $C = 0$. Therefore $y = \tan\left(\frac{x^2}{2}\right)$, so $y(4) = \tan 8 = -6.80$

3. [2 points, 7.4 #19] Find P(ln2) if $\frac{dP}{dt}$= 2P - 8 and P(0)=5

A. 7 B. 9 **C. 8** D. 3ln(2) E. 3ln2+6 F. 1

Solution: Writing the separable equation $\frac{dP}{dt} = 2P - 8$ in differential form and integrating both sides, we have

$$\frac{dP}{2P-8} = dt \Leftrightarrow \int \frac{dP}{2P-8} = \int dt \Leftrightarrow \frac{\ln|2P-8|}{2} = t + C.$$

Since $P(0) = 5$, we get $C = \frac{\ln 2}{2}$. Hence $\ln|2P - 8| = 2t + \ln 2$ so $|2P - 8| = e^{2t+\ln 2} = 2e^{2t} \Rightarrow |P - 4| = e^{2t}$. Then $|P(\ln 2) - 4| = e^{2\ln 2} = 4$ and since $P > 0$, we obtain $P(\ln 2) = 8$.

4. [2 points, 8.2 #15] Determine whether the series is convergent or divergent. If it is convergent, find its sum.
$\sum_{n=1}^{\infty} 8^{n+1}\, 9^{-n}$

A.∞ **B.** 64 C. 1 D.8 E. 9 F. 25.51

Solution: We have

$$\sum_{n=1}^{\infty} 8^{n+1}9^{-n} = \sum_{n=1}^{\infty} 8\cdot\left(\frac{8}{9}\right)^n = 8\cdot\frac{8}{9} + 8\cdot\left(\frac{8}{9}\right)^2 + 8\cdot\left(\frac{8}{9}\right)^3 + \dots$$

This series is a geometric series with first term $a = \frac{64}{9}$ and common ratio $r = \frac{8}{9}$. Since $|r| < 1$, the series converges and we have

$$\sum_{n=1}^{\infty} 8\cdot\left(\frac{8}{9}\right)^n = \frac{a}{1-r} = \frac{\frac{64}{9}}{\frac{1}{9}} = 64.$$

5. [2 points, 8.4 #11,21,23] Which series among three series below are convergent?

(1) $\sum_{n=1}^{\infty} 2^n$ n^{-4} (2) $\sum_{n=1}^{\infty}$ (-1)$^{n+1}$ /$\sqrt{n}$ (3) $\sum_{n=1}^{\infty}$ n$^{-1.1}$
A.(1) and (2) **B. (2) and (3)** C. only (2) D. all E. only (1) F.(1) and (3)

Solution:
(1) The series $\sum_{n=1}^{\infty} 2^n n^{-4}$ diverges. Indeed, let $a_n = \frac{2^n}{n^4}$. Then

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^4}}{\frac{2^n}{n^4}} = \lim_{n\to\infty} 2\frac{(n+1)^4}{n^4} = 2$$

Since $L > 1$, the series diverges by the Ratio Test.
(2) The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges. Indeed, if $b_n = \frac{1}{\sqrt{n}}$ for all n, then we have:

$$b_{n+1} = \frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} = b_n, \text{ for all } n, \text{ and } \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Therefore the series converges by the Alternating Series Test.
(3) The series $\sum_{n=1}^{\infty} n^{-1.1}$ converges. Indeed, this series is the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 1.1$. Since $p = 1.1 > 1$, this series converges.

6. [2 points, 8.5 #6,10,17] Find the radius of the convergence R of the following power series.

(1) $\sum_{n=1}^{\infty} n^2($ $2x-1)^n/$ 5^n
A. ∞ **B.5/2** C.3 D. 1 E 2/5 F 1/2

Solution: The series

$$\sum_{n=1}^{\infty} \frac{n^2(2x-1)^n}{5^n} = \sum_{n=1}^{\infty} \frac{n^2 2^n}{5^n}\left(x-\frac{1}{2}\right)^n$$

is a power series about $a = \frac{1}{2}$. Let $a_n = \frac{n^2 2^n}{5^n}\left(x - \frac{1}{2}\right)^n$. Then

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|(n+1)^2 \cdot \left(\frac{2}{5}\right)^{n+1} \cdot \left(x - \frac{1}{2}\right)^{n+1} \cdot \frac{1}{n^2} \cdot \left(\frac{5}{2}\right)^n \cdot \frac{1}{\left(x-\frac{1}{2}\right)^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{2}{5}\left(x - \frac{1}{2}\right)\frac{(n+1)^2}{n^2}\right| = \left|\frac{2}{5}\left(x - \frac{1}{2}\right)\right|.\end{aligned}$$

By applying the Ratio test, the series converges if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Leftrightarrow \left|x - \frac{1}{2}\right| < \frac{5}{2}.$$

It then follows that the radius of convergence is $\frac{5}{2}$.

7. [3 points, 7.5 #3,8] Some fish population P(t) in a lake can be modelled by the logistic equation. The maximal population
for the fish of this species in that lake is 6000. The lake was stocked with 500 fish of this species and this number doubled in 6 months
Knowing that the solution of the logistic equation has a form: P(t)= K/(1+A e $^{-kt}$) :
a) Find the constants K and A.
b) Find the constant k using a year as a unit of time .
c) Find the estimated fish population in the lake after two years rounded to the closest integer.
(*Provide a detailed solution and draw a box around the final answers.*)
Solution:
a)K is the carrying capacity, so $K = 6000$. Since $P(0) = 500$, we have

$$500 = \frac{6000}{1+A} \Leftrightarrow A = 11.$$

b) We know that

$$P\left(\frac{1}{2}\right) = 1000 \Leftrightarrow \frac{6000}{1+11e^{-\frac{k}{2}}} = 1000 \Leftrightarrow e^{-\frac{k}{2}} = \frac{5}{11} \Leftrightarrow k = -2\ln\left(\frac{5}{11}\right) = 1.576$$

c)The fish population in the lake after two years will be

$$P(2) = \frac{6000}{1+11e^{-3.152}} \simeq 4080$$