YORK UNIVERSITY

Faculty of Science and Engineering
MATH 1505 6.0 G
Term-Test #4

February 20, 2008

Surname (print): ____________________ Given Name: ______________

Student No: ______________________ Signature: ______________

Solutions

INSTRUCTIONS:

1. Please write your name, student number and final answers in ink.
2. This is a closed-book test, duration – 50 minutes.
3. Non-programmable, non-graphing calculators are permitted.
4. The test has six pages. It consists of five questions. Read the instructions carefully. Fill in answers in designated spaces, or in multiple choice questions, circle the correct answer(s). Your work must justify the answer you give. To get part marks show your work on the space provided. If you need more space, use the back of the page (indicate this fact on the original page).
5. Remain seated until we collect all the test papers.
6. Do the easiest questions first, GOOD LUCK!

Question	Points	Scored
1	6	
2	12	
3	7	
4	10	
5	10	
Total:	45	

Name: ____________________ Student No: ____________

1. (6 pts) Let $y = f(x)$ be a function whose derivative $f'(x) = (x-5)x^2(x+1)^3(2x-3)^5$. Classify each critical point of $y = f(x)$ as a relative maximum, relative minimum, or neither.

$f'(x) = 0 \iff x = -1$, or $x = 0$, or $x = \frac{3}{2}$, or $x = 5$.

ANSWER:

Sign of $f'(x)$: $- - -$ | -1 | $+ + +$ | 0 | $+ + +$ | $\frac{3}{2}$ | $- - -$ | 5 | $+ + +$

$f(x)$: $\downarrow$ st | -1 | $\uparrow$ st | 0 | $\uparrow$ st | $\frac{3}{2}$ | $\downarrow$ st | 5 | $\uparrow$ st

Hence, $(-1, f(-1))$ is a relative min point,
$(0, f(0))$ is neither,
$(\frac{3}{2}, f(\frac{3}{2}))$ is a relative max point,
and $(5, f(5))$ is a relative min point.

2. (4 + 4 + 4 pts) Given a function $y = f(x) = \dfrac{x^2+2x+1}{x^2}$, and its first and second derivatives: $f'(x) = -\dfrac{2x+2}{x^3}$ and $f''(x) = \dfrac{4x+6}{x^4}$.

(a) Determine intervals on which the function is strictly increasing, strictly decreasing. Find the coordinates of all relative maximum and minimum points (if any).

$f'(x)$ is undefined when $x = 0$,
but $x = 0$ is not in the domain of $f(x)$.
$f'(x) = 0 \iff 2x + 2 = 0 \iff x = -1$ is a critical point. $f(-1) = 0$.
So, $(-1, 0)$ is a possible extremum point of $f(x)$.

ANSWER:

Sign of $f'(x)$: $- - -$ | -1 | $+ + +$ | 0 | $- - -$

$f(x)$: $\downarrow$ st | -1 | $\uparrow$ st | 0 | $\downarrow$ st

Thus, $f(x)$ $\downarrow$ st on $(-\infty, -1) \cup (0, \infty)$
and $f(x)$ $\uparrow$ st on $(-1, 0)$.
Therefore, $(-1, 0)$ is a relative min point.

Continues...

Name: ____________ Student No: ____________

(b) Determine intervals on which the function is concave up, concave down. Find the coordinates of all inflection points (if any).

$f''(x)$ is undefined when $x=0$,

but $0 \notin Dom(f)$.

ANSWER: ____________

$f''(x) = 0 \iff 4x+6=0 \iff x = -\frac{3}{2}$.

So, $\left(-\frac{3}{2}, f(-\frac{3}{2})\right) = \left(-\frac{3}{2}, \frac{1}{9}\right)$ is a possible inflection point.

Sign of $f''(x)$	$- - -$	$+ + +$	$+ + +$
	$-\frac{3}{2}$		0
$f(x)$	CAP	CUP	CUP

Hence, $f(x)$ is concave down at $(-\infty, -\frac{3}{2})$,

concave up at $(-\frac{3}{2}, 0) \cup (0, \infty)$.

Therefore, $(-\frac{3}{2}, \frac{1}{9})$ is inflection point.

(c) Sketch the graph of the function indicating horizontal and vertical asymptotes, and intercept(s).

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = 1$$

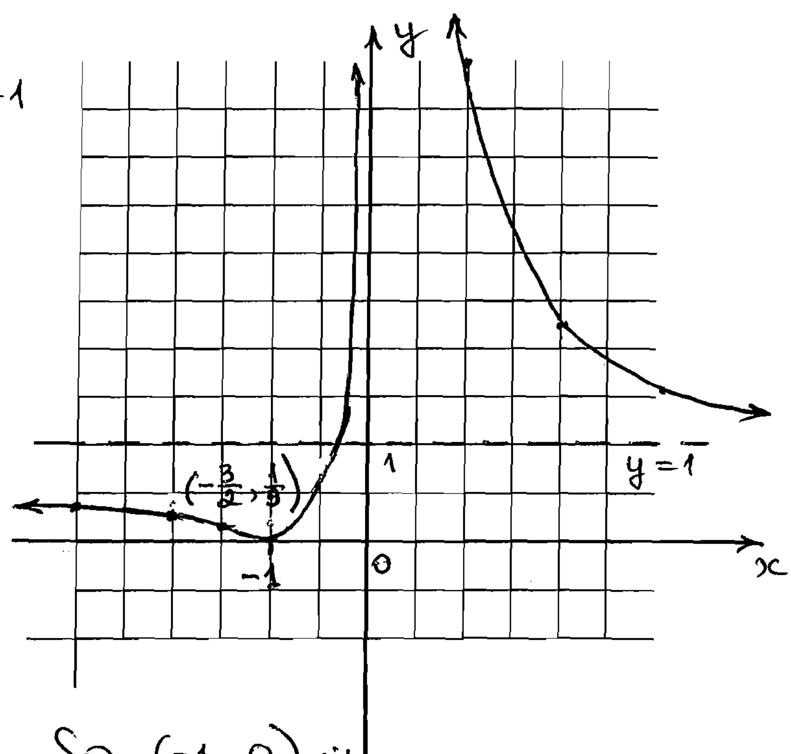
So, $y=1$ is a horizontal asymptote.

$$\lim_{x\to 0} f(x) = \infty,$$

So, $x=0$ is a vertical asymptote.

$f(0)$ is undefined.

So, there is no y-intercept.



$f(x)=0 \iff x=-1$. So, $(-1,0)$ is the x-intercept.

Continues...

Note that $(-1,0)$ is also an absolute minimum point.

Name: ____________________ Student No: ______________

3. (7 pts) Find the largest possible area of a right triangle whose hypothenuse is 16 cm long.

ANSWER: $64\ cm^2$

Let x be the base and h be the height of the right triangle.
Then the area of the triangle

$$A = \frac{1}{2}xh, \text{ where } x^2+h^2=16^2$$

So, $h^2 = 16^2 - x^2 = 256 - x^2$ and $h = \sqrt{256-x^2}$.

$$h>0 \Rightarrow 256-x^2>0 \Rightarrow 0<x<16.$$

Therefore, we have to maximize

$$A(x) = \frac{1}{2}x\sqrt{256-x^2} = \sqrt{64x^2 - \frac{x^4}{4}}$$

on the interval $0<x<16$.

$$A'(x) = \frac{128x - x^3}{2\sqrt{64x^2-\frac{x^4}{4}}} = 0 \Leftrightarrow x(128-x^2)=0$$

$$\Leftrightarrow x=0 \text{ or } x=\sqrt{128}=8\sqrt{2}.$$

So, $x=8\sqrt{2}$ is a critical value of $A(x)$ on the interval $(0,16)$.

Since $A'(x)>0 \quad \forall x\in(0,8\sqrt{2})$,

$A'(x)<0 \quad \forall x\in(8\sqrt{2},16)$

and $\lim_{x\to 0^+} A(x) = \lim_{x\to 16^-} A(x) = 0$,

$A(x)$ attains its absolute maximum value of

$$A(8\sqrt{2}) = \sqrt{64(8\sqrt{2})^2 - \frac{(8\sqrt{2})^4}{4}} = \sqrt{(64)^2\cdot 2 - (64)^2} = 64$$

at $x=8\sqrt{2}$.

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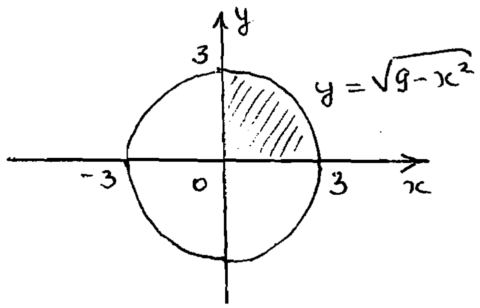
Name: ____________________ Student No: ______________

4. (3 + 3 + 4 pts)

(a) Use the geometric interpretation of definite integrals to evaluate $\int_0^3 \sqrt{9-x^2}\,dx$.



ANSWER: $\frac{9\pi}{4}$

$x^2+y^2=9 \Rightarrow y=\pm\sqrt{9-x^2}$

So, $\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\left(\begin{array}{c}\text{area of a circle}\\ \text{of radius } 3\end{array}\right)$

$= \frac{1}{4}\pi \cdot 3^2 = \frac{9\pi}{4}$.

(b) Given $\int_0^3 f(x)\,dx = -3$ and $\int_0^5 f(x)\,dx = 2$. Evaluate $\int_5^3 f(x)\,dx$.

ANSWER: -5

$\int_5^3 f(x)\,dx = -\int_3^5 f(x)\,dx = -\left(\int_0^5 f(x)\,dx - \int_0^3 f(x)\,dx\right)$

$= -(2-(-3)) = -5$

(c) Use the properties of definite integrals to proof the inequality

$$\int_0^{\frac{\pi}{4}} \tan^2 x\,dx \le \int_0^{\frac{\pi}{4}} \tan x\,dx\,.$$

$0 \le \tan x \le 1$, whenever $0 \le x \le \frac{\pi}{4}$.

So, $\tan^2 x \le \tan x$, whenever $0 \le x \le \frac{\pi}{4}$.

Therefore, $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx \le \int_0^{\frac{\pi}{4}} \tan x\,dx$.

Continues...

Name: ____________________ Student No: ______________

5. (3 + 3 + 4 pts)

(a) Find the derivative of $F(x)$, if $F(x) = \int\limits_{1}^{\ln x} e^{-t}\,dt, \;\; x > 0.$

Hint: Use Leibniz's Rule.

ANSWER: $\frac{1}{x^2}$

$$F'(x) = \frac{d}{dx}\left(\int_1^{\ln x} e^{-t}dt\right) =$$

$$= e^{-\ln x}\frac{d}{dx}(\ln x) = e^{\ln \frac{1}{x}} \cdot \frac{1}{x} = \frac{1}{x}\cdot\frac{1}{x} = \frac{1}{x^2}$$

(b) $\int \frac{2x-3}{x}\,dx =$ $2x - 3\ln|x| + C$

$$\int \frac{2x-3}{x}dx = \int\left(\frac{2x}{x} - \frac{3}{x}\right)dx = \int\left(2 - \frac{3}{x}\right)dx$$

$$= \int 2dx - \int \frac{3}{x}dx = 2\int dx - 3\int \frac{1}{x}dx$$

$$= 2x - 3\ln|x| + C$$

(c) $\int\limits_{\frac{\pi}{8}}^{\frac{\pi}{6}} \sec(2x)\tan(2x)\,dx =$ $1 - \frac{\sqrt{2}}{2}$

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \sec(2x)\tan(2x)dx = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \sec(2x)\tan(2x)\,2\,dx$$

$$= \frac{1}{2}\sec(2x)\Big]_{\frac{\pi}{8}}^{\frac{\pi}{6}} = \frac{1}{2}\left[\sec\left(\frac{\pi}{3}\right) - \sec\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left(2 - \frac{2}{\sqrt{2}}\right) = 1 - \frac{\sqrt{2}}{2}.$$

The end.