MAT 2377 3X (Spring 2011)

§3.5 Discrete Uniform Distribution

Definition : A random variable has a discrete uniform distribution if it has a finite range $R_X = \{x_1, \ldots, x_n\}$ and that

$$f(x_i) = \frac{1}{n}, \quad \text{pour } i = 1, \ldots, n.$$

Special Case : If X has a discrete uniform distribution on the consecutive integers $a, a+1, a+2, \ldots, b$, then

$$\mu = E[X] = \frac{a+b}{2}$$

and

$$\sigma^2 = V(X) = \frac{(b-a+1)^2 - 1}{12}.$$

Example 5 : Suppose that the number of employees that are absent for medical reasons in the month of January has a discrete uniform distribution on 0 to 5 employees. Let X the number of employees that are absent due to medical reasons in January. Compute the mean and the standard deviation of X.

§3.6 **Binomial Distribution**

Définition : A *Bernoulli trial* is a random experiment with two possible outcomes that we will call "success" and "failure". The probability that a success will occur is p.

Definition : A *binomial experiment* consists of n independent Bernoulli trials, each having the same probability of success.

Definition : Let X the number of successes for a binomial experiment of n Bernoulli trials, each with a probability p of success. We say that X has a binomial distribution with parameters n and p.

1. **Notation :** $X \sim B(n, p)$ means that X has a binomial distribution with the parameters n and p.
2. The p.m.f. for a binomial random variable is

$$f(x) = P(X = x) = \left(\begin{array}{c} n \\ x \end{array} \right) p^x \, (1 - p)^{n-x},$$

   for $x = 0, 1, \ldots, n$

3. $\mu = E[X] = n\,p$ and $\sigma^2 = V(X) = n\,p\,(1 - p)$
4. The c.d.f. for $X \sim B(n, p)$ is found in Table II for certain values n and p.

Example 6 : Example 1 : During one stage in the manufacture of integrated circuit chips, a coating must be applied. If 70% of chips receive a thick enough coating, then compute the probabilities that, among 15 chips :

(a) exactly 10 will have thick enough coatings ;

(b) at most 2 will have thick enough coatings ;

(c) at least 14 will have thick enough coatings ;

Example 7 : Consider the random experiment from Example 6.
(a) What is the expected number of chips with a thick enough coating among 15 chips ?
(b) Compute the standard deviation of the number of chips with a thick enough coating among 15 chips ?

§3.7.1 **Geometric Distribution**

Definition : Let X be the number of independent Bernoulli trials required to observe a success. We assume that each trial has the same probability of success. We say that X has a geometric distribution with parameter p.

1. The p.m.f. for the geometric distribution is

$$f(x) = P(X = x) = (1-p)^{x-1}\, p, \quad \text{for } x = 1, 2, 3, \ldots$$

2. Its mean is

$$\mu = E(X) = \frac{1}{p}$$

3. Its variance is

$$\sigma^2 = V(X) = \frac{1-p}{p^2}$$

4.

$$\begin{aligned} F(x) &= P(X \le x) \\ &= 1 - P(X > x) \\ &= 1 - P(\text{“the first } x \text{ trials are failures” }) \\ &= 1 - (1-p)^x, \end{aligned}$$

for $x = 1, 2, 3, \ldots$

Example 8 : Electronic components are shipped to a supplier in boxes of 10. The supplier will return a box to the manufacturer only if it contains more than one defective component. Suppose that 5% of the components are defective.

(a) What is the probability that the 10th box received by the supplier will be the first that is returned to the manufacturer ?

(b) What is the expected number of boxes that the manufacturer will have received when it first returns a box to the supplier ?

(c) What is the probability that more than 10 boxes will be received by the supplier before the first box is returned ?

§3.7.2 **Negative Binomial**

Definition : Let X be the number of independent Bernoulli trials required to observe r successes. Each trial has the same probability of success. We say that X has a negative binomial distribution with parameters p and r.

1. The p.m.f. of the negative binomial distribution is

$$f(x) = P(X = x) = \binom{x-1}{r-1} (1-p)^{x-r} \, p^r,$$

   for $x = r, r+1, r+2, \ldots$

2. Its mean is

$$\mu = E(X) = \frac{r}{p}$$

3. Its variance is

$$\sigma^2 = V(X) = \frac{r(1-p)}{p^2}$$

Example 9 : Electronic components are shipped to a supplier in boxes of 10. The supplier will return a box to the manufacturer only if it contains more than one defective component. Suppose that 2.5% of boxes contain more than one defective component. What is the probability that the 10th box received by the supplier will be the 3rd that is returned to the manufacturer ?

§3.9 **The Poisson Distribution**

Definition : A random variable X is said to have a Poisson distribution with parameter $\lambda > 0$ if it has the following probability mass function :

$$f(x) = P(X = x) = e^{\lambda}\,\frac{\lambda^x}{x!}, \quad \text{for } x = 0, 1, 2, \ldots$$

Mean and Variance : The mean and variance of a Poisson random variable X are equal to the parameter λ, that is

$$\mu = E[X] = \lambda \quad \text{and} \quad \sigma^2 = V(X) = \lambda.$$

Poisson approximation to the binomial distribution :
If X is a binomial random variable with n large and p small then the binomial p.m.f. may be approximated by the Poisson p.m.f. with $\lambda = n\,p$:

$$P(X = x) = \binom{n}{x} p^x\,(1-p)^{n-x} \approx e^{-\lambda}\,\frac{\lambda^x}{x!},$$

where $\lambda = n\,p$.

Rule of thumb :
The approximation is good, if $n \geq 20$ and $p \leq 0.05$.
It is very good if $n \geq 100$ and $np \leq 10$.

Example 10 : In a given city, 3% of all drivers get at least one parking ticket per year. Approximate the probabilities that among 80 drivers (randomly chosen in this city) :

(a) 3 will get at least one parking ticket in any given year ;

(b) at least 3 will get at least one parking ticket in any given year ;

(c) anywhere from 2 to 5, inclusive, will get at least one parking ticket in any given year.

Poisson Process

Note : We have already seen that the Poisson Distribution can be used to approximate a binomial distribution when n is large and p is small. Below we will encounter another application of the Poisson distribution.

Definition : Let the number of "changes" that occur in a continuous interval (of time or space) be counted. We have a **Poisson process** of rate α if :

1. The number of changes occuring in nonoverlapping intervals are independent.
2. The probability of exactly one change in a short interval of length h is approximately $\alpha\, h$.
3. The probability of two or more changes in a sufficiently short interval is essentially 0.

Counting "changes" in an interval of fixed length : Let X be the number of changes in an interval of length t in a Poisson process of rate α. Then, X has a Poisson distribution with parameter λ, where $\lambda = \alpha\, t$.

Example 11 : Assume that the calls to a call center occur according a Poisson process. If on average there are 6 calls per hour, what is the probability that

(a) there are no calls in one hour ?
(b) there are at most 2 calls in 30 minutes ?
(c) there are exactly 10 calls in 2 hours ?

Example 12 : Errors on a data tape occur at a rate of one error per 35,000 centimeters. Assume that errors occur as a Poisson process.

(a) Give the distribution of X, where X is the number of errors in 52,500 centimeters ?
(b) Compute the probability that there is at least on error in 52 500 centimeters ?