

## Probability - Part I

**Definition :** A *random experiment* is an experiment or a process for which the outcome cannot be predicted with certainty.

**Definition :** The *sample space* (denoted  $S$ ) of a random experiment is the set of all possible outcomes.

### Example 1:

Here are examples of random experiments. Give the corresponding sample space.

1. selection of a plastic component and verification of its compliance
2. lifetime of a computer
3. number of calls to a communication system during a fixed length interval of time
4. The selection of two tools **without replacement** among a box of three tools.
5. The selection of three tools **with replacement** among a box of two tools.

**Notation:** Let  $A$  be a set. The notation  $x \in A$  means that  $x$  belongs to  $A$ .

**Definition:** An event is a subset of the sample space. We say that  $E$  has occurred if the observed outcome  $x$  is an element of  $E$ , that is  $x \in E$ .

### Remarks [A few special events]:

$S$  is called the *certain* event.

$\emptyset$  (the empty set) is called the *impossible* event.

## Operations on events

**Note:** The following operations allow us to represent some events in terms of other events.

### Union:

- a)  $E_1 \cup E_2$  occurs means  $E_1$  occurs, or  $E_2$  occurs, or both occur.
- b)  $E_1 \cup E_2 \cup \dots \cup E_n$  occurs means that **at least** one of the events  $E_1, E_2, \dots, E_n$  occurs.

### Intersection:

- a)  $E_1 \cap E_2$  occurs means  $E_1$  occurs **and**  $E_2$  occurs.
- b)  $E_1 \cap E_2 \cap \dots \cap E_n$  occurs means that **all** of the events  $E_1, E_2, \dots, E_n$  occur.

### Complement:

$E'$  occurs means that  $E$  **does not occur**.

### DeMorgan Laws:

a)

$$\begin{aligned} & (E_1 \cup E_2 \cup \dots \cup E_n)' \text{ occurs} \\ &= \text{none of the events } E_1, E_2, \dots, E_n \text{ occur} \\ &= E'_1 \cap E'_2 \cap \dots \cap E'_n \text{ occurs} \end{aligned}$$

b)

$$\begin{aligned} & (E_1 \cap E_2 \cap \dots \cap E_n)' \\ &= \text{at least one of the events } E_1, E_2, \dots, E_n \text{ occurs} \\ &= E'_1 \cup E'_2 \cup \dots \cup E'_n \text{ occurs} \end{aligned}$$

## Mutually exclusive events:

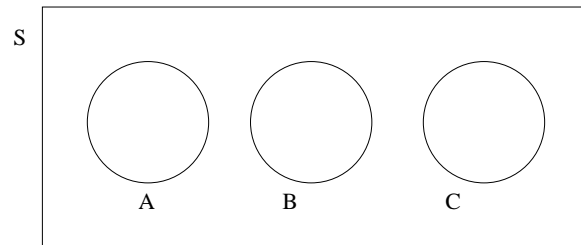
**Definition :** The events in the sequence  $E_1, E_2, \dots$  are said to be *mutually exclusive*, if

$$E_i \cap E_j = \emptyset, \text{ for all } i \neq j,$$

where  $\emptyset$  represents the empty set.

**Note:** In other words, the events are said to be mutually exclusive if they do not have any outcomes (elements) in common, i.e. they are pairwise disjoint.

Here is an illustration of three mutually exclusive events.

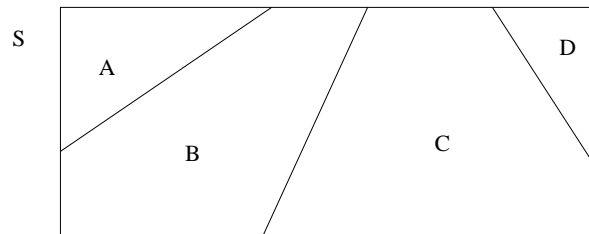


## Exhaustive Events:

**Definition:** The events  $E_1, E_2, \dots, E_n$  are said to be **exhaustive** if

$$E_1 \cup E_2 \cup \dots \cup E_n = S.$$

Here is an illustration of four *mutually exclusive* and *exhaustive* events.



### Interpretation of a Probability:

**Goal:** Define a measure (i.e. to quantify) of the likelihood or the chances that an event  $E$  will occur.

**Note:** We introduce three interpretations of  $P(E)$ , that is the probability that the event  $E$  occurs:

1. Subjective Probability;
2. Equally Likely Model;
3. Relative Frequency Model.

**Subjective Probability:** We associate a real number  $P(E)$  between 0 and 1 in a subjective manner to the event  $E$ . Numbers near 0 are interpreted as less likely and numbers near 1 are interpreted as highly likely.

### Classical Approach :

The concepts of probability have existed for millennia, but the theory of probability arose as a discipline of mathematics only in the 17th century. The first definition of a probability is found below the model is called the equally likely model.

**[Equally Likely Model]:** Consider a random experiment with a finite sample space  $S$  such that each result has the same chance to occur. The probability that  $E$  will occur is

$$P(E) = \frac{N(E)}{N(S)} = \frac{\# \text{ favourable outcomes}}{\# \text{ total possible outcomes}}$$

where  $N(E) = \#$  of outcomes in  $E$ .

**Remark:** We consider a **random selection** of an object among  $N$  objects as an experiment with equally likely outcomes.

**Problems with this model:** We must decompose the experiment into equally likely outcomes.

- It is not always possible to accomplish this decomposition.
- It is not always obvious to recognize if the outcomes should be considered as equally likely.
- It is possible to construct experiments where we should not consider the results as equally likely.

**Frequency Approach:** Consider the selection of a screw. We note if it is compliant to specification or not. There are two possible results: compliant, non-compliant.

**Question:** Should we consider the results as equally likely?

If not, how can we measure the probability that the screw is compliant.

**A solution:** Select a large number of screws and count the number of compliant screws. The proportion of compliant screws among this sample can be used as an approximation of the probability that a screw will be compliant.

**[Relative Frequency Model]:** Consider a random experiment with a sample space  $S$ . We repeat the experiment  $n$  times. The probability that the event  $E$  will occur is

$$P(E) = \lim_{n \rightarrow \infty} \frac{f_n(E)}{n},$$

where  $f_n(E)$  is the number of times (the frequency) that event  $E$  occurs among the  $n$  trials of the experiment.

**Note 1:** This approach allows us to consider experiments with outcomes which are not necessarily equally likely.

**Remark 2:** James Bernoulli has shown that if the outcomes are equally likely, then the equally likely model and the relative frequency model are compatible. (This is known as the law of large numbers).

**Problems:**

- To obtain a probability, we must repeat the experiment an infinite number of times or at least a large number times to obtain a good approximation.
- The definition is not simple enough. We must interpret this result as a result and not a definition.

**Remark:** The modern theory of probability is based on three fundamental principles that are called the axioms of probability theory. The equally likely model and the relative frequency model satisfy these axioms. Thus, the axiomatic approach is more general and any consequences of the axioms are true for both models.

**Axioms of Probability:** Consider an experiment with the sample space  $S$ . For each event  $E$ , we can associate a real number  $P(E)$  such that:

**Positivity :**

(a)  $P(E) \geq 0$ ,

**Certainty :**

(b)  $P(S) = 1$ ,

**Additivity :**

(c) For each sequence of events  $E_1, E_2, \dots$  that are mutually exclusive (that is  $E_i \cap E_j = \emptyset$  (the empty set), if  $i \neq j$ ), we have

$$P(E_1 \cup E_2 \cup \dots) = P(E_1) + P(E_2) + \dots$$

**N.B.**  $P(E)$  is called the probability that  $E$  occurs.

**Theorem:**

1.  $P(\emptyset) = 0$

2.  $E_1 \subseteq E_2 \Rightarrow P(E_1) \leq P(E_2)$ .

3.  $0 \leq P(E) \leq 1$ .

4.  $P(E') = 1 - P(E)$ .

**Addition Rules:**

a)

$$P(A) = P(A \cap B) + P(A \cap B')$$

b)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

c)

$$\begin{aligned} & P(A \cup B \cup C) \\ = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

d)

$$\begin{aligned} & P(E_1 \cup E_2 \cup \dots \cup E_n) \\ = & 1 - P[(E_1 \cup E_2 \cup \dots \cup E_n)'] \\ = & 1 - P(E_1' \cap E_2' \cap \dots \cap E_n') \end{aligned}$$

**Example 2:** The probability that a piece of integrated circuit will have a defective etching is 0.12, the probability that it will have a defective slot is 0.29 and the probability that it will have both defects is 0.07.

(a) What is the probability that a piece of integrated circuit will have a defective etching but not a defective slot.

(b) What is the probability that a piece of integrated circuit will have a defective etching or a defective slot?

(c) What is the probability that it will have neither a defective etching nor a defective slot?