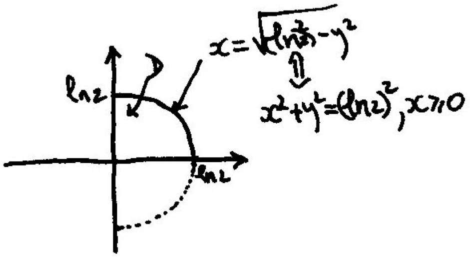
1. [5 marks] Use polar coordinates to evaluate

$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} \exp\left(\sqrt{x^2+y^2}\right)dxdy.$$

Solution. We first sketch the region D of integration determined by the limits of the iterated integrals.



$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} \exp\left(\sqrt{x^2+y^2}\right)dxdy = \iint_D \exp\left(\sqrt{x^2+y^2}\right)dA = \int_0^{\pi/2}\int_0^{\ln 2} e^r r\,drd\theta$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\ln 2} \underbrace{r}_{u}\underbrace{e^r\,dr}_{dv}\right) = \frac{\pi}{2}\left(re^r\Big|_0^{\ln 2} - \int_0^{\ln 2} e^r\,dr\right) = \frac{\pi}{2}\left(2\ln 2 - e^r\Big|_0^{\ln 2}\right)$$

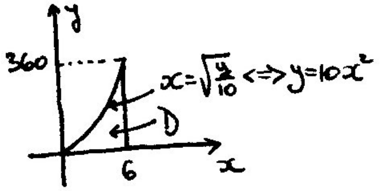
$$= \frac{\pi}{2}(2\ln 2 - 1).$$

□

2. [5 marks] Evaluate the following integral

$$\int_0^{360} \int_{\sqrt{y/10}}^{6} \frac{\sin\left(x^2\right)}{x} dxdy.$$

Solution. We need to change the order of integration. We first sketch the region D of integration determined by the limits of the iterated integrals.



$$\int_0^{360} \int_{\sqrt{y/10}}^{6} \frac{\sin(x^2)}{x}\, dxdy = \iint_D \frac{\sin(x^2)}{x}\, dA = \int_0^6 \int_0^{10x^2} \frac{\sin(x^2)}{x}\, dydx = \int_0^6 \frac{\sin(x^2)}{x} y\Big|_{y=0}^{y=10x^2}\, dx$$

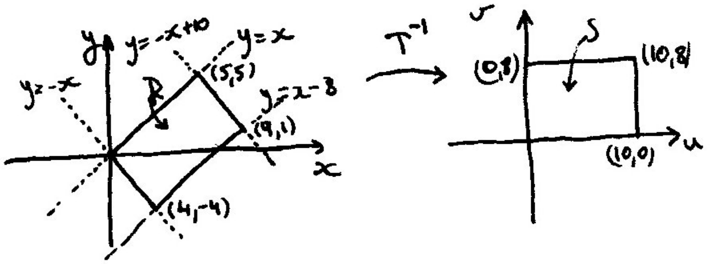
$$= \underbrace{\int_0^6 10x\sin(x^2)\, dx}_{u=x^2} = -5\cos u\Big|_0^{36} = 5(1-\cos 36).$$

□

3. [5 marks] Use the transformation $u = x + y,\ v = x - y$ to evaluate $\iint_R (x + y)\sin(x - y)dxdy$, where R is the parallelogram bounded by the lines $y = x,\ y = x - 8,\ y = -x$, and $y = -x + 10$.

Solution. By finding the points of intersection for the lines determining R we conclude that the vertices of R are $(0, 0)$, $(4, -4)$, $(9, 1)$, $(5, 5)$. We are given that $(u, v) = T^{-1}(x, y) = (x + y, x - y)$. Since T^{-1} is linear we can conclude that $S = T^{-1}(R)$ is a polygon with vertices

$$T^{-1}(0,0) = (0,0) \qquad T^{-1}(4,-4) = (0,8)$$
$$T^{-1}(9,1) = (10,8) \qquad T^{-1}(5,5) = (10,0).$$



To make the change of variables we also need to compute the Jacobian:

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\frac{1}{2}.$$

Alternatively, one can solve for x, y in terms of u, v and compute the Jacobian in the usual way.

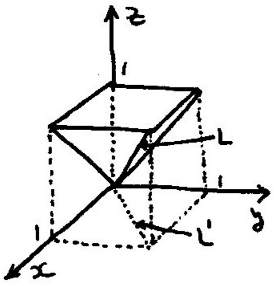
$$\iint\limits_R (x+y)\sin(x-y)\,dA = \iint\limits_S u\sin v\left|-\frac{1}{2}\right|\,dA = \int_0^8\int_0^{10}\frac{1}{2}u\sin v\,dudv$$
$$= \frac{1}{2}\left(\int_0^8 \sin v\,dv\right)\left(\int_0^{10} u\,du\right) = 25(1-\cos 8)$$

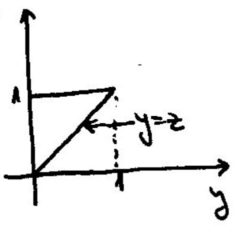
□

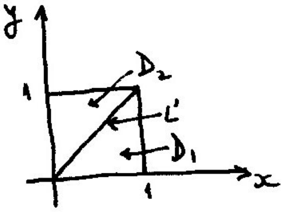
4. [5 marks] Reverse the order of integration in the following integral

$$\int_0^1 \int_0^z \int_0^z f(x,y,z)dxdydz.$$

Solution. We first sketch the solid E, corresponding to the limits of integration, together with its projections onto the yz and xy planes. Note that L is the line of intersection between the planes $z = x$ and $z = y$, and L' is its projection onto the xy plane.







$$\begin{aligned}
&\int_0^1 \int_0^z \int_0^z f(x,y,z)\,dxdydz \\
&\qquad = \iiint_E f(x,y,z)\,dV = \iint_{D_1} \left(\int_x^1 f(x,y,z)\,dz \right) dA + \iint_{D_2} \left(\int_y^1 f(x,y,z)\,dz \right) dA \\
&\qquad\qquad = \int_0^1 \int_0^x \int_x^1 f(x,y,z)\,dzdydx + \int_0^1 \int_x^1 \int_y^1 f(x,y,z)\,dzdydx.
\end{aligned}$$

□

5. [5 marks] Use cylindrical coordinates to evaluate $\iiint_E y dV$, where E is the solid enclosed by the planes $z = 0$, $z = y + 3$ and by the cylinders $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.

Solution.

$$\iiint\limits_E y dV = \int_0^{2\pi} \int_1^2 \int_0^{3+r\sin\theta} (r\sin\theta) r\, dz dr d\theta = \int_0^{2\pi} \int_1^2 r^2 \sin\theta \left(z \Big|_{z=0}^{z=3+r\sin\theta} \right) dr d\theta$$

$$= \int_0^{2\pi} \int_1^2 \left(3r^2 \sin\theta + r^3 \sin^2\theta\right) dr d\theta$$

$$= 3 \underbrace{\left(\int_0^{2\pi} \sin\theta\, d\theta\right)}_{=0} \left(\int_1^2 r^2\, dr\right) + \left(\int_0^{2\pi} \sin^2\theta\, d\theta\right) \left(\int_1^2 r^3\, dr\right)$$

$$= \left(\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta\right) \left(\frac{r^4}{4}\Big|_1^2\right) = \frac{15}{4}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} = \frac{15\pi}{4}.$$

□

6. [5 marks] Use spherical coordinates to evaluate $\iiint_E xyzdV$, where E lies between the spheres $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$ and below the cone $z=-\sqrt{x^2+y^2}$.

Solution.

$$\begin{aligned}
\iiint\limits_E xyzdV &= \int_{3\pi/4}^{\pi}\int_0^{2\pi}\int_1^2 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\, d\rho d\theta d\phi \\
&= \underbrace{\left(\int_{3\pi/4}^{\pi}\sin^3\phi\cos\phi\, d\phi\right)}_{u=\sin\phi}\left(\int_0^{2\pi}\underbrace{\cos\theta\sin\theta}_{=(\sin 2\theta)/2}\, d\theta\right)\left(\int_1^2\rho^5\, d\rho\right) \\
&= \left(-\frac{u^4}{4}\bigg|_{-\sqrt{2}/2}^{0}\right)\underbrace{\left(-\frac{\cos 2\theta}{4}\bigg|_0^{2\pi}\right)}_{=0}\left(\frac{\rho^6}{6}\bigg|_1^2\right) = 0.
\end{aligned}$$

□