MAT 134 A F14 Test 1 sol V1

1. Parametric equations of the line containing $(-5,\ 0,\ 1)$ and which is parallel to the two planes $2x - 4y + z = 0$ and $x - 3y - 2z = 1$ are:

A. $x = 5 + 11t,\ y = 3t,\ z = 1 + 2t,\ t \in \mathbf{R}$
B. $x = -5 + 11t,\ y = -3t,\ z = 1 + 2t,\ t \in \mathbf{R}$
C. $x = 5t,\ y = 0,\ z = t,\ t \in \mathbf{R}$
D. ~~$x = -5 + 5t,\ y = -5t,\ z = 1 - 10t$~~, $t \in \mathbf{R}$
E. $x = -5 + 11t,\ y = 5t,\ z = 1 - 2t,\ t \in \mathbf{R}$
F. $x = -5t,\ y = 0,\ z = t,\ t \in \mathbf{R}$

A direction vector for this line must be perpendicular to both normals above, and hence parallel to their cross product. But

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 1 \\ 1 & -3 & -2 \end{vmatrix} = (11, -(-5), -2) = (11, 5, -2).$$

Since the line must also contain $(-5, 0, 1)$, E is correct.

2. Which two of the following are vector parametric descriptions for the plane with equation $x + y - 2z = 4$?

✗ I. ~~$v = (0,0,0) + s\ (0,2,1) + t\ (2,0,1); s,t \in \mathbf{R}$~~.
✓ II. $v = (4,0,0) + s\ (1,-1,0) + t\ (0,2,1); s,t \in \mathbf{R}$.
III. ~~$v = (4,0,0) + s\ (1,1,1) + t\ (1,1,0); s,t \in \mathbf{R}$~~.
IV. $v = (0,0,-2) + s\ (1,-1,0) + t\ (2,0,1); s,t \in \mathbf{R}$.

the plane does not contain $(0,0,0)$.

A. I & II
B. I & III
C. I & IV
D. II & III
E. II & IV
F. III & V

When $v = P + s v_1 + t v_2$ is the vector parametric description of this plane, both vector v_1 & v_2 must be perpendicular to any normal — $(1,1,-2)$ in this case. Simple computations show that $(1,-1,0)$, $(0,2,1)$, $(1,1,1)$ and $(1,-1,0)$ are $\perp$ to $(1,1,-2)$, but $(1,1,0)$ is not. Hence II & IV are correct

3. An equation for the plane parallel to the x–axis and passing through the points $(2, 1, -1)$ and $(3, 2, 1)$ is:

A. $-3x + 7y - 2z = 3$
B. $x - y = 1$
C. $2y - z = 3$
D. $2x - z = 5$
E. $x + y - z = 4$
F. $x + y + z = 2$

Such a plane, being parallel to the x-axis, has a normal which is $\perp$ to $(1,0,0)$ and the only plane represented here with a normal of the form $(0, a, b)$ is (C). Hence, (C) is correct (you can check its normal is $\perp$ to both $(2,1,-1)$ and $(3,2,1)$ and that it contains $(2,1,-1)$)

4. Find an equation of the plane which passes through the point $(1,\ -7,\ 8)$ and which is perpendicular to the line whose (scalar) parametric equations are:

$$x = 2 + 2t, \quad y = 7 - 4t, \quad z = -3 + t; \quad t \in \mathbf{R}.$$

A. $2x - 4y + z = -38$
B. $-4x + 2y + z = -10$
C. $-4x + 2y + z = 10$
D. $2x - 4y + z = 38$
E. $2x + 7y - 3z = -71$
F. $2x - 4y + z = -28$

A normal for this plane will be the direction vector of the line above, namely $(2,-4,1)$. Since $(1,-7,8)$ belongs to the plane, the RHS of the equation must be

$$(2,-4,1)\cdot(1,-7,8)$$

$$= 2(1)+(-4)(-7) + 1.8 = 38$$

Hence (D) is correct

5. One of the following is an equation for the plane with vector parametric description

$$v = (2,0,3) + s(1,0,1) + t(0,-1,0);\, s,t \in \mathbf{R}.$$

Which is it?

A. $4x - 9y + z = 18$

B. $x + y - 2z = 14$

C. $x - 2y + 2z = 0$

D. $x + 2y - z = 0$

E. $x - z = -1$

F. $9x - 11y + 18z = -40$

Such a plane has normal

$(1,0,1) \times (0,-1,0) = (1,0,-1)$.

Hence the only possibility is (E),

(which indeed is an equation

for a plane containing $(2,0,3)$).

6. Find the polar form of

$$\frac{z_1}{z_2} = \frac{1-\sqrt{3}i}{-1+i} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

A. $\sqrt{2}(\cos(-7\pi/12) + i\sin(-7\pi/12))$

B. $\sqrt{2}(\cos(5\pi/12) + i\sin(5\pi/12))$

C. $\sqrt{2}(\cos(-\pi/12) + i\sin(-\pi/12))$

D. $\sqrt{2}(\cos(\pi/12) + i\sin(\pi/12))$

E. $\sqrt{2}(\cos(-5\pi/12) + i\sin(-5\pi/12))$

F. $\sqrt{2}(\cos(11\pi/12) + i\sin(11\pi/12))$

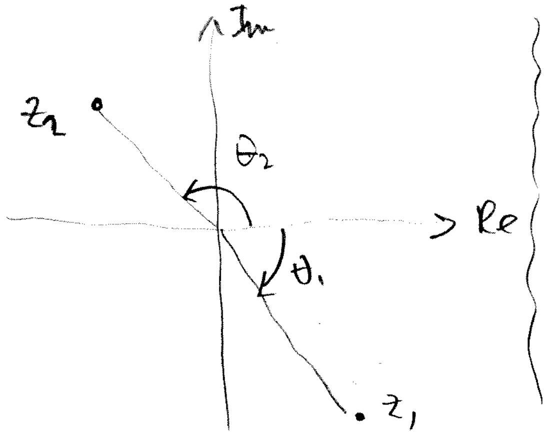
$r_1 = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\cos\theta_1 = \frac{1}{2}$; $\sin\theta_1 = \frac{-\sqrt{3}}{2}$; $\therefore \theta_1 = -\pi/3$

$r_2 = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\cos\theta_2 = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$; $\sin\theta_2 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\therefore \theta_2 = 3\pi/4$.



Hence $\theta_1 - \theta_2 = \pi\left(-\frac{1}{3} - \frac{3}{4}\right) = \frac{-13\pi}{12}$.

To put the argument between $-\pi$ & π, add 2π to obtain $\frac{11\pi}{12}$; As $\frac{r_1}{r_2} = \sqrt{2}$, (F) is correct.

7. What is the area of the triangle with vertices $\overset{A}{(3,\ 0,\ -2)}$, $\overset{B}{(5,\ 2,\ -1)}$ and $\overset{C}{(5,\ 9,\ 0)}$?

A. 13/2
B. 15/2
C. 17/2
D. 10
E. 13
F. 15

This area is $\frac{1}{2}\|(B-A)\times(C-A)\|$

$$= \frac{1}{2}\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 2 & 9 & 2 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\|(-5,-2,14)\|$$

$$= \frac{1}{2}\sqrt{25+4+196}$$

$$= \frac{1}{2}\sqrt{225}$$

$$= \frac{15}{2}$$

$$\begin{array}{r} 14 \\ 14 \\ \hline 56 \\ 140 \\ \hline 196 \end{array}$$

$225 = 5\cdot 45$
$= \underbrace{5\cdot 5}_{5^2}\cdot \underbrace{9}_{3^2}$
$= 15^2$

8. Let L be the line passing through $\overset{A}{(1,\ 1,\ 0)}$ and $\overset{B}{(2,\ 3,\ 1)}$. The point of intersection of L with the plane $x+y-z=1$ is:

A. (1/2, 1/2, 0)
B. (0, 1/2, −1/2)
C. (0, 1, 0)
D. (1/2, 0, −1/2)
E. (1, 0, 0)
F. (−1, 0, −1)

A direction vector for this line is $B-A = (1,2,1)$. Hence scalar parametric eqns for L are

$$\begin{aligned} x &= 1+t \\ y &= 1+2t \ ; \ t\in\mathbb{R} \\ z &= 0+t \end{aligned}$$

Hence, $x+y-z=1 \iff (1+t)+(1+2t)-t=1$
$\iff 2t=-1 \iff t=-1/2$; so $(x,y,z)=\left(\frac{1}{2},0,-\frac{1}{2}\right)$

9. Express the following complex numbers in the form $a + bi$:

$\frac{1}{3} = \frac{\bar{3}}{|3|^2}$ $(3+0)$

$z_1 = \dfrac{i}{-1+i}$ $= \dfrac{i \cdot (-1-i)}{2} = \dfrac{1-i}{2}$

$= \frac{1}{2} - \frac{1}{2}i$

$z_2 = (2+i)(1+i)$

$= (2-1) + 3i = 1+3i.$

A. $z_1 = \dfrac{1}{2} + \dfrac{1}{2}i$; $z_2 = 1 - 3i$

B. $z_1 = \dfrac{1}{2} - \dfrac{1}{2}i$; $z_2 = 1 + 3i$

C. $z_1 = 1 - i$; $z_2 = 2 + 2i$

D. $z_1 = -1 + i$; $z_2 = 1 + 2i$

E. $z_1 = 2 - \dfrac{1}{4}i$; $z_2 = 3 - i$

F. $z_1 = 1 - i$; $z_2 = 2$

Hence B is correct.

10. If $v = (2,\ 2,\ 2)$ and $u = (-1, 0, -1)$ then $\text{proj}_u\, v =$

$\dfrac{u \cdot v}{\|u\|^2}\, u$

$= \dfrac{-4}{2} \cdot (-1, 0, -1)$

$= (2, 0, 2)$

A. $\frac{4}{9}(2, 1, 2)$

B. $\frac{12}{7}(3,\ 3,\ 3)$

C. $\frac{4}{3}(2, 1, 2)$

D. $(2, 0, 2)$

E. $\frac{\sqrt{2}}{2}(1, 0, 1)$

F. $\frac{11}{7}(3,\ 3,\ 3)$

11. Find the volume of the parallelepiped determined by the vectors $u = (1,\ 1,\ -1)$, $v = (2,\ 0,\ 1)$ and $w = (1,\ -1,\ 3)$.

A. 6
B. 8
C. 16
D. 2
E. 4
F. -2

This volume is $|u \cdot v \times w|$

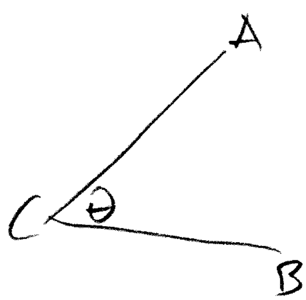
But $v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{vmatrix} = (1, -5, -2).$

Hence vol $= |(1,1,-1) \cdot (1,-5,-2)|$

$= |1 - 5 + 2| = |-2| = 2$

12. If $A = (1, 2, 1)$, $B = (2, 2, 1)$ and $C = (2, 2, 2)$, find the angle $\angle ACB$.

A. $\pi/4$
B. $\pi/6$
C. $3\pi/4$
D. $4\pi/3$
E. $\pi/2$
F. $\pi/3$



$$\cos\theta = \frac{(A-C) \cdot (B-C)}{\|A-C\|\ \|B-C\|} = \frac{(-1,0,-1) \cdot (0,0,-1)}{\sqrt{2} \cdot 1} = \frac{\sqrt{2}}{2}$$

$\therefore \theta = \pi/4.$